November 25, 2019

Last time:

Formally, we can write the solution to \( \dot{x} = Ax \) as \( \tilde{x}(t) = e^{At} \tilde{v} \), for any constant \( \tilde{v} \). The goal is to pick \( \tilde{v} 's \) such that \( e^{At} \tilde{v} \) is easy to evaluate, and also generate a linearly independent solutions.

Note: \( e^{At} \tilde{v} = e^{Ae} e^{At} \tilde{v} \)

\( \Rightarrow \) Choose \( \tilde{v} \) such that \( (A - \lambda I) \tilde{v} = 0 \). (with \( \lambda \) an eigenvalue)

If \( m = 1 \), then \( \tilde{v} \) is an eigenvector.

If \( m > 1 \), then using the power series definition of \( e^{At} \), we see that \( e^{Ae} e^{At} \tilde{v} \) terminates after \( m \) terms (and therefore is easy to evaluate).

Next up: Qualitative solutions to systems of O.E.'s

Example:

\( \tilde{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{x} \)

\( \Rightarrow \) Eigenvalue \( \lambda = -1, 1 \), and eigenvectors are \( \tilde{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), \( \tilde{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), respectively.

General solution is:

\( \tilde{x}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

Solution trajectories are drawn.

\( \tilde{x}(t) = 0 \) is an unstable equilibrium.
General systems take the form:

\[ \dot{x} = f(t, x) \]

No systematic method for solving this, can we use ideas from \( \dot{x} = Ax \) to obtain qualitative results?

We must also restrict our attention to autonomous systems in which \( \dot{f}(t, x) = \dot{f}(x) \), i.e., \( \dot{f} \) does not explicitly depend on \( t \), otherwise stability and equilibrium solutions may be time dependent.

**Definition (Stability)** A solution \( \phi(t) \) to \( \dot{x} = f(x) \) is stable if all other solutions \( \psi(t) \) which start sufficiently close to \( \phi \) remain sufficiently close.

More precisely: for any \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon) > 0 \) such that if \( \| \phi(0) - \psi(0) \| < \delta \) then \( \| \phi(t) - \psi(t) \| < \epsilon \) for all \( t > 0 \).

**Definition (Equilibrium)** \( \dot{x} = 0 \) is an equilibrium point if \( \dot{f}(x) = 0 \), i.e., if at \( x_0 \), \( \dot{x}(t) = 0 \). (Nothing changing).

Back to our example: \( \dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x \).

The point \( x_0 = (0, 0) \) is an equilibrium point (since \( A\dot{x} = 0 \)).

It is unstable since any other solution that starts near it goes off to infinity.