Integral Equations: Continuous Theory

Freddy Law

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10.1 Reducing the dimension of the computational domain

- Our model problem will be the Laplace equation with Dirichlet data:

\[
\begin{align*}
-\Delta u &= 0, \quad \text{in } \Omega \\
u &= f, \quad \text{on } \Gamma := \partial \Omega
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \) is simply connected, open, with smooth boundary \( \Gamma \).
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\]

where \( \Omega \subset \mathbb{R}^2 \) is simply connected, open, with smooth boundary \( \Gamma \).

- We want a solution \( u \) of the form

\[
u(x) = \int_{\Gamma} \phi(x - y) \sigma(y) \, ds(y), \quad x \in \Omega
\]

where \( \phi(x) = -\frac{1}{2\pi} \log(|x|) \) is the free space Green’s function for \(-\Delta\) in 2D.

- This expression for \( u \) looks like a superposition of \( \phi \) weighted by \( \sigma \). So we formally expect \(-\Delta u = 0\) since \( \phi \) is harmonic away from 0.
To match the boundary condition, we solve the *Boundary Integral Equation* (BIE) formulation of our original problem:

\[
\int_{\Gamma} \phi(x - y) \sigma(y) \, ds(y) = f(x), \quad x \in \Gamma
\]

(1)

From a numerics standpoint, this formulation requires *fewer degrees of freedom* since discretizing \( \Gamma \) is much easier than discretizing \( \Omega \).

---

1 more details in chapters 6, 7 of *Linear Integral Equations* by R. Kress
10.1 Reducing the dimension of the computational domain

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From a numerics standpoint, this formulation requires *fewer degrees of freedom* since discretizing \( \Gamma \) is much easier than discretizing \( \Omega \).

Our *single-layer operator* \( S \) is:

\[
[S\sigma](x) = \int_{\Gamma} \phi(x - y) \sigma(x) \, ds(y) = \int_{\Gamma} -\frac{1}{2\pi} \log(|x - y|) \sigma(y) \, ds(y)
\]

Existence and uniqueness of solutions \( \sigma \) to (1) require some technical assumptions (primarily \( f \in C^{1,\alpha}(\Gamma) \) + geometric condition on \( \Omega \))^1, but formal manipulations typically hold.

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10.2 Obtaining a well-conditioned mathematical equation

- The BIE (1) leads to linear systems with condition number $O(h^{-1})$ using a grid size $h$. This beats $O(h^{-2})$ from FD or FEM discretizations.
- The approach in this section will give a BIE leading to condition number converging to a finite number as $h \to 0$. 
The BIE (1) leads to linear systems with condition number $O(h^{-1})$ using a grid size $h$. This beats $O(h^{-2})$ from FD or FEM discretizations.

The approach in this section will give a BIE leading to condition number converging to a finite number as $h \to 0$.

For $y \in \Gamma$, define

$$d(x, y) = n(y) \cdot \nabla_y \phi(x - y) = \frac{n(y) \cdot (x - y)}{2\pi |x - y|^2}$$

for $x \in \Omega$. This is just the normal derivative of $\phi$ at $y$.

Now seek solutions of the form

$$u(x) = \int_{\Gamma} d(x, y) \sigma(y) \, ds(y)$$
10.2 Obtaining a well-conditioned mathematical equation

- Just like before, we expect \( u \) to satisfy \(-\Delta u = 0\) in \( \Omega \), so we just need to worry about matching the boundary condition.

- The singularity from \( d(x, y) \) is stronger than from \( \phi(x - y) \). Turns out \([S\sigma](x)\) is continuous as you approach \( \Gamma \), but when using \( d(x, y) \) we pick up an extra term.

\[
\frac{1}{2} \sigma(x) + \int_\Gamma d(x, y) \sigma(y) \, ds(y) = f(x), \quad x \in \Gamma
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Our BIE formulation now becomes

$$-\frac{1}{2}\sigma(x) + \int_\Gamma d(x, y)\sigma(y) \, ds(y) = f(x), \quad x \in \Gamma$$

Our double-layer operator $D$ is

$$[D\sigma](x) = \int_\Gamma d(x, y)\sigma(y) \, ds(y) = \int_\Gamma \frac{n(y) \cdot (x - y)}{2\pi|x - y|^2} \sigma(y) \, ds(y)$$

Note $[D\sigma]$ is defined on $\bar{\Omega}$, but has a jump as you approach $\Gamma$. 
10.2 Obtaining a well-conditioned mathematical equation

- The BIE \((-\frac{1}{2}I + D)\sigma = f\) is a Fredholm equation of the second kind, and technical results regarding compact operators tell us that discretizations of this BIE lead to exceedingly well-conditioning systems.

- Even better, the eigenvalues for discretizations of \((-\frac{1}{2}I + D)\sigma = f\) are clustered near \(-1/2\), so we can expect iterative solvers to converge rapidly (with \# of iterations independent of grid size).
What about exterior problems?

\[
\begin{align*}
-\Delta u &= 0, \quad \text{in } \Omega \\
u &= f, \quad \text{on } \Gamma \\
\lim_{|x| \to \infty} \left( u(x) + \frac{Q}{2\pi} \log |x| \right) &= 0, \quad \text{for some } Q \in \mathbb{R}
\end{align*}
\]

where now \( \Omega \) is the domain \textit{exterior} to the smooth close contour \( \Gamma \). The third line is a growth condition at \( \infty \).

The computational domain \( \Omega \) is \textit{unbounded}, so if we used FD or FEM methods, we would have to artificially truncate the domain and impose artificial boundary conditions.
10.3 External domain and B.C. at $\infty$ for Laplace equation

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where now $\Omega$ is the domain *exterior* to the smooth close contour $\Gamma$. The third line is a growth condition at $\infty$.

- The computational domain $\Omega$ is *unbounded*, so if we used FD or FEM methods, we would have to artificially truncate the domain and impose artificial boundary conditions.

- Conversion to a BIE makes the computational domain $\Gamma$ which is *bounded*. With a single-layer potential, the solution is $u(x) = [S\sigma](x)$ in $\Omega$, where $\sigma(y)$ solves the BIE:

\[
[S\sigma](x) = \int_\Gamma \phi(x - y)\sigma(y) \, ds(y) = f(x), \quad x \in \Gamma \quad (3)
\]
10.3 External domain and B.C. at $\infty$ for Laplace equation

- The single-layer solution $u$ automatically satisfies $-\Delta u = 0$ in $\Omega$ like before, and also automatically satisfies the growth condition since

$$\inf_{y \in \Gamma} |\phi(x - y)| \lesssim |u(x)| \lesssim \sup_{y \in \Gamma} |\phi(x - y)|, \quad \forall x \in \Omega$$

- Similar to the interior Dirichlet problem, the single layer formulation upon discretization gives linear systems whose condition number grows with $\#$ of points describing $\Gamma$. 

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Similar to the interior Dirichlet problem, the single layer formulation upon discretization gives linear systems whose condition number grows with # of points describing $\Gamma$.

If we want to use a double-layer formulation, we need to correct for the growth condition, since

$$[D\sigma](x) = \int_{\Gamma} d(x, y)\sigma(y)\,ds(y) = \int_{\Gamma} \frac{n(y) \cdot (x - y)}{2\pi |x - y|^2} \sigma(y)\,ds(y)$$

should now decay like $|x|^{-1}$. 

\[\int_{\Gamma} \frac{n(y) \cdot (x - y)}{2\pi |x - y|^2} \sigma(y)\,ds(y)\]
To manually correct for the decay of the double-layer, we fix $z$ interior to $\Gamma$, and look for solutions of the form

$$u(x) = [D\sigma](x) + \phi(x - z) \int_{\Gamma} \sigma(y) \, ds(y)$$

These solutions now satisfy the growth condition and still satisfy $-\Delta u = 0$ in $\Omega$ since $z \not\in \Omega$. The resulting BIE for $\sigma$ is then

$$\frac{1}{2} \sigma(x) + \int_{\Gamma} [d(x, y) + \phi(x - z)] \sigma(y) \, ds(y) = f(x), \quad x \in \Gamma \quad (4)$$

note that in the exterior problem we pick up a term $+\frac{1}{2} \sigma(x)$ while in the interior problem we picked up $-\frac{1}{2} \sigma(x)$. 
10.4 The Helmholtz equation

- Other PDE can be also be solved using a BIE formulation. Consider the interior Dirichlet problem for the Helmholtz equation with positive wave number $\kappa$.

$$
\begin{align*}
-\Delta u - \kappa^2 u &= 0, \quad \text{in } \Omega \\
u &= f, \quad \text{on } \Gamma
\end{align*}
$$

- The free space Green’s function for the Helmholtz operator is given by the zeroth order Hankel function: $\phi_{\kappa}(x) = \frac{i}{4} H_0^{(1)}(\kappa|x|)$. 

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- We can repeat the exact same process as for $-\Delta$ and get the single and double-layer operators:

\[
[S_{\kappa}\sigma](x) = \int_{\Gamma} \phi_\kappa(x - y)\sigma(y) \, ds(y)
\]

\[
[D_{\kappa}\sigma](x) = \int_{\Gamma} d_{\kappa}(x, y)\sigma(y) \, ds(y)
\]

where $d_{\kappa}(x, y) = n(y) \cdot \nabla_y \phi_\kappa(x - y)$. 
The function $\phi_\kappa(x)$ has a log-singularity near the origin, just like $\phi(x)$, hence we expect the layer operators to behave similarly.

If we try to use a double-layer formulation and look for solutions of the form $u(x) = [D_\kappa \sigma](x)$, we get the BIE $\left(-\frac{1}{2}I + D_\kappa\right) \sigma = f$ on $\Gamma$ which is not well defined for all $\kappa$. 
The function $\phi_\kappa(x)$ has a log-singularity near the origin, just like $\phi(x)$, hence we expect the layer operators to behave similarly.

If we try to use a double-layer formulation and look for solutions of the form $u(x) = [D\kappa\sigma](x)$, we get the BIE $(-\frac{1}{2}I + D\kappa)\sigma = f$ on $\Gamma$ which is not well defined for all $\kappa$.

To remedy this, the combined field formulation uses a linear combination of $S\kappa$ and $D\kappa$. We look for solutions of the form $u(x) = [(D\kappa + i\eta S\kappa)\sigma](x)$ where $\eta = \pm\kappa$. The resulting BIE is

$$\left[\left(-\frac{1}{2}I + D\kappa + i\eta S\kappa\right)\sigma\right](x) = f(x), \quad x \in \Gamma$$

(5)
Just like for the Laplace equation, we can also apply a BIE formulation for exterior Helmholtz problems:

\[
\begin{align*}
-\Delta u - \kappa^2 u &= 0, \quad \text{in } \Omega \\
 u &= f, \quad \text{on } \Gamma \\
 \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u(rz)}{\partial r} - i\kappa u(rz) \right) &= 0, \quad \text{for every unit vector } z
\end{align*}
\]

where the last term is a condition at \( \infty \). This exterior equation is useful in modeling certain types of scattering problems.

Using the combined field formulation and guessing solutions like \( u(x) = [(D_\kappa + i\eta S_\kappa)\sigma](x) \), the corresponding BIE for \( \sigma \) is

\[
\left[ \left( \frac{1}{2} I + D_\kappa + i\eta S_\kappa \right) \sigma \right](x) = f(x), \quad x \in \Gamma
\]

(6)
Here, we derive a direct method of reformulating Laplace’s equation as a BIE. Let \( s(x, y) = \phi(x - y) \) and \( d(x, y) = n(y) \cdot \nabla_y \phi(x - y) \).

**Theorem**

Let \( \Gamma \) be a smooth, bounded domain in \( \mathbb{R}^2 \). For any \( u \) such that \(-\Delta u = 0\) in \( \Omega \), then for \( x \in \mathbb{R}^2 \):

\[
\theta(x) u(x) = \int_{\Gamma} \left( s(x, y) \frac{\partial u(y)}{\partial n} - d(x, y) u(y) \right) \, ds(y), \tag{7}
\]

where

\[
\theta(x) = \begin{cases} 
1 & \text{for } x \in \Omega \\
1/2 & \text{for } x \in \Gamma \\
0 & \text{for } x \in \Omega^c
\end{cases}
\]
Given boundary conditions on $\Gamma$, we can use (7) to immediately convert the PDE to a BIE:

- **Dirichlet data**: $u = f$ on $\Gamma$. Then (7) gives the BIE for $\frac{\partial u}{\partial n}|_{\Gamma}$:

$$
\int_{\Gamma} s(x, y) \frac{\partial u(y)}{\partial n} \, ds(y) = \frac{1}{2} f(x) + \int_{\Gamma} d(x, y) f(y) \, ds(y), \quad x \in \Gamma
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If we solve this BIE for $\frac{\partial u}{\partial n}|_{\Gamma}$, we can use (7) to recover $u$ in $\Omega$. 
Given boundary conditions on $\Gamma$, we can use (7) to immediately convert the PDE to a BIE:

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  If we solve this BIE for $\frac{\partial u}{\partial n} \bigg|_{\Gamma}$, we can use (7) to recover $u$ in $\Omega$.

- **Neumann data**: $\frac{\partial u}{\partial n} = f$ on $\Gamma$. Then (7) gives the BIE for $u \bigg|_{\Gamma}$:

  $$\frac{1}{2} u(x) + \int_{\Gamma} d(x, y) u(y) \, ds(y) = \int_{\Gamma} s(x, y) f(y) \, ds(y), \quad x \in \Gamma$$

  If we solve this BIE for $u \bigg|_{\Gamma}$, we can use (7) to recover $u$ on $\Omega$. 
**10.6 ”Direct” derivation of BIE for harmonic potentials**

- Equation (7) also tells us why we pick up a factor of $\frac{1}{2}$ in the double layer formulation. Applying (7) with $u \equiv 1$ gives

$$\int_{\Gamma} d(x, y) \, ds(y) = \begin{cases} -1, & \text{for } x \in \Omega \\ -1/2, & \text{for } x \in \Gamma \\ 0, & \text{for } x \in \Omega^c \end{cases}$$
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-1, & \text{for } x \in \Omega \\
-1/2, & \text{for } x \in \Gamma \\
0, & \text{for } x \in \overline{\Omega}^c
\end{cases}$$

Then for continuous $\sigma$ defined on $\Gamma$ and $x \in \Gamma$, we get

$$\lim_{x' \to x} [D\sigma](x') = -\frac{1}{2}\sigma(x) + [D\sigma](x), \quad x' \in \Omega$$

Proof sketch: $[D\sigma](x') = \int_{\Gamma} d(x', y)(\sigma(y) - \sigma(x)) \, ds(y) - \sigma(x)$, assume some regularity on $\sigma$, swap limit with integral.
10.6 "Direct" derivation of BIE for harmonic potentials

- Proof outline for Theorem. For a fixed $x \in \mathbb{R}^2$, set $v(y) = \phi(x - y)$. Green's 2nd identity says

$$
\int_{\Omega} u \Delta v - v \Delta u = \int_{\Gamma} d(x, y)u(y) - s(x, y)\frac{\partial u(y)}{\partial n} \, ds(y) \tag{8}
$$

- **Case 1**: $x \in \overline{\Omega}^c$. Then $u, v$ harmonic in $\Omega$, so LHS of (8) is 0.
Case 2: $x \in \Omega$. Now $v$ is not harmonic in $\Omega$. Let $B_\varepsilon(x)$ be ball of radius $\varepsilon$ centered at $x$. Apply (8) to $\Omega \setminus B_\varepsilon(x)$ and show

$$\lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(x)} u \frac{\partial v}{\partial n} = u(x), \quad \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(x)} v \frac{\partial u}{\partial n} = 0$$
Case 3: $x \in \Gamma$. Nearly same argument as in Case 2, but the cut boundary is slightly different. Apply (8) to $\Omega \setminus B_\varepsilon(x)$

$$\lim_{\varepsilon \to 0^+} \int_{\Lambda_\varepsilon} u \frac{\partial v}{\partial n} = \frac{1}{2} u(x), \quad \lim_{\varepsilon \to 0^+} \int_{\Lambda_\varepsilon} v \frac{\partial u}{\partial n} = 0$$

$\Omega \setminus B_\varepsilon(x)$

locally, $\Lambda_\varepsilon$ looks like a semicircle with radius $\varepsilon$, on $\Lambda_\varepsilon$, $v \sim \log \varepsilon$
Now consider a body load $g$ for Laplace’s equation:

$$\begin{cases}
-\Delta u = g, & \text{in } \Omega \\
u = f, & \text{on } \Gamma
\end{cases}$$

We can first compute a particular solution $u_p$ which satisfies $-\Delta u_p = g$ on $\Omega$, ignoring boundary conditions. Analytically:

$$u_p(x) = \int_{\Omega} \phi(x - y)g(y) \, dy$$

which will satisfy $-\Delta u = g$ in $\Omega$.

Then set $u_h = u - u_p$ which solves (via BIE formulation):
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\]

Computing $u_p$ can be challenging, due complicated $\Omega$ and singular $\phi$. That said, there are methods of extending $\Omega$ and $g$ to be simpler computationally (e.g. put $\Omega$ inside a big box and smoothly extend $g$). Then specialized methods like FMM or FFT can evaluate $u_p$ fast.
11.2 Variable coefficient PDE; Lippmann-Schwinger equation

- For variable coefficient PDE, integral formulations are still possible but typically they are *volume integral equations*.
- While these formulations lose the benefit of reducing the dimension of the computational domain, they still retain the benefits of finite computational domain + well-conditioned systems.

\[
\begin{align*}
-\Delta u(x) - \kappa^2 (1 - b(x)^2) u(x) &= -\kappa^2 b(x) u(x) \quad \text{in } \mathbb{R}^2, \\
\partial u(x) / \partial r - i\kappa u(x) &= o(r - 1/2) \quad \text{as } r = |x| \to \infty.
\end{align*}
\]

This models acoustic wave propagation in a medium with variable wave speed. Assume \( b \) is smooth, vanishes outside \( \Omega \), and bounded by 1, and that \( u \) solves Helmholtz in \( \Omega \) with constant \( \kappa \).
For variable coefficient PDE, integral formulations are still possible but typically they are *volume integral equations*.

While these formulations lose the benefit of reducing the dimension of the computational domain, they still retains the benefits of finite computational domain + well-conditioned systems.

As an example, consider the free space, variance coefficient Helmholtz equation:

\[
\begin{aligned}
-\Delta u(x) - \kappa^2 (1 - b(x)^2) u(x) &= -\kappa^2 b(x) u_{in}(x), \quad \text{in } \mathbb{R}^2 \\
\frac{\partial u(x)}{\partial r} - i \kappa u(x) &= o(r^{-1/2}), \quad \text{as } r = |x| \to \infty.
\end{aligned}
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\[
\begin{aligned}
-\Delta u(x) - \kappa^2 (1 - b(x)^2)u(x) &= -\kappa^2 b(x)u_{\text{in}}(x), & \text{in } \mathbb{R}^2 \\
\frac{\partial u(x)}{\partial r} - i\kappa u(x) &= o(r^{-1/2}), & \text{as } r = |x| \to \infty.
\end{aligned}
\]

Here, \( b \) indicates how much wave speed in \( \Omega \) differs compared to free space wave speed.
11.2 Variable coefficient PDE; Lippmann-Schwinger equation

- Free space Green’s function for Helmwoltz with radiating BC is $G_{\kappa}(x, y) = \frac{i}{4} H_0^{(1)}(|x - y|)$, where $H_0^{(1)}$ is the zeroth order Hankel function.
- Search for solutions of the form

$$u(x) = \int_{\Omega} G_{\kappa}(x, y) \sigma(y) \, dy, \quad x \in \mathbb{R}^2$$
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- Search for solutions of the form
  \[ u(x) = \int_\Omega G_\kappa(x, y) \sigma(y) \, dy, \quad x \in \mathbb{R}^2 \]

- This leads to the BIE for \( \sigma \):
  \[ \sigma(x) + \kappa^2 b(x) \int_\Omega G_\kappa(x, y) \sigma(y) \, dy = -\kappa^2 b(x) u_{\text{in}}(x), \quad x \in \Omega \]

- Computational domain is now \( \Omega \) which is bounded (instead of \( \mathbb{R}^2 \)), and the above BIE leads to well-conditioned systems (just like double-layer formulation).
Integral equations serve as a powerful, alternative modeling tool to PDE. Benefits include

- Reducing dimension of computational domain ($\Omega$ down to $\Gamma$).
- Well-conditioned systems upon discretization (e.g. double-layer formulation)
- Can handle exterior problems with a finite computational domain.

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Different BIE formulations with different properties can be found for the same PDE.

With extra work/challenges, can be extended to other types of models (e.g. linear elasticity, Stokes flow, time-Harmonic Maxwell).

3D is possible, but $\Gamma$ harder to treat as a surface + kernels are more singular.