New topic: Two-point boundary value problems:

Consider:

\[ \begin{align*}
  y'' + \lambda y &= 0 \\
  y(0) + by(1) &= 0 \\
  cy(1) + dy'(1) &= 0
\end{align*} \]

ODE

boundary conditions (B.C.'s)

For what values of \( \lambda \) are there non-trivial (non-zero) solutions?

This is an eigenvalue problem:

\[ y'' = -\lambda y + \text{B.C.'s}. \]

Very different than an initial value problem.

**Ex:** \( y'' + \lambda y = 0 \)

\( y(0) = 0 \)

\( y(1) = 0 \)

**Case 1:**

If \( \lambda = 0 \), then \( y = c_1 x + c_2 \)

\( \Rightarrow y = 0 \) is only solution

**Case 2:**

If \( \lambda < 0 \), then \( y = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x} \)

Applying B.C.'s:

\( c_1 + c_2 = 0 \)

\( c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} = 0 \)

\( \Rightarrow c_1 = -c_2 \)

\( \Rightarrow c_1 (e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}) = 0 \)

\( \Rightarrow c_1 = c_2 = 0 \)

\( \Rightarrow 0 \text{ for } \lambda < 0 \)
If \( \lambda > 0 \), then 
\[
y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x
\]

\[
y(0) = 0 \Rightarrow c_1 = 0
\]
\[
y(L) = 0 \Rightarrow c_2 \sin \sqrt{\lambda} L = 0 \Rightarrow \text{Either } c_2 = 0, \text{ and } y = 0
\]


Therefore the 2-point BVP 
\[
y'' + \lambda y = 0 \quad y(0) = y(L) = 0
\]

non-trivial solutions if 
\[
\lambda = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, \ldots
\]

and in which case the solution is 
\[
y(x) = \sin \frac{n \pi}{L} x
\]

The set of eigenvalues and eigenfunctions to this problem are 
\[
\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad \psi_n(x) = \sin \frac{n \pi}{L} x
\]

Rewritng: 
\[
- y'' = - \frac{d^2}{dx^2} y = \lambda y = \lambda y
\]

Physics application:

\[
\begin{align*}
\text{spring} & \text{ satisfies an ODE, at which certain frequencies will then be "standing waves"?}
\end{align*}
\]

Other applications come from PDEs...
Partial Differential Equations

Relate changes in multiple variables (time, space, etc.).

Three main PDEs:

\[
\begin{align*}
\Delta u &= 0 \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
\frac{\partial u}{\partial t} &= \Delta u \\
\frac{\partial^2 u}{\partial t^2} &= \Delta u
\end{align*}
\]

Laplace Eqn | Heat Equation | Wave Equation

Each equation must be augmented with boundary conditions (in the spatial variables) and initial condition (in the temporal variable).

Example: Heat equation

\[
\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + u(x,0) = f(x) \quad \text{Initial condition}
\]

\[
\begin{align*}
\frac{u(0,t)}{u(L,t)} = 0 & \quad \text{Boundary condition}
\end{align*}
\]

One ansatz for the solution: \( u(x,t) = X(x)T(t) \)

Separation of variables.

Inserting into the equation we have:

\[
X T' = \alpha^2 X'' T
\]

\[
\frac{T'}{T} = \frac{X''}{X} \quad \Rightarrow \quad \frac{T'}{T} = -\lambda = \frac{X''}{X}
\]

only depends on \( t \) \quad only depends on \( x \)

\[
\Rightarrow T(t) = e^{-\lambda t} \\
X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)
\]
The initial condition implies: \( u(x,0) = X(x) \Theta(t) = f(x) \).
The B.C. implies: \( \begin{align*}
    u(0,t) &= X(0)\Theta(t) = 0 \quad \Rightarrow \quad X(0) = 0 \\
    u(L,t) &= X(L)\Theta(t) = 0 \quad \Rightarrow \quad X(L) = 0
\end{align*} \)
Thus, \( u(x,t) = X(x)\Theta(t) \) only if
\[
X'' + 2X = 0, \quad X(0) = 0, \quad X(L) = 0 \quad \Rightarrow BVP, \quad \text{only has non-trivial solution if } \lambda \text{ is eigenval.}
\]

\[
\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad X_n(x) = \sin \frac{n \pi x}{L}
\]

Using these values of \( \lambda_n \), it also must be that \( T_n(t) = e^{-x^2 n^2 \pi^2 t / L^2} \).

So \( u \) consists of any linear combination of \( u_n = X_n T_n \)
\[
\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t)
\]
\[
= \sum_{n=1}^{\infty} a_n \sin \frac{n \pi x}{L} e^{-x^2 n^2 \pi^2 t / L^2}
\]
This automatically satisfies the B.C.'s \( X(0) = X(L) = 0 \).

The initial condition now means that
\[
u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n \pi x}{L}
\]

Goal: Determine \( a_n \) such that the above expression is true.

\( f \) could be an arbitrary function on \([0,L]\) so long as \( f(0) = f(L) = 0 \) in this case. Can arbitrary functions be written as infinite linear comb. of sinuosids? (or cosines, or \( e^{\pm n \pi L x} \) is?)
\[
\Rightarrow \text{Fourier series}
\]