Theory of Probability

Oct 14, 2020

Bernoulli random variables (ex: a coin flip)

$X \sim \text{Bernoulli}(p)$

"Success" $P[X=1] = p = p(1)$

"failure" $P[X=0] = 1-p = p(0)$

$\Big\}$ probability mass function

Next consider n independent trials or flips $X_1, X_2, \ldots, X_n$.

$Y = X_1 + X_2 + \cdots X_n$

$$P[Y=k] = \binom{n}{k} p^k (1-p)^{n-k} \qquad k=0, \ldots, n$$

$Y \sim \text{binomial}(n,p)$.

Note: $\displaystyle\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$.

Let $X \sim \text{Bernoulli}(p)$.

$$E[X] = 1 \cdot p + 0 \cdot (1-p)$$
$$= p$$

$$Var[X] = E[X^2] - (E[X])^2$$
$$= p - p^2$$
$$= \boxed{p(1-p)}$$

Binomial R.V. expected value and variance:

$Y \sim \text{binomial}(n,p)$.

$$E[Y] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$k\binom{n}{k} = \frac{kn!}{k!(n-k)} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$$

$$= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \underbrace{\binom{n-1}{k} p^k (1-p)^{n-1-k}}_{\text{probability mass function for binomial}(n-1,p)\text{ random variable.}}$$

$$= np$$

$E[Y]$ = first moment of Y.

More generally, let $Y_n \sim \text{binomial}(n,p)$

$$\underbrace{E[Y_n^m]}_{m^{th}\text{ moment}} = np\, E[(Y_{n-1}+1)^{m-1}]$$

Y_{n-1}: binomial$(n-1,p)$.

In particular, $E[Y_n^2] = np((n-1)p+1)$

$$Var[Y_n] = E[Y_n^2] - (E[Y_n])^2$$

$$= np((n-1)p+1) - n^2p^2$$

$$= \boxed{np(1-p).}$$

Lastly: Examine the probability mass function for $X \sim \text{binomial}(n,p)$.

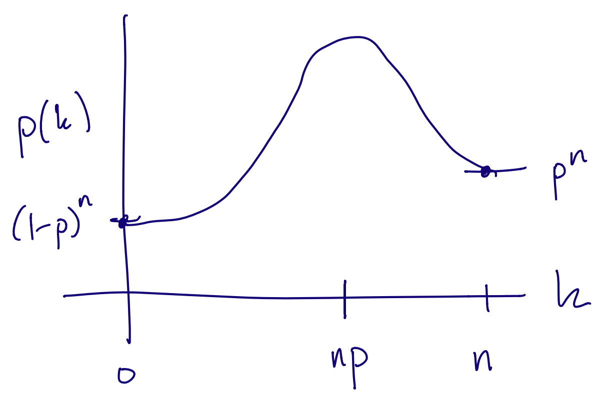
$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

$$p(0) = (1-p)^n$$

$$p(n) = p^n.$$

Proposition:

$p(k)$ first increases monotonically and then decreases monotonically, the largest value occurs when k is the largest integer less than or equal to $(n+1)p$ $(> E[X] = np)$.



Pf: examine $\dfrac{p(k)}{p(k-1)}$