1. $X \sim \text{Geometric} \ (p)$ if
\[
P[X = k] = (1-p)^{k-1}p \quad \text{for } k = 1, 2, \ldots
\]
"$k-1$ failures, before a success on the $k$th trial"

\[E(X) = \frac{1}{p}, \quad \text{Var}(X) = \frac{1-p}{p^2}.
\]

2. $X \sim \text{Negative Binomial} \ (r, p)$

\[
\begin{array}{c}
\text{HITITITITITITITIT} \\
\text{n flips}
\end{array}
\]
\[
\begin{array}{c}
r \text{ heads} \\
(n-r) \text{ tails.}
\end{array}
\]

\[
P(X=n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}
\]

\[E[X] = \frac{r}{p}, \quad \text{Var}(X) = \frac{r(1-p)}{p^2}.
\]

3. $X \sim \text{Hypergeometric} \ (N, m, n)$

Imagine an urn with $N$ balls, $m$ are white, $N-m$ are blue, randomly choose $n$ of them.

\[
P[W=k] = \binom{m}{k} \binom{N-m}{n-k} \binom{N}{n}
\]

\[E[W] = \frac{mn}{N}, \quad \text{Var}(W) = \frac{mn}{N} \left( \frac{(m-1)(n-1)}{N-1} + \frac{N-mn}{N} \right).
\]
Sums of Random Variables (§4.9)

Consider two random variables $X, Y$.
Then $Z = X + Y$ is also a random variable.

Let $s \in S$ be an individual outcome in the sample space,

$X(s) =$ the value that $X$ takes when $s$ occurs,

$Y(s) = \ldots$

$\Rightarrow Z(s) = X(s) + Y(s)$.

$$E[X] = \sum_i x_i \cdot P[X = x_i]$$

For each $i$, let

$S_i =$ set of outcomes for which $X = x_i$

$= \{ s \in S : X(s) = x_i \}$

$p(s) = P[S]$}

$$= \sum_i \sum_{s \in S_i} x_i \cdot P(s)$$

$$= \sum_{i} \sum_{s \in S_i} x(s) \cdot p(s)$$

$$= \sum_{s \in S} x(s) \cdot p(s)$$

Since one could show that $S_i$ are mutually exclusive.

By the previous calculation,

$$E[Z] = \sum_{s \in S} Z(s) \ p(s) = \sum_{s \in S} (X(s) + Y(s)) \ p(s) = \sum_{s \in S} X(s) \ p(s) + \sum_{s \in S} Y(s) \ p(s) = E[X] + E[Y].$$

---

Cumulative Distribution Function

$$F(x) = P[X \leq x].$$

1. $F$ is non-decreasing $\Rightarrow$ if $a \leq b$, then $F(a) \leq F(b)$.

2. $\lim_{b \to \infty} F(b) = \lim_{b \to \infty} P[X \leq b] = 1$

3. $\lim_{a \to -\infty} F(a) = 0$

4. $F$ is right continuous.

If $b_n \to b$, with $b_{n+1} \leq b_n$, then $\lim_{n \to \infty} F(b_n) = F(b)$.

$P[X=0] = \frac{1}{2}$

$P[X=1] = \frac{1}{2}$