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Qualitative solutions of systems of D.E's:

Since, in general, the system
$$\vec{x}'(t) = \vec{f}(t, \vec{x}(t)) \quad \text{is } \underline{\text{not}} \text{ analytically}$$
solvable, we would like some qualitative statements of the solution.

<u>Equilibrium</u>: $\vec{x}_0$ is an equilibrium solution if $\vec{f}(\vec{x}_0) = 0$

<u>Autonomous</u>: The system $\vec{x}' = \vec{f}$ is autonomous if $\vec{f} = \vec{f}(\vec{x}(t))$, i.e., $\vec{f}$ does not have an explicit t-dependence.

<u>Example</u>:
$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -x - xy^2 \\ -y - yx^2 \\ 1 - z + x^2 \end{pmatrix}}_{\vec{f}(\vec{x})} \qquad \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Equilibrium solution satisfies $\vec{f}(\vec{x}) = \vec{0}$.

$$\Rightarrow \begin{array}{lll} -x - xy^2 = 0 & \rightarrow -x(1+y^2) = 0 & \rbrace \rightarrow x = y = 0 \\ -y - yx^2 = 0 & \rightarrow -y(1+x^2) = 0 & \\ 1 - z + x^2 = 0 & \rightarrow z = 1 + x^2 & \rbrace \rightarrow z = 1 \end{array}$$

Equilibrium at $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

<u>Stability</u>: A solution $\vec{\varphi}(t)$ to $\vec{x}' = \vec{f}(\vec{x})$ is stable if any other solution $\vec{\psi}$ such that $\|\vec{\varphi}(0) - \vec{\psi}(0)\| < \delta$ implies that $\|\vec{\varphi}(t) - \vec{\psi}(t)\| < \epsilon$ for all $t > 0$ (in general, δ depends on ϵ).

Next, what can we say about the general problem, $\vec{x}' = \vec{f}(\vec{x})$?

Recall, $\vec{f}(\vec{x}) = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix}$

Expand $\vec{f}(\vec{x})$ near an equilibrium point $\vec{x}^0$:

$$\vec{f}(\vec{x}) = \underbrace{\vec{f}(\vec{x}^0)}_{=\vec{0}} + \underbrace{\begin{pmatrix} \partial f_1/\partial x_1 (x_1 - x_1^0) + \frac{\partial f_1}{\partial x_2}(x_2 - x_2^0) + \dots \\ \vdots \\ \partial f_n/\partial x_1 (x_1 - x_1^0) + \dots \end{pmatrix}}_{\text{First-order terms in multivariate Taylor series of } f_j.}$$

$$+ \underbrace{\begin{pmatrix} \sum_{i,j} \frac{\partial^2 f_1}{\partial x_i \partial x_j}(x_i - x_i^0)(x_j - x_j^0) \\ \vdots \\ \sum_{i,j} \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x_i - x_i^0)(x_j - x_j^0) \end{pmatrix}}_{\text{Quadratic terms for } \vec{f}.} + \underbrace{\dots}_{\text{Higher order terms.}}$$

Rewrite:

$$\vec{f}(\vec{x}) = \overbrace{\underbrace{(\nabla \vec{f})}_{\substack{\text{Gradient} \\ \text{(matrix)}}}(\vec{x} - \vec{x}^0)}^{\substack{\text{The linearization} \\ \text{of } \vec{f} \text{ at } \vec{x} = \vec{x}^0}} + (\vec{x} - \vec{x}^0)^T \underbrace{(\nabla \nabla \vec{f})}_{\substack{\text{Hessian} \\ \text{(matrix)}}}(\vec{x} - \vec{x}^0) + \dots$$

$$= A(\vec{x} - \vec{x}_0) + g(\vec{x} - \vec{x}^0)$$

Let $\vec{y} = \vec{x} - \vec{x}^0$,

then $\vec{y}' = \vec{x}'$, so equivalent to $\vec{x}' \approx A\vec{x} + g(\vec{x})$.

Stability for $\vec{x}' = A\vec{x}$

Stability is a consequence of the eigenvalues of A:

Thm:

(a) If all λ's have $\mathrm{Re}(\lambda) < 0$ $\Rightarrow$ all solutions to $\vec{x}' = A\vec{x}$ are stable solutions

(b) If at least one λ has $\mathrm{Re}(\lambda) > 0$ $\Rightarrow$ all solutions are unstable.

(c) All $\mathrm{Re}(\lambda) \leq 0$, n linearly independent eigenvectors $\Rightarrow$ all solutions are stable

See Text for proof:

If $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \dots + c_n e^{\lambda_n t}\vec{v}_n$ then all λ must have $\mathrm{Re}(\lambda) < 0$ to be decaying.

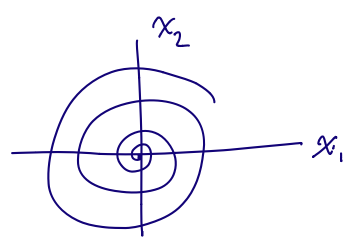
If $\vec{x}(t) \sim (a v_1 + bt v_2 + \dots)e^{\lambda t}$ then unstable.

Next definition <u>Asymptotically stable</u> $\vec{\varphi}$ is asymptotically stable if it is stable, <u>and</u> any solution which starts sufficiently close also approaches $\vec{\varphi}$, i.e. $\vec{\psi} \to \vec{\varphi}$ as $t \to \infty$.

Example:

$$\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$\vec{x}_0 = \vec{0}$ is asymptotically stable:



Stability for $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$

Assume that $\vec{g}(\vec{0}) = \vec{0}$ (as in the Hessian before)

$\Rightarrow$ Then $\vec{x}^{\circ} = 0$ is an equilibrium solution. Stable or unstable?

Thm If $\dfrac{\vec{g}(\vec{x})}{\|\vec{x}\|} \to 0$ as $\vec{x} \to \vec{0}$, then basically have the stability results of $\vec{x}' = A\vec{x}$.

(a) $\mathrm{Re}(\lambda) < 0 \Rightarrow \vec{x}(t) = \vec{0}$ is asymptotically stable.

(b) At least one λ with $\mathrm{Re}(\lambda) > 0$ $\Rightarrow$ $\vec{x} = 0$ unstable.

(c) $\mathrm{Re}(\lambda) \leq 0 \Rightarrow$ stability cannot be determined from the stability of $\vec{x} = \vec{0}$ for $\vec{x}' = A\vec{x}$.

Proof: (a) Generalized variation of parameters

(b) Too detailed...

(c) By counterexample.

Stable

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2 - x_1(x_1^2 + x_2^2) \\ -x_1 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{Linearization}} + \begin{pmatrix} -x_1(x_1^2 + x_2^2) \\ -x_2(x_1^2 + x_2^2) \end{pmatrix}$$

$\lambda = \pm i$

What happens to $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$?

Note:

$$\underbrace{x_1 x_1' + x_2 x_2'}_{= \frac{1}{2}\frac{d}{dt}(x_1^2 + x_2^2)} = -x_1^2 \|\vec{x}\|^2 - x_2^2 \|\vec{x}\|^2 = -\|\vec{x}\|^4$$

$$\Rightarrow \frac{d}{dt}\|\vec{x}\|^2 = \underbrace{-2\|\vec{x}\|^4}_{\text{Always } < 0 \text{ unless } \vec{x} = 0}$$

$\Rightarrow \|\vec{x}\| \to 0$

$\Rightarrow \vec{x} = \vec{0}$ is asymptotically stable.

On the other hand, look at:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2 + x_1(x_1^2 + x_2^2) \\ -x_1 + x_2(x_1^2 + x_2^2) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{Same linearization}} + \begin{pmatrix} x_1(x_1^2+x_2^2) \\ x_2(x_1^2+x_2^2) \end{pmatrix}$$

Same linearization, but... $\frac{d}{dt}\|\vec{x}\|^2 = 2\|\vec{x}\|^4 \Rightarrow \|\vec{x}\| \to \infty$ unless $\vec{x} = \vec{0}$

And therefore $\vec{x} = \vec{0}$ is unstable.

In general, to examine the stability of an equilibrium solution to a general system $\vec{x}' = \vec{f}(\vec{x})$:

⓪ Find equilibrium $\vec{x}^0$ such that $\vec{f}(\vec{x}) = 0$.

① Linearize $\vec{f}(\vec{x} - \vec{x}^0)$:

$$\vec{x}' = \underline{A(\vec{x} - \vec{x}^0)} + g(\vec{x} - \vec{x}^0)$$

② Find eigenvalues of A

③ Apply previous theorem.

Example:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \log(1 + x + y^2) \\ -y + x^3 \end{pmatrix}$$

Clearly $\vec{x} = \vec{0}$ is an equilibrium.

Linearize:

$$\log(1+x+y^2) \approx 0 + \frac{1}{1+x+y^2}\Big|_{\vec{x}=\vec{0}} x + \frac{1}{1+x+y^2}\cdot 2y\Big|_{\vec{x}=\vec{0}} y + \ldots$$

So $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{} \begin{pmatrix} x \\ y \end{pmatrix} + \text{higher order terms}$$

$\lambda = \pm 1 \Rightarrow \vec{x} = \vec{0}$ is unstable.