November 18, 2019

Last time:

Systems of DE's:

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0$$

Eigenvalue/vector solution method:

If $\vec{x} = e^{\lambda t}\vec{v}$, $\vec{v}$ a constant vector, then it must be the case that

$$\vec{x}' = \lambda e^{\lambda t}\vec{v} = A e^{\lambda t}\vec{v}$$

$$\Rightarrow \quad \lambda\vec{v} = A\vec{v}$$

$$\Rightarrow \quad \lambda, \vec{v} \text{ is an eigenpair for } A.$$

Therefore if A has n linearly independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$, then the general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + \dots + c_n e^{\lambda_n t}\vec{v}_n$$

To then satisfy the initial condition $\vec{x}(0) = \vec{x}_0$,

solve $\vec{x}(0) = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{x}_0$

$$\Rightarrow \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{x}_0$$

$$\Rightarrow \quad V\vec{c} = \vec{x}_0 \quad \text{an } n\times n \text{ linear system}$$

Eigenvalues are found as solution to $\underbrace{\det(A - \lambda I)}_{\text{polynomial in } \lambda} = 0$

Then eigenvectors are the nullspace of $A - \lambda I$.

Example:
$$\vec{x}' = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$p(\lambda) = (1-\lambda)(1-\lambda) - 36 = 0$$
$$\lambda^2 - 2\lambda - 35 = 0$$
$$(\lambda-7)(\lambda+5) = 0$$
$$\lambda = -5, 7$$

$$\begin{pmatrix} 6 & 12 \\ 3 & 6 \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 12 \\ 3 & -6 \end{pmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

General solution: $\vec{x} = c_1 e^{-5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \vec{c} \Rightarrow \vec{c} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Ex: $\vec{x} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}_{\text{Not invertible}} \vec{x}$

Clearly: $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \vec{x} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

Complex Roots

$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$$

$$p(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

Eigenvectors:
$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

And therefore we know that
$$\vec{v}_2 = \overline{\vec{v}_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

The complex-valued general solution is then
$$\vec{x} = c_1 e^{it} \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Rewriting this in terms of real & imaginary parts we see:

$$\vec{x} = c_1 (\cos t + i\sin t)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$+ c_2 (\cos t - i \sin t) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right)$$

$$= c_1 \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - c_1 \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - c_2 \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ i \left(c_1 \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_1 \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - c_2 \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - c_2 \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= \underset{c_1 + c_2}{\underset{\uparrow}{d_1}} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i \underset{c_1 - c_2}{\underset{\uparrow}{d_2}} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

So two, real-valued linearly independent solutions are $\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ and $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

More generally, we have:

is a solution to $\vec{x}' = A\vec{x}$

If $\vec{x}(t) = e^{(\alpha + i\beta)t} (\vec{v}_1 + i\vec{v}_2)$, with $\alpha, \beta, \vec{v}_1, \vec{v}_2$ real, then

$$\vec{x}(t) = \vec{y}(t) + i\vec{z}(t) \quad \text{with}$$

$$\vec{y}(t) = e^{\alpha t} (\cos \beta t \, \vec{v}_1 - \sin \beta t \, \vec{v}_2)$$

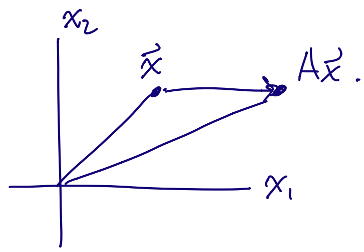
$$\vec{z}(t) = e^{\alpha t} (\sin \beta t \, \vec{v}_1 + \cos \beta t \, \vec{v}_2)$$

two real-valued, linearly independent solutions.

Another example:

$$\vec{x}' = \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}_{\text{shear transformation}} \vec{x}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ x_2 \end{pmatrix}$$



Compute eigenvalues:

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 \Rightarrow \lambda = 1 \text{ (repeated root)}$$

Next compute eigenvectors:

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

ONLY 1 linearly independent eigenvector, even though $\lambda = 1$ has <u>algebraic</u> multiplicity 2.

Clearly one solution is $\vec{x}_1(t) = e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. How do we find another linearly independent solution?

To do this we first need to discuss the <u>matrix exponential</u>.

In 1D, for the ODE $x' = ax$ we have that the general solution is $x(t) = Ce^{at}$ for <u>any</u> <u>constant</u> C. It would be nice if we could write the general solution to $\vec{x}' = A\vec{x}$ as $\vec{x} = e^{At}\vec{v}$, for <u>any</u> constant vector $\vec{v}$.

This begs the question, how do we define e^{At}?

In 1D, $e^{at} = 1 + (at) + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$ by powerseries expansion

Therefore, define e^{At} to be:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

This expansion can be shown to converge for <u>any A</u> and <u>any t</u>.

Aside: If A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ and set $V = (\vec{v}_1 \cdots \vec{v}_n)$, then $A = V \lambda V^{-1}$

$$\lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

And therefore, $A^p = (V\lambda V^{-1})^p = \underbrace{(V\lambda V^{-1}) \cdots (V\lambda V^{-1})}_{p \text{ times}}$

$$= V \lambda^p V$$

So in this case,

$$e^{At} = VV^{-1} + V\lambda V^{-1} t + \frac{1}{2!} V\lambda^2 V^{-1} t^2 + \dots$$

$$= V\left(I + \lambda + \frac{\lambda^2 t}{2!} + \frac{\lambda^3 t^3}{3!} + \dots\right)V^{-1}$$