Theory of Probability Oct 19, 2020

Poisson random variables:

$X \sim \text{Poisson}(\lambda)$

$$P[X=k] = e^{-\lambda}\frac{\lambda^k}{k!}$$

Key approximation: Poisson(λ) is approximately the same as binomial(n,p) when n is large and p is small, and $\lambda = np$ is $O(1)$.

If $X \sim \text{binomial}(n,p)$ then

$$P[X=k] = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$\lim_{n\to\infty}\left(1-\lambda/n\right)^n = e^{-\lambda}$$

$$= \frac{n!}{(n-k)!\,n^k}\,\frac{\lambda^k}{k!}\,\frac{(1-\lambda/n)^n}{(1-\lambda/n)^k}$$

$$= \underbrace{\frac{n(n-1)(n-2)\cdots(n-k+1)}{n\;n\;n\cdots n}}_{\approx 1}\,\frac{\lambda^k}{k!}\,\frac{(1-\lambda/n)^n}{\underbrace{(1-\lambda/n)^k}_{\approx 1}}$$

$$\approx e^{-\lambda}\frac{\lambda^k}{k!}$$ Poisson probability mass function.

If $X \sim \text{Binomial}(n, p)$

$$E[X] = np$$
$$Var[X] = np(1-p)$$

Conjecture: If n is large, p is small, $\lambda = np \sim O(1)$,

then if $Y \sim \text{Poisson}(\lambda)$, then

$$E[Y] \approx E[X] = \lambda$$
$$Var[Y] \approx Var[X] = np(1-p) = \lambda(1-p) \simeq \lambda.$$

Calculate:

$$E[Y] = \sum_{k=0}^{\infty} k \, e^{-\lambda} \frac{(\lambda)^k}{k!}$$
$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$
$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$
$$= e^{-\lambda} \lambda \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{e^{\lambda}}$$
$$= e^{-\lambda} \lambda e^{\lambda} = \boxed{\lambda}$$

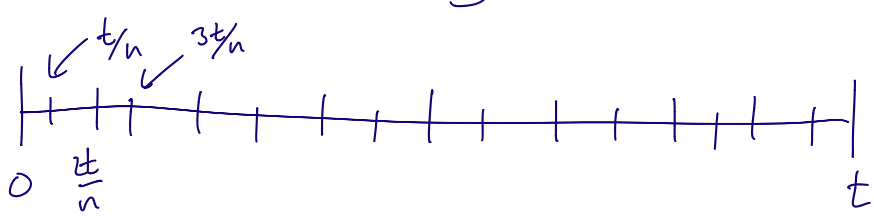
If $P[E|F] = P[E]$ then E, F are independent

If $P[E|F] \approx P[E]$, then we say E, F are only "weakly dependent".

Poisson Paradigm

Consider n trials, with $P[E_i] = p_i$. If n is large, and <u>all</u> the p_i are small, and either the E_i are independent or "weakly dependent", then the sum $E = E_1 + E_2 + \dots + E_n$ is approximately Poisson with parameter $\lambda = \sum_{i=1}^{n} p_i$.

Rare Event Modelling



We are interested in $P[N(t) = k]$?

<u>Assumption 1</u>: n is large enough so that either 1 or 0 events happen in each interval.

$$\Rightarrow P[N = k] = P\left[\begin{matrix} k \text{ subintervals have 1 event} \\ n-k \text{ subintervals have 0 events} \end{matrix}\right]$$

Next, assume that the probability of an event occuring in a subinterval is $\lambda h = \frac{\lambda t}{n} = p$.

$$\Rightarrow P\left[\begin{matrix} \text{any subinterval} \\ \text{having 0 events} \end{matrix}\right] = 1 - p = 1 - \lambda h.$$

<u>Assumption 2</u>: Independence between subintervals.

$\Rightarrow N \sim \text{binomial}(n, p)$.

$\Rightarrow N \sim \text{Poisson}(np) \sim \text{Poisson}(\lambda t)$.