Theory of Probability

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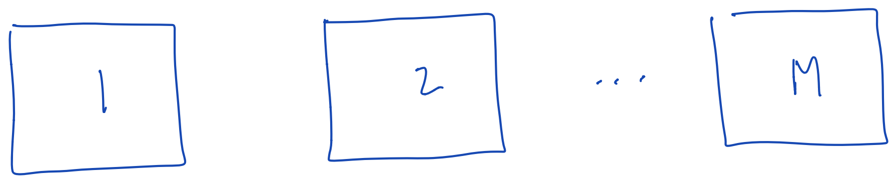
Imagine we have N distinct objects:

$$O_1,\ O_2 \ \ldots\ O_N$$

sort into M bins



$n_1 \qquad n_2 \qquad n_M \qquad \Rightarrow \boxed{\sum_{i=1}^{M} n_i = N}.$

n_i objects go into bin i. Order within each bin does <u>not</u> matter.

Filling up bin 1 $\quad$ Filling up bin 2 $\quad$ Filling up bin 3 $\quad$ Fill up bin M

$$\binom{N}{n_1} \times \binom{N-n_1}{n_2} \times \binom{N-n_1-n_2}{n_3} \times \cdots \times \underbrace{\binom{\overbrace{N-n_1-\cdots-n_{M-1}}^{=\,n_M}}{n_M}}_{1}$$

$$= \frac{N!}{n_1!\,\cancel{(N-n_1)!}} \cdot \frac{\cancel{(N-n_1)!}}{n_2!\,\cancel{(N-n_1-n_2)!}} \cdot \frac{\cancel{(N-n_1-n_2)!}}{n_3!\,\cancel{(N-n_1-n_2-n_3)!}} \cancel{\cdots} \frac{\cancel{n_M}}{n_M! \cdot 1!}$$

$$= \frac{N!}{n_1!\, n_2!\, n_3! \cdots n_M!} = \binom{N}{n_1, n_2, n_3, \ldots, n_M}$$

Multinomial coefficient = number of ways to sort N distinct objects into bins, each having n_i objects.

Extension to Multinomial Theorem

$$(x_1 + x_2 + \dots + x_r)^n = \underset{\uparrow}{\sum} \binom{n}{n_1, n_2, \dots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

sum over $n_1, n_2, \dots, n_r$
such that $n_1 + n_2 + \dots + n_r = n$.

Sample Spaces and Events

Sample space S is the set of possible outcomes of an experiment.

Ex: Roll two dice

$$S = \{ (1,1), \underbrace{(2,1)}_{\text{outcome}}, (3,1), \dots \}$$

A collection of outcomes is called an event.
An event is a subset of S.

Let E & F be two events defined on the sample space S.

Define a new event $G = E \underset{\text{union}}{\cup} F$ = all outcomes that are contained in E or F.

$E = \{(2,1), (2,2) \dots (2,6)\}$

$F = \{(\cdot,1), (\cdot,2)\}$

$E \cup F = \{(2,1), (2,2), \dots, (2,6), (1,1), (3,1), (4,1), (5,1), (6,1)$
$(1,2), (3,2)\ (4,2)\ (5,2)\ (6,2)\}$

$H = E \cap F$ = outcomes in $E \cap F$

$\uparrow$ intersect

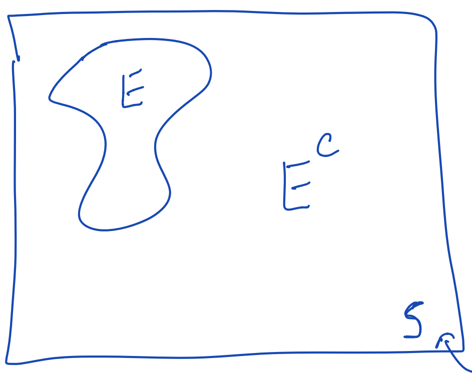
$= \{(2,1), (2,2)\}$

We say E and F are mutually exclusive if
$E \cap F = \{\}$ the empty set
$= \phi$ the null set

= Other properties of set theory useful for probability

Multiple events: $G = \bigcup_{i=1}^{N} E_i$

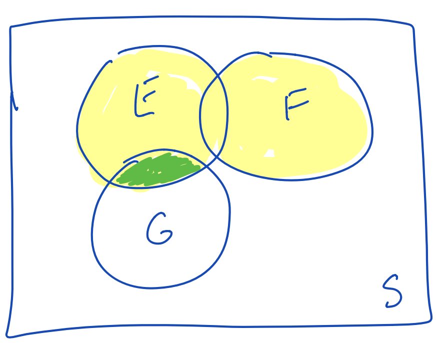
Complement: E^c = all outcomes that are <u>not</u> in E
$= S \setminus E$



sample space

Venn Diagram

$S = E \cup E^c$, $E \cap E^c = \phi$



$E \cap G =$

$E \cup F =$

$F \cap G = \phi$

Set operations obey the following laws:

Commutative Law: $E \cup F = F \cup E$ $\qquad \underbrace{E \cap F}_{EF} = F \cap E$

Associative Law: $(E \cup F) \cup G = E \cup (F \cup G)$

$(E \cap F) \cap G = E \cap (F \cap G)$

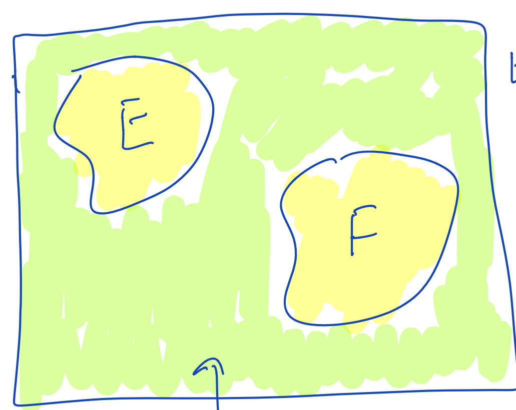
Distributive Law: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

i.e. $(x+y)z = xz + yz$.

Combining the laws with complements, we get DeMorgan's Laws:

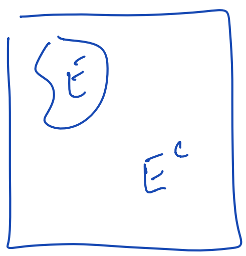
① $\left(\bigcup_{i=1}^{n} E_i \right)^c = \bigcap_{i=1}^{n} E_i^c$

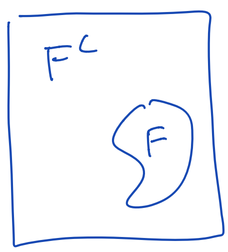
② $\left(\bigcap_{i=1}^{n} E_i \right)^c = \bigcup_{i=1}^{n} E_i^c$



$E \cup F$

$(E \cup F)^c$





$= E^c \cap F^c =$