December 4, 2019

Last time:

General system of linear DE's:

$$\vec{x}'(t) = \vec{f}(\vec{x}) \qquad (\text{autonomous})$$

Qualitative features:

At equilibrium point, $\vec{x}' = 0 \Rightarrow \vec{f}(\vec{x}^0) = \vec{0}$

Linearize $\vec{f}$ ~~near~~ about this point:

$$\vec{f}(\vec{x}) = \underset{(\nabla\vec{f})(\vec{x}^0)}{A}(\vec{x}-\vec{x}^0) + \underbrace{\vec{g}(\vec{x}-\vec{x}^0)}_{\text{Higher order terms}}$$

Then, study systems:

$$\vec{x}'(t) = A\vec{x} + \vec{g}(\vec{x})$$

If $\vec{g}$ contains only higher order terms, then it is small near $\vec{x} = \vec{0}$ and $A\vec{x}$ dominates.

Stability of equilibrium $\vec{x} = \vec{0}$ (mostly) depends on eigenvalues of A (see last lecture's notes).

The Phase-Plane

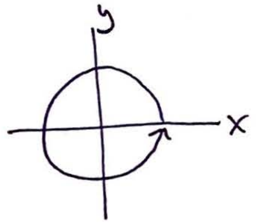
Model Problem: $\vec{x}' = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$

Solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ traces out <u>orbit</u> or <u>trajectory</u> in

$\mathbb{R}^2$. $\rightarrow$ This is called the phase-plane.

<u>Ex:</u> $\frac{dx}{dt} = -y$ $\quad$ (solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$)

$dy/dt = x$



<u>Periodic Orbit</u>

<u>Ex:</u> $x' = -x - y$

$y' = x - y$ $\quad \Rightarrow \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-t} \cos t \\ e^{t} \sin t \end{pmatrix}$

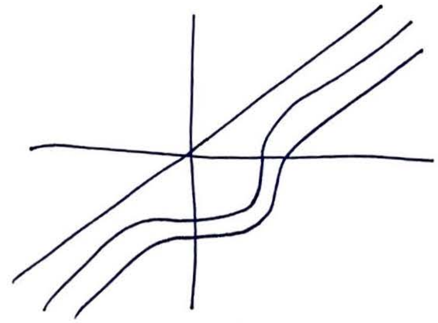
<u>Stable Spiral</u>

<u>Local orbits / trajectories</u>

Note that $\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ $\Big\}$ local ODE assuming $f(x,y) \neq 0$.

Example

$$x' = y^2$$
$$y' = x^2$$



$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y^2}$$

Separable, solution is: $\underbrace{y(x) = (x^3 - c)^{1/3}}_{\text{these are the orbits}}$

Note Equilibrium points and orbits

Ex: $x' = y(1-x^2-y^2)$
$y' = -x(1-x^2-y^2)$ $\Rightarrow$ $\dfrac{dy}{dx} = -\dfrac{x}{y}$ $\Big\}$ circles are orbits.

But when $x^2+y^2=1$, $x'=y'=0$, $\Rightarrow$ equilibrium point.

So every point on $x^2+y^2=1$ is an equilibrium point,

and not an ~~[illegible]~~ orbit.

$x^2+y^2=1$ is

Furthermore the orbits do not tell you how fast the solution moves, ~~speed~~ since $dy/dx = h(x,y)$ contains no t variable.

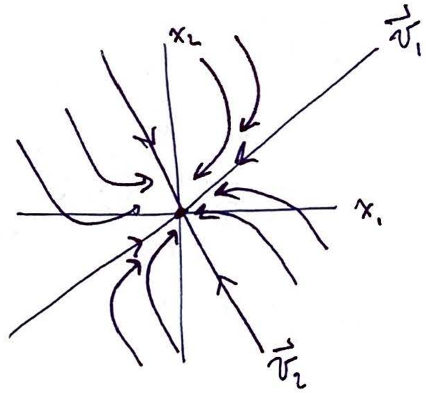
Phase Portraits of Linear Systems

Phase Portrait is a picture of <u>all</u> orbits of the system $\vec{x}' = A\vec{x}$. Let $\vec{x}$ be a point in $\mathbb{R}^2$.

Recall, for $\vec{x}' = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \vec{x}$, the solution

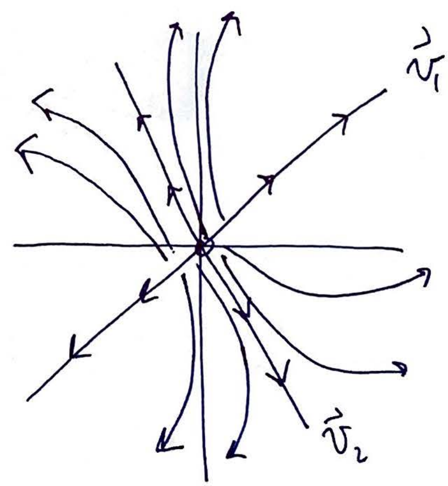
$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 \qquad (\text{if two distinct eigenvectors}).$$

First, assume <u>real λ</u>:



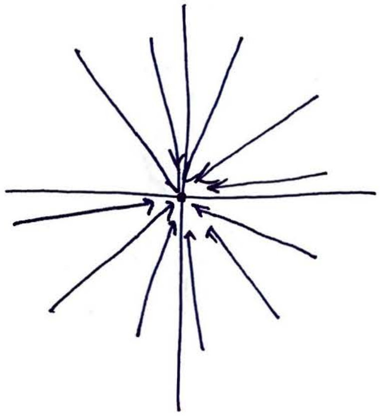
$\underline{\lambda_2 < \lambda_1 < 0}$

$\vec{x} = \vec{0}$ is stable.



$\underline{0 < \lambda_1 < \lambda_2}$

$\vec{0}$ is unstable.

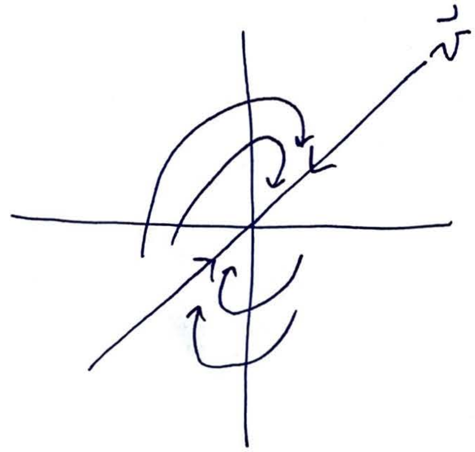


$\underline{\lambda_1 = \lambda_2 < 0}$

$\vec{0}$ is stable

2 lin. indep. eigenvectors

$\vec{x}(t) = e^{\lambda t}(c_1\vec{v}_1 + c_2\vec{v}_2)$

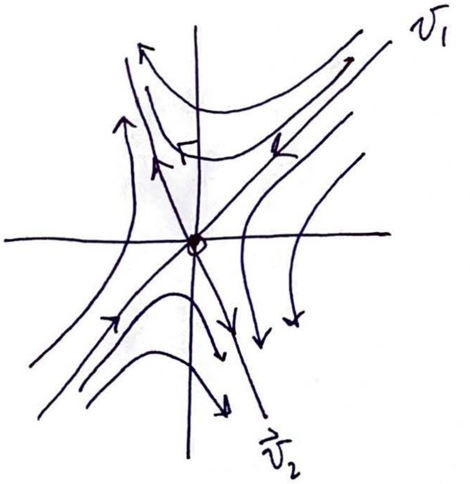


$\underline{\lambda_1 = \lambda_2 < 0}$

$\vec{0}$ is stable

1 linearly indep. eigenvector

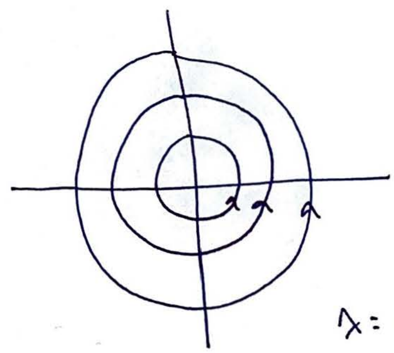
$\vec{x}(t) = e^{\lambda t}(c_1\vec{v} + c_2(\vec{u} + kt\vec{v}))$



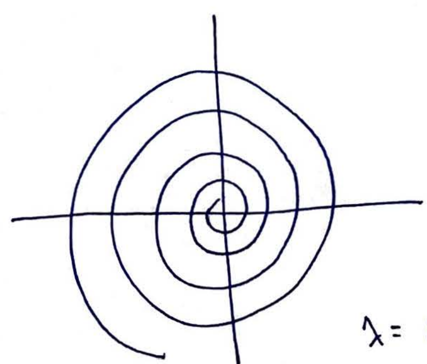
$\underline{\lambda_1 < 0 < \lambda_2}$

"saddle point"

$\vec{0}$ is unstable.



$\lambda = i\beta$



$\lambda = \alpha + i\beta$

$\alpha > 0 \rightarrow$ unstable

$\alpha < 0 \rightarrow$ stable.

Ex: $\vec{x}' = \begin{pmatrix} 0 & -1 \\ 8 & -6 \end{pmatrix} \vec{x}$

Ex: $\vec{x}' = \begin{pmatrix} 4 & -1 \\ -2 & 5 \end{pmatrix} \vec{x}$