Theory of Probability

Poisson random variables:

\( X \sim \text{Poisson} (\lambda) \)

\( P[X = k] = e^{-\lambda} \frac{\lambda^k}{k!} \)

Key approximation: \( \text{Poisson}(\lambda) \) is approximately the same as binomial \((n, p)\) when \( n \) is large and \( p \) is small, and \( X = np \) is \( \Theta(1) \).

If \( X \sim \text{binomial} (n, p) \) then

\[
P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}
\]

\[
= \frac{n!}{k! (n-k)!} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k}
\]

\[
= \frac{n!}{(n-k)! n^k \frac{\lambda^k}{k!} \left( 1 - \frac{\lambda}{n} \right)^{n-k}}
\]

\[
= \frac{n(n-1)(n-2) \cdots (n-k+1)}{n^n \cdots n} \frac{\lambda^k}{k!} \left( 1 - \frac{\lambda}{n} \right)^{n-k}
\]

\[
\approx e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{Poisson probability mass function.}
\]
If $X \sim \text{Binomial}(n, p)$

$E(X) = np$
$\text{Var}(X) = np(1-p)$

**Conjecture:** If $n$ is large, $p$ is small, $X = np \sim O(1)$, then

If $Y \sim \text{Poisson}(\lambda)$, then

$E(Y) \approx E[X] = \lambda$
$\text{Var}(Y) \approx \text{Var}[X] = np(1-p) = \lambda(1-p) \approx \lambda$.

**Calculate:**

\[
E(Y) = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}(\lambda)^k}{k!} \\
= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \\
= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
= e^{-\lambda} \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
= e^{-\lambda} \lambda e^\lambda = \sqrt{\lambda}
\]

If $P(E|F) = P(E)$ then $E, F$ are independent.

If $P(E|F) \approx P(E)$, then we say $E, F$ are only "weakly dependent."
Poission Paradigm

Consider $n$ trials, with $P(E_i) = p_i$. If $n$ is large, and all the $p_i$ are small, and either the $E_i$ are independent or “weakly dependent”, then the sum $E = E_1 + E_2 + \ldots + E_n$ is approximately Poisson with parameter $\lambda = \sum p_i$.

Rare Event Modelling

\[ \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \]

\[ 0 \leq t \leq n \]

We are interested in $P[N(t) = k]$?

Assumption 1: $n$ is large enough so that either $1$ or $0$ events happen in each interval.

$\Rightarrow P[N(t) = k] = P[ k \text{ subintervals have } 1 \text{ event } ] P[ n-k \text{ subintervals have } 0 \text{ events } ]$

Next, assume that the probability of an event occurring in a subinterval is $\lambda h = \frac{\lambda t}{n} = \rho$.

$\Rightarrow P[ \text{any subinterval} ] = 1 - \rho = 1 - \lambda h$.

Assumption 2: Independence between subintervals.

$\Rightarrow N \sim \text{binomial}(n, \rho)$. 

$\Rightarrow N \sim \text{Poisson}(np) \sim \text{Poisson}(\lambda t)$.