November 20, 2019

Last time:

1. When solving $\ddot{x} = A \dot{x}$, one may find a complex valued $\lambda, \vec{v}$. In this case, if
   
   $\lambda = \alpha + e^B$
   
   $\vec{v} = \vec{v}_1 + e^B \vec{v}_2$

   then two real-valued linearly independent solutions are:
   
   $\vec{y}(t) = e^t \left( \cos Bt \vec{v}_1 - \sin Bt \vec{v}_2 \right)$
   
   $\vec{z}(t) = e^t \left( \sin Bt \vec{v}_1 + \cos Bt \vec{v}_2 \right)$

2. On the other hand, not all matrices have a full set of linearly independent eigenvalues:

   Ex: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda = 1 \\ \vec{v} = (0) \end{pmatrix}$ (algebraic mult = 2, geometric mult = 1)

To solve $\ddot{x} = A \dot{x}$ in this case, turn to matrix exponential:

We wish to write solution to $\ddot{x} = A \dot{x}$ as

$\vec{x} = e^{At} \vec{z}$ for any vector $\vec{z}$

Why? $e^{At}$ is invertible for any $A$

$\Rightarrow$ general solution is linear combination of the columns of $e^{At}$

Recall: $e^{At} = \sum_{n=0}^{\infty} \frac{A^t n^t}{n!}$
Why is $e^{At}$ useful in computing solutions with repeated roots?

$\Rightarrow$ $e^{At}v$ is a solution to $\dot{x} = Ax$ for every $v$:

$$\frac{d}{dt} e^{At}v = \left( \frac{d}{dt} e^{At} \right) v = Ae^{At}v$$

$$= A(e^{At}v)$$

Properties of $e^{At}$:

$$e^{(A+B)t} = \sum_{n=0}^{\infty} \frac{(A+B)^n t^n}{n!}$$

$$= I + (A+B)t + \frac{(A^2 + AB + BA + B^2)t^2}{2!} + \ldots$$

$$= \ ? \ A \text{e} Bt \ ?$$

Obviously $A + B = B + A$, so is it true that $e^{(A+B)t} = e^{(B+A)t}$?

$$= \ ? \ A \text{e} Bt \ ?$$

Only if $A$ and $B$ commute, i.e., if $AB = BA$. (to see multiply)

Example: $e^{0t} = I$

$$= e^{(A-A)t}$$

$$= \sum_{n=0}^{\infty} \frac{(A-A)^n t^n}{n!}$$

Obviously $A$ and $-A$ commute,

so...

$$= e^{At}e^{-At} = e^{At}e^{At}$$

$$\Rightarrow I = e^{At}e^{At} = (e^{At})^{-1} = e^{-At}$$

I.e. $e^{At}$ is always invertible.

The goal is to evaluate $e^{At}$.\[\]
A solution of the form $e^{At} \vec{v}$ is only useful if we can compute it, i.e., sum the infinite series $e^{At}$.

One can show that
\[ e^{At} \vec{v} = e^{(A-\lambda I)t} \vec{v} \] for any constant $\lambda$.

Furthermore:
1. $e^{(A-\lambda I)t} \vec{v} = e^{At} \vec{v}$ (easy to show since $I\vec{v} = \vec{v}$)
2. If $(A-\lambda I)^n \vec{v} \neq 0$, then the series for $e^{(A-\lambda I)t} \vec{v}$ only contains $n$ terms,
   \[ e^{(A-\lambda I)t} \vec{v} = e^{At} \vec{v} \]
   \[ = e^{\lambda t} \left( \vec{v} + (A-\lambda I)t \vec{v} + \frac{(A-\lambda I)^2}{2!} t^2 \vec{v} + \ldots + \frac{(A-\lambda I)^{n-1}}{(n-1)!} t^{n-1} \vec{v} \right) \]

An algorithm for finding general solution $d \vec{x} = A \vec{x}$:

1. Find all eigenvalues $\lambda$, and as many eigenvectors as possible.
   - Linearly independent
   - This generates solutions of the form $e^{\lambda t} \vec{v}$.

2. If $\lambda$ has algebraic multiplicity $k$ but fewer than $k$ linearly independent eigenvectors, then:
   - Find all such vectors $\vec{v}$ such that:
     1. $(A-\lambda I)^2 \vec{v} = 0$
     2. $(A-\lambda I)^3 \vec{v} = 0$
     3. and so on.

If $(A-\lambda I)^n \vec{v} \neq 0$, but $(A-\lambda I)^{n+1} \vec{v} = 0$, then
\[ \vec{x}(t) = \left( \vec{v} + (A-\lambda I)t \vec{v} + \frac{1}{2!} (A-\lambda I)^2 t^2 \vec{v} + \ldots + \frac{1}{m!} (A-\lambda I)^m t^m \vec{v} \right) \]
is a solution, \( ( \text{Exactly analogous to the method of reduction of order, which yields solution } e^{\lambda t}, t e^{\lambda t} \text{ works, but it does not) } \to y'' + 2y' + y = 0 \).