Two events $E, F \subset S$, then the conditional probability of $E$ given $F$ is

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Fix the event $F$, and consider the function $Q(E) = P(E|F)$.

It turns out that $Q(E) = P(E|F)$ satisfies the three axioms of probability:

1. $0 \leq Q(E) \leq 1$
   
   **Proof:**
   
   $$0 \leq P(E|F) \leq 1$$
   
   $$0 \leq \frac{P(EF)}{P(F)} \leq 1$$

   $$\Rightarrow 0 \leq P(EF) \leq P(F)$$

   satisfied because $EF \subset F$

2. $P(S|F) = 1 = Q(S)$

   $$P(S|F) = \frac{P(SF)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

3. Let $E_i$ be mutually exclusive events:

   $$Q\left(\bigcup_i E_i\right) = \sum_i Q(E_i)$$

   $$P\left(\bigcup_i E_i|F\right) = \frac{P\left(\bigcup_i E_i \cap F\right)}{P(F)} = \frac{P\left(\bigcup_i E_i \cap F\right)}{P(F)} = \sum_i \frac{P(E_i|F)}{P(F)} = Q\left(\bigcup_i E_i\right)$$

The events $E_i \cap F$ are mutually exclusive.
Define a new probability function:

\[ Q(E) = P(E|F) \]

Consider conditional probabilities under \( Q \):

\[
Q(E_1 | E_2) = \frac{Q(E_1 E_2)}{Q(E_2)} = \frac{P(E_1 E_2 | F)}{P(E_2 | F)} = \frac{P(E_1 E_2 | F)}{P(F)} \cdot \frac{P(F)}{P(E_2 | F)} = \frac{P(E_1 E_2 F)}{P(E_2 F)}
\]

**Conditional Independence**

**Definition**: \( E_1 \) and \( E_2 \) are conditionally independent with respect to \( F \) if:

\[ P(E_1 | E_2 F) = P(E_1 | F) \]

Independence means: \( P(A | B) = P(A) \).

An equivalent definition of conditional independence is:

\[ P(E_1 E_2 | F) = P(E_1 | F) P(E_2 | F) \]

\((\text{compare with } P(AB) = P(A) P(B) \text{ if } A, B \text{ are independent})\).

\[ P \text{ of equivalence:} \]

\[
P(E_1 | E_2 F) = \frac{P(E_1 E_2 F)}{P(E_2 F)} = \frac{P(E_1 E_2 | F) P(F)}{P(E_2 F)}
\]
Since $P(E, E_2 | F) = P(E, 1 | F)$, we have that

$$P(E, 1 | F) = \frac{P(E, E_2 | F) P(F)}{P(E_2 | F)}$$

$$P(E, E_2 | F) = P(E, 1 | F) \frac{P(E_2 | F)}{P(F)} = P(E_2 | 1 F)$$

$$P(E, E_2 | F) = P(E, 1 | F) P(E_2 | 1 F)$$