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Just to re-cap:

(1) $y' = f(t,y)$

(2) $y'' + p(t)\,y' + q(t)\,y = g(t)$

The obvious generalizations are:

Higher-order: $y^{(n)} + a_{n-1}(t)\,y^{(n-1)} + \dots + a_0(t)\,y = g(t)$

Coupled systems: $y_1' = f(t, y_1, y_2)$
$y_2' = g(t, y_1, y_2)$

Next Topic: Systems of Differential Equations (§3.1→)

Similar to solving $A\vec{x} = \vec{b}$, we can solve coupled systems of differential equations:

$$x_1'(t) = \frac{dx_1}{dt} = f_1(t, x_1, \dots, x_n)$$
$$\vdots$$
$$x_n'(t) = \frac{dx_n}{dt} = f_n(t, x_1, \dots, x_n)$$

There are n functions x_j, each a function of t.
In general, the f_j's may be linear or nonlinear, and each equation may be homogeneous or inhomogeneous.

Systems may directly be a model for some physical process, or they may result from decomposing a higher order DE.

Ex:

$$y'' + py' + qy = f$$

Let $x_1 = y$ Then $x_1' = y' = x_2$
$x_2 = y'$ $x_2' = y'' = f - py' - qy$
$= f - px_2 - qx_1$

$\Rightarrow$ $\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= f - px_2 - qx_1 \end{aligned}}$

Linear Systems of DE:

The most general linear system of DE is:

$$x_1' = a_{11}(t)\,x_1 + \dots + a_{1n}(t)\,x_n + g_1$$
$$\vdots$$
$$x_n' = a_{n1}(t)\,x_1 + \dots + a_{nn}(t)\,x_n + g_n$$

We can write this using concise notation:

Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, then $\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$

Let $A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & a_{22}(t) & \vdots \\ & \ddots & \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$

Then we have $\vec{x}' = A\vec{x} + \vec{g}$

If $\vec{g} = \vec{0}$, then <u>homogeneous</u> system.

If $\vec{A}(t) = A$ (does not depend on t), then constant-coefficient.

<u>Example:</u>

$$x_1' = 3x_1 - 7x_2 + 9x_3$$
$$x_2' = 15x_1 + x_2 - x_3$$
$$x_3' = 7x_1 + 6x_3$$

$\Rightarrow \vec{x}' = \begin{pmatrix} 3 & -7 & 9 \\ 15 & 1 & -1 \\ 7 & 0 & 6 \end{pmatrix} \vec{x}$, specify initial condition as: $x(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

If $\vec{x}, \vec{y}$ are solutions to $\vec{x}' = A\vec{x}$ (no initial condition),

then: $\vec{x} + \vec{y}$ is also a solution

$c\vec{x}$ is also a solution

We will use ideas from linear algebra (a pre-requisite):

- Vector spaces
- Null space
- dimension of a vector space
- span, basis
- linear dependence
- invertibility of matrices

See sections 3.2 and 3.3 for a review.

Applications of linear algebra to DE:

Thm Existence & Uniqueness:

There exists exactly one solution to the IVP:

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0,$$

Furthermore, $\vec{x}(t)$ exists for all t. (Analogous to result for $au'' + bu' + cu = 0$.)

(Will not prove.)

Recall: Solutions to $\vec{x}' = A\vec{x}$ form a vector space (closed under linear combinations).

Thm: The dimension of V, the space of all solutions to $\vec{x}' = A\vec{x}$, is n. I.e., there are n linearly independent solutions.

Proof: Let $\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th}$ row.

Then let $\vec{\varphi}_j' = A\vec{\varphi}_j$, with $\varphi_j(0) = \vec{e}_j$.

The $\vec{\varphi}_j$ are linearly dependent since:

$$c_1\vec{\varphi}_1(0) + \ldots + c_n\vec{\varphi}_n(0) = \vec{0} \quad \Rightarrow \quad c_1\vec{e}_1 + \ldots + c_n\vec{e}_n = \vec{0}$$

$\Rightarrow$ $c_1 = \ldots = c_n = 0$ since we know that $\vec{e}_1, \ldots, \vec{e}_n$ are lin. indp.

$\Rightarrow$ $\vec{\varphi}_1, \ldots, \vec{\varphi}_n$ are lin. indp.

Do $\vec{\varphi}_1, \ldots, \vec{\varphi}_n$ span V? I.e. can any solution to $\vec{x}' = A\vec{x}$ be written as: $\vec{x} = c_1\vec{\varphi}_1 + \ldots + c_n\vec{\varphi}_n$?

Take any solution $\vec{x}$. Let set $\vec{c} = \vec{x}(0)$. Then construct $\vec{\varphi} = c_1\vec{\varphi}_1 + \ldots + c_2\vec{\varphi}_n$, Clearly $\vec{\varphi}' = A\vec{\varphi}$, and furthermore:

$$\begin{aligned} \vec{\varphi}(0) &= c_1\vec{\varphi}_1(0) + \ldots + c_n\vec{\varphi}_n(0) \\ &= c_1\vec{e}_1 + \ldots + c_n\vec{e}_n \\ &= \vec{c} \\ &= \vec{x}(0) \end{aligned}$$

Therefore $\vec{\varphi} = \vec{x}$ by the existence and uniqueness result.

$\Rightarrow$ $\varphi_1, \ldots, \varphi_n$ span V, are linearly independent

$\Rightarrow$ $\dim V = n$.