Last time:

Heat Equation:
\[
\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}
\]

Also must specify initial condition: \( u(x,0) = f(x) \)
and boundary condition: \( u(0,t) = u(L,t) = 0 \)

Separation of variables solution:

Assume \( u(x,t) = X(x)T(t) \), insert into equation:
\[
\frac{T'}{\alpha^2 T} = -\lambda = \frac{X''}{X}
\]

\( \lambda \) constant

This results in the boundary value problem:

\[
X'' + \lambda X = 0
\]

non-trivial solutions obtained when

\( X(0) = X(L) = 0 \)

\( \lambda_n = \frac{n^2 \pi^2}{L^2} \), \( X_n(x) = \sin \frac{n\pi}{L} x \)

The equation for \( T \) is then:

\[
T' + \alpha^2 \lambda T = 0
\]

\( \Rightarrow T(t) = e^{-\alpha^2 \lambda \frac{t^2}{L^2}} \)

Define \( u_n(x,t) = X_n(x)T_n(t) = \sin \frac{n\pi}{L} x e^{-\alpha^2 \lambda \frac{t^2}{L^2}} \)

In \( (x) \), \( \lambda \) is allowed to be any of the \( \lambda_n \)'s, and therefore \( u \)

is an arbitrary linear combination:
\[
u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x e^{-\alpha^2 \lambda \frac{t^2}{L^2}}
\]

The initial condition then implies that:
\[
u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x
\]
**Fourier Series**

**Question:** Can an arbitrary function $f$ on the interval $[L, L]$ be written as:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n \pi x}{L} + \sum_{n=0}^{\infty} b_n \sin \frac{n \pi x}{L}$$  \hspace{1cm} (**) \\

**Note:** The functions $\cos \frac{n \pi x}{L}$ and $\sin \frac{n \pi x}{L}$ are orthogonal on the interval $[L, L]$:

$$\left( \cos \frac{m \pi x}{L}, \sin \frac{n \pi x}{L} \right) = \int_{-L}^{L} \cos \frac{m \pi x}{L} \sin \frac{n \pi x}{L} \, dx$$

$$= 0 \quad \text{(show on your own)}.$$  

Furthermore, if $m \neq n$, then

$$\int_{-L}^{L} \cos \frac{m \pi x}{L} \cos \frac{n \pi x}{L} \, dx = 0$$

$$\int_{-L}^{L} \sin \frac{m \pi x}{L} \sin \frac{n \pi x}{L} \, dx = 0$$

So the set of functions $\left\{ \cos \frac{m \pi x}{L}, \sin \frac{n \pi x}{L} \right\}$ are mutually orthogonal.

**Therefore:** Assume that $f$ can be written as in (**), then multiplying by these functions and integrating allows us to solve for the $a$'s and $b$'s:

$$a_n = \frac{1}{\| \cos \frac{n \pi x}{L} \|_{L^2}} \left( f(x), \cos \frac{n \pi x}{L} \right)$$

$$= \frac{1}{\int_{-L}^{L} (\cos \frac{n \pi x}{L})^2 \, dx} \int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \, dx, \quad n \neq 0$$

$$= \begin{cases} L & \text{if } n \neq 0 \\ 2L & \text{if } n = 0 \end{cases}$$
And likewise:

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n > 1. \]

This is merely solving for \( a_n, b_n \)'s, (by taking inner products, i.e. projections).

**Complex version:**

Since \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \), \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \).

We can also expand:

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{and} \quad \left( i \times x \right) \]

\[ c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-inx} \, dx. \]

Do these series converge?

**Then:** Let \( f, f' \) be piecewise continuous on \([-L, L]\). Then, the series \( i \times x \) and \( i \times x \) converge to \( f(x) \) if \( f \) is continuous, and to \( \frac{1}{2} (f(x) + f(x')) \) if \( f \) is discontinuous at \( x \). At \( x = \pm L \), the series converge to \( \frac{1}{2} (f(-L) + f(L)) \).

**Graphically:**

The highlighted portion is known as the Runge Effect (and happens at discontinuity.)

Finite Fourier series approximation to step function.
Example: \( f(x) = \begin{cases} 0 & x \in [-1, 0) \\ 1 & x \in [0, 1] \end{cases} \)

Then \( a_n = \int_{-1}^{1} f(x) \cos n\pi x \, dx \)
\[
= \int_{0}^{1} \cos n\pi x \, dx = \begin{cases} 1 & \text{if } n = 0 \\ \frac{\sin n\pi}{n\pi} \bigg|_{0}^{1} & \text{otherwise} \end{cases}
\]
\[
= \begin{cases} 1 & \text{if } n = 0 \\ \frac{\sin n\pi}{n\pi} = 0 & \text{otherwise} \end{cases}
\]

\( b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx \)
\[
= \int_{0}^{1} \sin n\pi x \, dx = -\cos n\pi x \bigg|_{0}^{1} = -\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi}
\]
\[
= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\]

So the Fourier series is:

\[
f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2 \sin (2n+1)\pi x}{(2n+1)\pi}
\]

Note that at points of discontinuity \((-1, 0, 1)\) we have that

\[
f(-1) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2 \sin (2n+1)\pi}{(2n+1)\pi} = \frac{1}{2}
\]

and likewise \( f(0) = f(1) = \frac{1}{2} \).

Ex 2: \( f(x) = \begin{cases} 1 & x \in [-2, 0) \\ x & x \in (0, 2] \end{cases} \)

Ex 3: \( f(x) = \cos^2 x \) on \([-\pi, \pi]\)
Even & Odd Functions

Recall:  
Even function: \( f(x) = f(x) \)
Odd function: \( f(-x) = -f(x) \)

Also  
\( \cos \frac{\pi x}{L} \) is EVEN  
\( \sin \frac{\pi x}{L} \) is ODD

Therefore, if \( f \) is even, its Fourier series only contains \( \cos \)'s  
If \( f \) is odd, its Fourier series only contains \( \sin \)'s

\[
\sin x \cdot \text{ODD}(x) \cdot \text{EVEN}(x) = \text{ODD}(x) \\
\text{ODD}(x) \cdot \text{ODD}(x) = \text{EVEN}(x) \\
\text{EVEN}(x) \cdot \text{EVEN}(x) = \text{EVEN}(x)
\]

Also, if \( f \) is odd, then
\[
\int_{-L}^{L} f(x) \, dx = \int_{-L}^{0} f(x) \, dx + \int_{0}^{L} f(x) \, dx
\]
\[
= \int_{0}^{L} -f(x) \, dx + \int_{0}^{L} f(x) \, dx
\]
\[
= 0
\]

Example: If \( f \) is odd:

Then  
\( a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx = 0 \) since \( f \cdot \cos \) is odd,

\( b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx = \frac{1}{L} \left( \int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \, dx + \int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \, dx \right) \)
\[
= \frac{1}{L} \left( \int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \, dx + \int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \, dx \right) = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \, dx
\]
Any function on \([0, L]\) can be expanded as either a sine or cosine series by computing an extension. Let \(f\) be defined on \([0, L]\).

Then the odd extension of \(f\) to \([-L, 0)\) is:

\[
F(x) = \begin{cases} 
  f(x), & x \in [0, L] \\
  -f(-x), & x \in [-L, 0)
\end{cases}
\]

The even extension of \(f\) to \([-L, 0)\) is:

\[
G(x) = \begin{cases} 
  f(x), & x \in [0, L] \\
  f(-x), & x \in [-L, 0)
\end{cases}
\]

Since \(F\) is odd, it can be expanded in a sine series. Since \(G\) is even, it can be expanded in a cosine series. And on \([0, L]\), \(F(x) = G(x)\).