Fast solvers: An introduction

The algorithms of the course will address solving elliptic PDEs: E.g. $\Delta u = f$ Poisson equation.

Other examples: Fluid flow: incompressible (Stokes flow) Electro-magnetics: Maxwell's Equations Elasticity..

Example: Calculate the gravitational potential due to the moon: $\Delta u = g$ mass density of the moon.

Gravitational Field: $F = \nabla u$

Replace the moon with a single point with the same total mass:

$$\Delta u_0 = M \delta(x - x_0)$$

$|u - u_0|$ is very small when evaluated on earth.

Boundary Value Problem:

$\Delta u = 0$ in $\Omega$

$u = f$ on $\partial \Omega$

Interior Dirichlet Problem
A PDE discretization looks like:

$$\Delta u(x) = u_{xx} + u_{yy}$$

$$= \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

5 point stencil

The unknowns in my linear system are the values of \( u \).

The right hand side is mostly zeros.

$$\begin{pmatrix}
   -4 & 1 & 1 & 1 \\
   1 & -4 & 1 & 1 \\
   1 & 1 & -4 & 1 \\
   1 & 1 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
   u_1 \\
   u_2 \\
   u_3 \\
   u_4
\end{pmatrix}
= \begin{pmatrix}
   0 \\
   1 \\
   f
\end{pmatrix}$$

$$A \approx A$$

add in constraints for \( u = f \) on the boundary.

The matrix \( A \) is sparse. Only \( O(1) \approx 5 \) entries per column are non-zero.

\( A \) can be applied very fast \( \Rightarrow \) Good for iterative solvers, \( O(n) \).

Also, there exists a fast algorithm (nested dissection) for recording the columns and directly computing \( A^{-1} \), \( O(n \log n) \).
Integral Equations discretization

A Green's Function for the differential operator \( L \) is the function \( G \) such that

\[ L G = \delta \]

A Dirac delta function \( \delta \)

Example: \( \Delta G = \delta \). is such that

\[ G = \Delta^* \delta \quad \text{so} \quad f(x) = \int_{\mathbb{R}^2} f(y) \delta(x - y) \, dy \]

In 2D for Laplace:

\[ \Delta G = \frac{1}{2\pi} \log ||x - y|| = \delta(x - y) \]

In 3D:

\[ \Delta G = \frac{1}{4\pi ||x - y||} = \delta(x - y) \]

One consequence of a Green's Function:

Free space Poisson problem: \( \Delta U = g \)

The solution can immediately written down by convolving both sides with the Green's Function.

\[ \int G(x-y) \Delta \tilde{u}(y) \, dy = \int G(x-y) g(y) \, dy \]

By integration by parts

\[ \tilde{u}(x) = \int G(x-y) g(y) \, dy \]

Green's Functions give us ways to represent solutions to PDEs.

Green's Third Identity: If \( \Delta U = 0 \) in \( \mathbb{R}^n \), then

\[ u(x) = \int \frac{2u}{\partial \eta} (x - y) \tilde{u}(y) \, dy + \int u(x-y) \frac{\partial \eta}{\partial n} \, dy \]
In general, this fact might lead us to represent solutions to PDEs in terms of integrals of the Green's Function.

**Poisson BVP:**

\[
\nabla^2 u = f \quad \text{in } \Omega
\]
\[u = g \quad \text{on } \Gamma
\]

Represent \( u \) by an integral equation involving the Green's Function:

\[
\int_{\Omega} G(\mathbf{x},\mathbf{y}) \left[ u(\mathbf{y}) = f(\mathbf{x}) - \int_{\Gamma} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_y} \sigma(\mathbf{y}) \, dy \right] \, d\mathbf{x}
\]

This automatically satisfies the PDE, but not the boundary condition.

To derive an integral equation, insert (x) into the boundary condition:

\[
\frac{1}{2} \sigma(x) + \int_{\Gamma} \frac{\partial G(x,y)}{\partial n_y} \sigma(y) \, dy = f - \int_{\Gamma} G(x,y) g(y) \, dy
\]

When discretized, we get

\[
\left( \frac{1}{2} \mathbf{I} + L \right) \sigma = \mathbf{RHS}
\]

Apply: \( O(n^2) \).

Invert: \( O(n^3) \). where \( n \) is the number of points on the boundary.

**Advantages of IEs:**
- Mathematically better: \( \theta \)-well-conditioned (small condition number).
- Excellent for exterior problems.

**Disadvantages:**
- Linear systems are dense.
- Requires singular integrals.