November 25, 2019

Last time:

Formally, we can write the solution to $\vec{x}' = A\vec{x}$ as $\vec{x}(t) = e^{At}\vec{v}$, for any constant $\vec{v}$. The goal is to pick $\vec{v}$'s such that $e^{At}\vec{v}$ is easy to evaluate, and also generates n linearly independent solutions.

Note $e^{At}\vec{v} = e^{\lambda t} e^{(A-\lambda I)t}\vec{v}$

$\Rightarrow$ Choose $\vec{v}$ such that $(A-\lambda I)^m \vec{v} = \vec{0}$. (with λ an eigenvalue)

If $m=1$, then $\vec{v}$ is an eigenvector.

If $m>1$, then using the power series definition of e^{At}, we see that $e^{(A-\lambda I)t}\vec{v}$ terminates after m terms (and therefore is easy to evaluate).

Next up: Qualitative solutions to Systems of D.E.'s

Example:

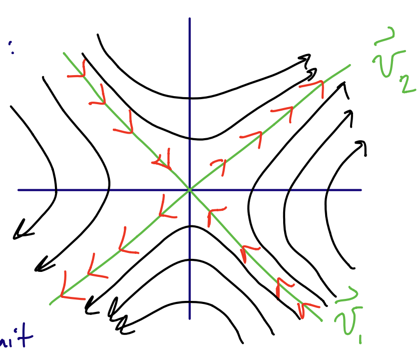
$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x}$$

$\Rightarrow$ Eigenvalues are $\lambda = -1, 1$, and eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively.

General solution is:

$$\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Plot:



Phase Portrait

Solution trajectories are drawn.

$\vec{x}(t) = \vec{0}$ is an unstable equilibrium.

General systems take the form:

$$\vec{x}' = \vec{f}(t, \vec{x})$$

No systematic method for solving this, can we use ideas from $\vec{x}' = A\vec{x}$ to obtain qualitative results?

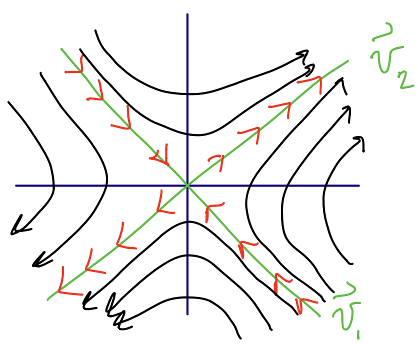
We must also restrict our attention to <u>autonomous</u> systems in which $\vec{f}(t,\vec{x}) = \vec{f}(\vec{x})$, i.e. $\vec{f}$ does not explicitly depend on t, otherwise stability and equilibrium solutions may be time dependent.

<u>Definition</u> (Stability) A solution $\vec{\varphi}(t)$ to $\vec{x}' = \vec{f}(\vec{x})$ is <u>stable</u> if all other solutions $\vec{\psi}(t)$ which start sufficiently close to $\vec{\varphi}$ remain sufficiently close.

More precisely: for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $\| \vec{\psi}(0) - \vec{\varphi}(0) \| < \delta$ then $\| \vec{\psi}(t) - \vec{\varphi}(t) \| < \epsilon$ for all $t > 0$.

<u>Definition</u> (Equilibrium) $\vec{x}_0$ is an equilibrium point if $\vec{f}(\vec{x}_0) = 0$, i.e. if at $\vec{x}_0$, $\vec{x}'(t) = \vec{0}$. (Nothing changing).

<u>Back to our example</u>: $\vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x}$.



The point $\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ is an equilibrium point (since $A\vec{0} = 0$).

It is <u>unstable</u> since any other solution that starts near it goes off to infinity.