Theory of Probability

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Old interpretation of probability:
"long term frequency of events"

If E is an event, then

$$P(E) = \text{probability that } E \text{ occurs} = \lim_{n\to\infty} \frac{n(E)}{n}$$

$n(E)$ = number of times E occurs in n trials.

Axioms of Probability

$P(E)$ = probability of event E

Axiom 1

$$0 \le P(E) \le 1$$

Axiom 2

$$P(S) = 1$$

S = sample space.

Axiom For any sequence of mutually exclusive events $E_1, E_2, \ldots, E_\infty$ (i.e. $E_i E_j = \phi$ when $i \neq j$), we have that

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

Some simple Propositions

$\boxed{\text{Prop 1}}$ $\quad P(E^c) = 1 - P(E)$

Pf: $S = E \cup E^c$, and $E \cap E^c = \phi$

$\Rightarrow P(S) = 1 = P(E \cup E^c) = P(E) + P(E^c)$

Prop 2 If $E \subset F$, then $P(E) \le P(F)$. (Each outcome in E is also an outcome in F).

Proof Since $E \subset F$, we can write

$$F = E \cup (E^c F)$$

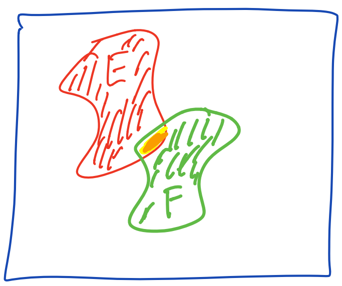
mutually exclusive since E & E^c are mutually exclusive.

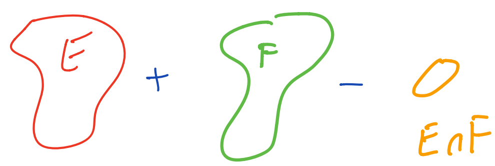
$$\Rightarrow \quad P(F) = P(E) + P(E^c F)$$

$$\Rightarrow \quad P(F) \geq P(E) \quad \text{since } P(E^c F) \geq 0.$$

Prop 3. $P(E \cup F) = P(E) + P(F) - P(EF)$

(also known as the principle of inclusion-exclusion)



$P(E \cup F) =$ 

Prop 4 General inclusion exclusion

$$P(E_1 \cup E_2 \cup E_3 \cdots E_n)$$
$$= \sum_{i=1}^{n} P(E_i) - \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k)$$
$$- \ldots,$$
$$= \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < i_2 < i_3 < \cdots < i_r} P(\overbrace{E_{i_1} E_{i_2} \cdots E_{i_r}}^{r \text{ events}}).$$

When $n = 3$