Recursive HODLR Inversion

function $\mathbf{C} = \texttt{HODLR\_invert}(\mathbf{A})$

  if $(\mathrm{dim}(\mathbf{A}) < 2k)$ **then**

    Invert by brute force: $\mathbf{C} = \mathbf{A}^{-1}$.

  else

    Split $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$.

    $\mathbf{X}_{22} = \texttt{HODLR\_invert}(\mathbf{A}_{22})$.

    $\mathbf{X}_{11} = \texttt{HODLR\_invert}(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{X}_{22}\mathbf{A}_{21})$.

    $\mathbf{C} = \begin{bmatrix} \mathbf{X}_{11} & -\mathbf{X}_{11}\,\mathbf{A}_{12}\,\mathbf{X}_{22} \\ -\mathbf{X}_{22}\,\mathbf{A}_{21}\,\mathbf{X}_{11} & \mathbf{X}_{22} + \mathbf{X}_{22}\,\mathbf{A}_{21}\,\mathbf{X}_{11}\,\mathbf{A}_{12}\,\mathbf{X}_{22} \end{bmatrix}$.

    Recompress the lower right block of $\mathbf{C}$.

  end if

end function

Computational Complexity of Nonrecursive HODLR Inversion

(1) **function** $\mathbf{C} = \texttt{HODLR\_invert}(\mathbf{A})$

(2)   **for** $\tau = N_{\text{boxes}} : (-1) : 1$

(3)   **if** (τ is a leaf) **then**

(4)   Invert by brute force: $\mathbf{C}_\tau = \big(\mathbf{A}(I_\tau, I_\tau)\big)^{-1}$.

(5)   **else**

(6)   Let $\{\alpha, \beta\}$ denote the children of τ.

(7) $\mathbf{C}_\tau = \begin{bmatrix} \mathbf{I} & \mathbf{C}_\alpha \mathbf{A}_{\alpha\beta} \\ \mathbf{C}_\beta \mathbf{A}_{\beta\alpha} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C}_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\beta \end{bmatrix}$.

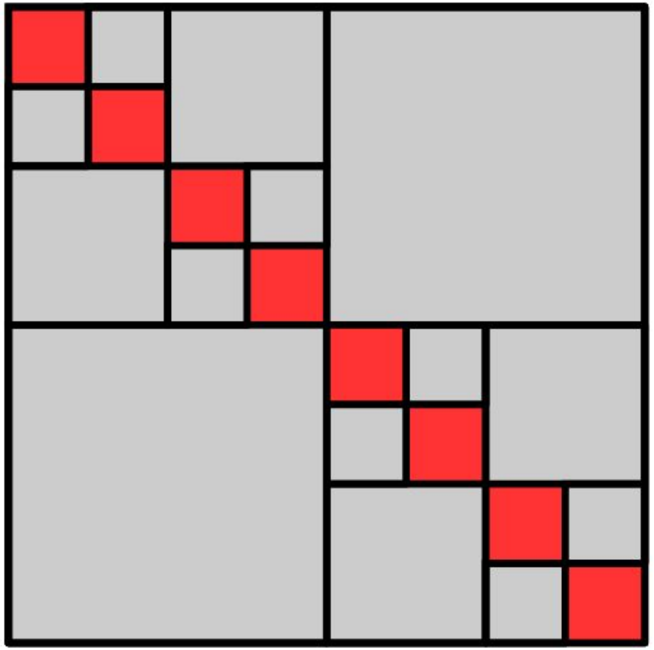
(8)   Recompress $\mathbf{C}_\tau$ to combat potential increase in ranks of off-diagonal blocks on line (6).

(9)   **end if**

(10)   **end for**

(11) **end function**

(4) Invert by brute force: $\mathbf{C}_\tau = \left(\mathbf{A}(I_\tau, I_\tau)\right)^{-1}$.



The algorithm starts with inversion of the smaller blocks on the diagonals.

$N/(2k)$ many inversions of [2k,2k] matrices

$\sim k^3 N/k \sim Nk^2$

At ith depth we have blocks of size $N_i = 2^{-i}N$.

$i = 1, 2, \ldots, L$

At the final depth L, $N_L = 2^{-L}N = k$

(6) Let $\{\alpha, \beta\}$ denote the children of τ.

(7) $$\mathbf{C}_\tau = \begin{bmatrix} \mathbf{I} & \mathbf{C}_\alpha \mathbf{A}_{\alpha\beta} \\ \mathbf{C}_\beta \mathbf{A}_{\beta\alpha} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C}_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\beta \end{bmatrix}.$$

(8) Recompress $\mathbf{C}_\tau$ to combat potential increase in ranks of off-diagonal blocks on line (6).

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}_{11}^{-1} \mathbf{A}_{12} \\ \mathbf{A}_{22}^{-1} \mathbf{A}_{21} & \mathbf{I} \end{bmatrix}.$$

$$\begin{aligned} \mathbf{A}^{-1} &= \begin{bmatrix} \mathbf{I} & \mathbf{A}_{11}^{-1} \mathbf{A}_{12} \\ \mathbf{A}_{22}^{-1} \mathbf{A}_{21} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{C}_1 \mathbf{A}_{12} \\ \mathbf{C}_2 \mathbf{A}_{21} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix}. \end{aligned}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{C}_\alpha \mathbf{A}_{\alpha\beta} \\ \mathbf{C}_\beta \mathbf{A}_{\beta\alpha} & \mathbf{I} \end{bmatrix}^{-1}$$

Off-diagonals are already compressed.

$A_{\alpha,\beta} = E_{A_{\alpha,\beta}} F^*_{A_{\alpha,\beta}} = [N_i, k] * [k, N_i]$

We need to get compressed forms of $C_\alpha A_{\alpha,\beta}$ and $C_\beta A_{\beta,\alpha}$ then follow the lemma below

Lemma 5.3:

$\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{F}_1$, *and* $\mathbf{F}_2$ *have* k *columns each.*

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{E}_1 \mathbf{F}_2^* \\ \mathbf{E}_2 \mathbf{F}_1^* & \mathbf{I} \end{bmatrix} \qquad \mathbf{B}^{-1} = \mathbf{I} - \begin{bmatrix} \mathbf{E}_1 \mathbf{X}_{11} \mathbf{F}_1^* & \mathbf{E}_1 \mathbf{X}_{12} \mathbf{F}_2^* \\ \mathbf{E}_2 \mathbf{X}_{21} \mathbf{F}_1^* & \mathbf{E}_2 \mathbf{X}_{22} \mathbf{F}_2^* \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I} & \mathbf{F}_2^* \mathbf{E}_2 \\ \mathbf{F}_1^* \mathbf{E}_1 & \mathbf{I} \end{bmatrix} \qquad \mathbf{Z}^{-1} = \begin{bmatrix} \mathbf{X}_{12} & \mathbf{X}_{11} \\ \mathbf{X}_{22} & \mathbf{X}_{21} \end{bmatrix}.$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{C}_\alpha \mathbf{A}_{\alpha\beta} \\ \mathbf{C}_\beta \mathbf{A}_{\beta\alpha} & \mathbf{I} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{E}_1 \mathbf{F}_2^* \\ \mathbf{E}_2 \mathbf{F}_1^* & \mathbf{I} \end{bmatrix}$$

Acquiring $E_{1,2}, F_{1,2}$:

$$C_\alpha A_{\alpha,\beta} = (C_\alpha E_{A_{\alpha,\beta}}) F^*_{A_{\alpha,\beta}} = (E_1) F_2^*, \quad F_{A_{\alpha,\beta}} = F_2$$

$$C_\beta A_{\beta,\alpha} = (C_\beta E_{A_{\beta,\alpha}}) F^*_{A_{\beta,\alpha}} = (E_2) F_1^*, \quad F_{A_{\beta,\alpha}} = F_1$$

Two mat-mat multiplications of type $[N_i, N_i]_{\text{HODLR}} * [N_i, k]$

$\sim N_i \log(N_i) k^2$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I} & \mathbf{F}_2^*\mathbf{E}_2 \\ \mathbf{F}_1^*\mathbf{E}_1 & \mathbf{I} \end{bmatrix} \qquad \mathbf{Z}^{-1} = \begin{bmatrix} \mathbf{X}_{12} & \mathbf{X}_{11} \\ \mathbf{X}_{22} & \mathbf{X}_{21} \end{bmatrix}.$$

Building and inverting Z :

Two mat-mat multiplications of $[k, N_i] * [N_i, k]$

$[2k, 2k]$ matrix inversion

$\sim N_i k^2 + k^3 \sim N_i k^2$

$$\mathbf{B}^{-1} = \mathbf{I} - \begin{bmatrix} \mathbf{E}_1\mathbf{X}_{11}\mathbf{F}_1^* & \mathbf{E}_1\mathbf{X}_{12}\mathbf{F}_2^* \\ \mathbf{E}_2\mathbf{X}_{21}\mathbf{F}_1^* & \mathbf{E}_2\mathbf{X}_{22}\mathbf{F}_2^* \end{bmatrix}$$

$$B^{-1} = I - \begin{pmatrix} E_{11} F_{11}^* & E_{12} F_{12}^* \\ E_{21} F_{21}^* & E_{22} F_{22}^* \end{pmatrix}$$

E_{ij}, F_{ij} have k columns.

Computing B^{-1} :

Four mat-mat multiplications of $[N_i, k] * [k, k]$

Four mat-mat multiplications of $[k, k] * [k, N_i]$

$\sim N_i k^2$

$$\mathbf{C}_\tau = \begin{bmatrix} \mathbf{I} & \mathbf{C}_\alpha \mathbf{A}_{\alpha\beta} \\ \mathbf{C}_\beta \mathbf{A}_{\beta\alpha} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C}_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\beta \end{bmatrix} \tag{7}$$

$$C_\tau = B^{-1} \begin{pmatrix} C_\alpha & 0 \\ 0 & C_\beta \end{pmatrix} = \begin{pmatrix} C_\alpha - E_{11}(F_{11}^* C_\alpha) & -E_{12}(F_{12}^* C_\beta) \\ -E_{21}(F_{21}^* C_\alpha) & C_\beta - E_{22}(F_{22}^* C_\beta) \end{pmatrix}$$

Computing C_τ :

Four mat-mat multiplications of $[k, N_i] * [N_i, N_i]_{\text{HODLR}}$

$\sim N_i \log(N_i) k^2$

(8) Recompress $\mathbf{C}_\tau$ to combat potential increase in ranks of off-diagonal blocks on line (6).

Recompression of C_τ :

$$C_\tau = \begin{pmatrix} C_\alpha - E_{11}(F_{11}^* C_\alpha) & -E_{12}(F_{12}^* C_\beta) \\ -E_{21}(F_{21}^* C_\alpha) & C_\beta - E_{22}(F_{22}^* C_\beta) \end{pmatrix}$$

First off-diagonals $-E_{12}(F_{12}^* C_\beta)$ and $-E_{21}(F_{21}^* C_\alpha)$ are already compressed.

C_α is already HODLR

Hence off-diagonal blocks of $C_\alpha - E_{11}(F_{11}^* C_\alpha)$ have a special form:

For X off-diagonal block from $i+j$th level of depth

$X = [N_{i+j}, k] * [k, N_{i+j}] - [N_{i+j}, k] * [k, N_{i+j}]$

$$X = [N_{i+j}, k] * [k, N_{i+j}] - [N_{i+j}, k] * [k, N_{i+j}]$$

The multiplication $X\Omega = Y$ and SVD on Y:

$\sim N_{i+j} k^2$

There are 2^j many such off-diagonals:

$\sim \sum_{j=1}^{L-i} 2^j N_{i+j} k^2 \sim L N_i k^2$

In total we got

$N_i k^2 (L + \log(N_i)) \sim N_i k^2 \log(N)$

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$O(k^3)$ called $N/(2k)$ times

$O(\log(N) N_i k^2)$ called 2^i many times for $i = 1, 2, \ldots, L$

$$\sim k^3 N/k + \log(N) k^2 \sum_{i=1}^{L} N_i 2^i$$

$$\sim k^2 N + \log(N) k^2 N L$$

$$\sim N \log(N)^2 k^2$$