November 20, 2019

Last time:

(1) When solving $\vec{x}' = A\vec{x}$, one may find a complex valued $\lambda, \vec{v}$. In this case, if

$$\lambda = \alpha + i\beta$$
$$\vec{v} = \vec{v}_1 + i\vec{v}_2,$$

then two real-valued linearly independent solutions are:

$$\vec{y}(t) = e^{\alpha t}\left(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2\right)$$
$$\vec{z}(t) = e^{\alpha t}\left(\sin\beta t\,\vec{v}_1 + \cos\beta t\,\vec{v}_2\right)$$

(2) On the otherhand, not all matrices have a full set of linearly independent eigenvectors:

Ex: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} \lambda = 1 \\ \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{matrix}$ (algebraic mult = 2, geometric mult = 1)

To solve $\vec{x}' = A\vec{x}'$ in this case, turn to <u>matrix exponential</u>:

We wish to write solution to $\vec{x}' = A\vec{x}$ as

$$\vec{x} = e^{At}\vec{c} \quad \text{for } \underline{\text{any}} \text{ vector } \vec{c}.$$

Why? e^{At} is <u>invertible</u> for <u>any A</u>

$\Rightarrow$ general solution is linear combination of the columns of e^{At}.

<u>Recall</u>: $e^{At} = \displaystyle\sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$

Why is e^{At} useful in computing solutions with repeated roots?

$\Rightarrow$ $e^{At}\vec{v}$ is a solution to $\vec{x}' = A\vec{x}$ for every $\vec{v}$:

$$\frac{d}{dt} e^{At}\vec{v} = \left(\frac{d}{dt} e^{At}\right)\vec{v} = A e^{At}\vec{v}$$
$$= A\left(e^{At}\vec{v}\right)$$

Properties of e^{At}:

$$e^{(A+B)t} = \sum_{n=0}^{\infty} \frac{(A+B)^n t^n}{n!}$$
$$= I + (A+B)t + \frac{(A^2 + AB + BA + B^2)\,t^2}{2!} + \dots$$
$$=?\ e^{At} e^{Bt}\ ?$$

Obviously $A+B = B+A$, so is it true that $e^{(A+B)t} = e^{(B+A)t}$

$$= e^{At} e^{Bt} = e^{Bt} e^{At}\ ?$$

$\Rightarrow$ Only if A and B commute, i.e., if $AB = BA$. (to see multiply out $e^{At} e^{Bt} \dots$)

Example: $e^{0t} = I$

$$= e^{(A-A)t}$$
$$= \sum_{n=0}^{\infty} \frac{(A-A)^n t^n}{n!}$$ obviously A and $-A$ commute,

so...

$$= e^{At} e^{-At} = e^{-At} e^{At}$$

$\Rightarrow$ $I = e^{At} e^{-At}$ so therefore $\left(e^{At}\right)^{-1} = e^{-At}$

I.e. e^{At} is always invertible.

The goal is to evaluate e^{At}.

A solution of the form $e^{At}\vec{v}$ is only useful if we can compute it, i.e. sum the infinite series e^{At}.

One can show that

$$e^{At}\vec{v} = e^{(A-\lambda I)t}\, e^{\lambda I t} \qquad \text{for any constant } \lambda.$$

(since A, I commute)

Furthermore:

① $e^{\lambda I t}\vec{v} = e^{\lambda t}\vec{v}$ (easy to show since $I\vec{v} = \vec{v}$)

② If $(A-\lambda I)^m \vec{v} = 0$, then the series for $e^{(A-\lambda I)t}\vec{v}$ only contains m terms.

$$\Rightarrow e^{At}\vec{v} = e^{(A-\lambda I)t}\, e^{\lambda I t}\vec{v}$$

$$= e^{\lambda t}\left(\vec{v} + (A-\lambda I)t\vec{v} + \frac{(A-\lambda I)^2 t^2}{2!}\vec{v} + \dots + \frac{(A-\lambda I)^{m-1} t^{m-1}}{(m-1)!}\vec{v}\right)$$

An algorithm for finding general solution of $\vec{x}' = A\vec{x}$:

① Find all eigenvalues of A, and as many linearly independent eigenvectors as possible.

This generates solutions of the form $e^{\lambda t}\vec{v}$.

② If λ has algebraic multiplicity k but fewer than k linearly independent eigenvectors, then:

Find all such vectors $\vec{v}$ such that:

$$(A-\lambda I)^2 \vec{v} = 0$$

$$(A-\lambda I)^3 \vec{v} = 0$$

$$\vdots$$

and so on.

If $(A-\lambda I)^m \vec{v} \neq 0$, but $(A-\lambda I)^{m+1}\vec{v} = 0$, then

$$\vec{x}(t) = \left(\vec{v} + (A-\lambda I)t\vec{v} + \frac{1}{2!}(A-\lambda I)^2 t^2 \vec{v} + \dots + \frac{1}{m!}(A-\lambda I)^m t^m \vec{v}\right)$$

is a solution!

(Exactly analogous to the method of reduction of order, which yields solutions e^{t}, te^{t} to $y'' + 2y' + y = 0$).

(Will not prove this alg. works, but it does.)