Theory of Probability

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Sample spaces with equally likely outcomes:

$$S = \{O_1, O_2, O_3, \ldots, O_N\}$$

E.g. $S = \{1, 2, 3, 4, 5, 6\}$

If each outcome is equally likely, then

$$P(O_1) = P(O_2) = P(O_3) = \ldots = P(O_N)$$

$$\Rightarrow P(O_i) = \frac{1}{N}$$ ← this is the only such probability to assign to O_i such that the Axioms of Probability are satisfied.

Recall: Axioms

1. $0 \le P(O_i) \le 1$
2. $P(S) = 1$
3. If $O_1, \ldots, O_N$ are mutually exclusive, then

$$P(O_1 \cup \ldots \cup O_N) = P(S) = 1$$
$$= \underline{\sum_{i=1}^{N} P(O_i)}$$

E.g. Rolling a die, picking randomly from a deck of cards, etc.

Example From a group of 6 men and 9 women, a committee of 5 is to be formed. If the selection is random, what is the probability that the committee consists of 3 men and 2 women?

First: Assume each committee is equally likely to be formed.

Second: How many possible committees are there?

$$N = \binom{15}{5} \quad \Rightarrow \quad \text{each committee occurs with probability } 1/N.$$

Third: Count how many committees consist of 3 men and 2 women.

$$\binom{6}{3}\binom{9}{2}$$

$$\Rightarrow P(M=3, W=2) = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{\dfrac{\cancel{6}\cdot 5\cdot 4}{3\cdot\cancel{2}\cdot 1}\ \dfrac{9\cdot\cancel{8}^{\,4}}{\cancel{2}\cdot 1}}{\dfrac{\cancel{15}\cdot 14\cdot 13\cdot \cancel{12}^{\,3}\cdot 11}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1}} = \frac{240}{1001}$$

A standard deck of cards has 52 distinct cards.

- Each card has a "suit": clubs ♣
  hearts ♡
  diamonds ◇
  spades ♠
- Each card gets a value: 2, 3, ..., 10, Jack, Queen, King, Ace
- Some decks contain a "Joker" as a 53rd card.
- Lastly, Ace can be "low" or "high", i.e. < 2 or > King

- The numerical value of an Ace in Blackjack can be 1 or 11.
- The numerical value of J, Q, K is 10 in blackjack.

Example A poker hand consists of 5 cards, distributed randomly. If the 5 cards are consecutive, e.g.

3, 4, 5, 6, 7 or 8, 9, 10, J, Q, but not all of the same suit, then we say the hand is a straight.

Lastly A, 2, 3, 4, 5 and 10, J, Q, K, A are both straights.

Question: What is the probability that you are dealt a straight?

First The total number of 5 card hands is $\binom{52}{5}$.

Next: Count the number of straights that begin with an Ace:

$$\begin{array}{lccccc} \text{Value:} & A & 2 & 3 & 4 & 5 \\ \text{Suit:} & \left[\begin{array}{c} C \\ S \\ H \\ D \end{array}\right. & \begin{array}{c} C \\ S \\ H \\ D \end{array} & & & \end{array}$$

$= 4^5$ possible straights beginning with A.

4 of these 4^5 straights are straight flushes.

$\Rightarrow$ $4^5 - 4$ straights beginning with A.

$\Rightarrow$ $10(4^5 - 4)$ possible straights.

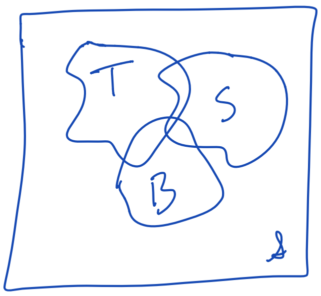
$$\Rightarrow P(\text{drawing a straight}) = \frac{10(4^5-4)}{\binom{52}{5}} \approx 0.0039$$

Example A recreation club has N members.

36 members play tennis
28 play squash
18 play badminton

22 play tennis and squash
12 play tennis and badminton
9 play squash and badminton
4 play all three sports.

How many members play at least one sport?



$T \cup S \cup B$

Example $P(T) = \dfrac{36}{N}$

Calculate $P(\underbrace{T \cup S \cup B}_{\text{the set of people who play at least one sport}})$ using inclusion - exclusion.

$$P(T \cup S \cup B) = P(T) + P(S) + P(B) - P(TS) - P(TB) - P(SB) + P(TSB).$$

$$= \frac{1}{N}\left(36 + 28 + 18 - 22 - 12 - 9 + 4\right)$$

$$= \frac{43}{N}$$

$\Rightarrow$ Probability of choosing someone at random that plays at least one sport is $43/N$

$\Rightarrow$ 43 members play atleast one sport.