Oct 21, 2019

Last time:

Model DE:

\[ Ly = P(t) y'' + Q(t) y' + R(t) y = 0 \]

If \( P, Q, R \) are polynomials in \( t \), then look for a solution \( y \) of the form

\[ y(t) = \sum_{n=0}^{\infty} a_n t^n. \]

\[ \Rightarrow (\sum p_n t^n) \left( \sum n(n-1) a_n t^{n-2} \right) + \left( \sum q_n t^n \right) \left( \sum a_n t^{n-1} \right) + \left( \sum r_n t^n \right) = 0 \]

Collect terms with \( t^n \) factor, determine recurrence relationship for the \( a_n \): Note: Initial conditions determine \( a_0, a_1, \)

\[ y(0) = \sum a_n (0)^n = a_0 \]
\[ y'(0) = \sum n a_n (0)^{n-1} = a_1 \]

Next topic: Singular Points

Euler's Equation: \( t^2 y'' + at y' + by = 0 \)

at \( t=0 \), these terms disappear. Can we still apply the series solution method?

If \( y = t^r \), then both \( ty' \sim t^r \) and \( t^2 y'' \sim t^r \).

Ansatz: \( y = t^r \).

\[ \Rightarrow r(r-1) t^r + a r t^r + b t^r = 0 \]
\[ (r^2 - r + a r + b) t^r = 0 \]
\[ (r^2 + (a-1)r + b) t^r = 0 \]
The solutions are:

\[ r = -\frac{1}{2} \left( (\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta} \right) \]

Once again, there are three cases:

**Case 1** \((\alpha-1)^2 - 4\beta > 0\)

\[ \Rightarrow \text{Two distinct real roots, solution is } y(t) = c_1 t^{r_1} + c_2 t^{r_2} \]

**Case 2** \((\alpha-1)^2 - 4\beta = 0\)

\[ \Rightarrow \text{Repeated roots, use Method of Reduction of Order to show that general solution is} \]

\[ y(t) = c_1 t^r + c_2 t^r \ln t \]

Alternative calculation to show that \(t^r \ln t\) is a solution:

Since the roots are repeated:

\[ \dot{t}^r = (r-r_1)^2 t^r \]

and

\[ \frac{d}{dr} (\dot{t}^r) = \dot{t} (\frac{d}{dr} t^r) = \dot{t} (\ln t t^r) \]

\[ = 2(r-r_1)t^r + (r-r_1)^2 \ln t t^r \]

\[ = 0 \text{ if } r = r_1, \]

\[ \Rightarrow \ln t t^r \text{ is a solution.} \]

**Case 3** \((\alpha-1)^2 - 4\beta < 0\)

\[ \Rightarrow \text{Two distinct complex roots } r = \lambda \pm \mu i. \]

What is \(t^{\lambda + \mu i}\)?

\[ t^{i\mu} : (e^{i\mu})^{\lambda} = e^{i\mu \lambda} = \cos(\mu \lambda t) + i \sin(\mu \lambda t) \]

\[ \Rightarrow \text{Real-valued general solution is } y(t) = c_1 t^{\lambda} \cos(\mu \lambda t) + c_2 t^{\lambda} \sin(\mu \lambda t). \]
Case of negative $t$ ($t \in (-\infty, 0)$).

$t^2 y'' + x t y' + \beta y = 0$ seems to make sense for $t < 0$, but often $t^2$ does not stay real-valued.

Ex: $r = \frac{1}{2} \Rightarrow (-1)^{\frac{1}{2}} = i$, not real valued

$r = \mu \Rightarrow \cos (\mu \log(-1))$ not defined (unless done very carefully).

These problems can be fixed with a change of variable:

Let $t = -x, \ x > 0$

Then \( \frac{dt}{dx} = -1 \Rightarrow \frac{du}{dt} = -\frac{du}{dx} \)

\( d^2 u \over dt^2 = \frac{d}{dt} (\frac{du}{dx}) = \frac{d}{dx} \left( \frac{du}{dx} \right) \frac{dx}{dt} = \frac{d^2 u}{dx^2} \)

Under this change of variable we have:

$t^2 u'' + x t u' + \beta u \rightarrow x^2 u'' + ax u' + \beta u = 0 \quad \text{Exactly the same equation!}$

$\Rightarrow$ Solutions $u(x)$ are the same.

Since $x = -t = t |t| \text{ if } t < 0$, we have that the solutions are of the form:

\[
\begin{align*}
(x-1)^2 - 4p &> 0 : u = c_1 |t|^{1/2} + c_2 |t|^{-1/2} \\
(x-1)^2 - 4p &< 0 : u = c_1 |t|^{1/2} \cos \mu \log |t| + c_2 |t|^{-1/2} \sin \log |t|
\end{align*}
\]

Next The Frobenius Method

More general class of singular ODEs than the Euler equation

$t^2 u'' + \beta |t| u' + \gamma |t| u = 0$

polynomials
Dividing by \( t^2 \) we have:

\[
\frac{u''}{t^2} + \left( p(t) \frac{u'}{t} + q(t) \right) u = 0 \quad (x)
\]

with \( p, q \) have expansions:

\[
p(t) = \frac{p_0 + p_1 t + p_2 t^2 + \cdots}{t}
\]

\[
q(t) = \frac{q_0 + q_1 t + q_2 + q_3 t + \cdots}{t}
\]

If this is the case, \((x)\) is said to have a **Regular Singular Point** at \( t=0 \).

**Example:** Bessel's equation:

\[
t^2 u'' + t u' + (t^2 - u) u = 0
\]

\[
= \left( \frac{u'}{t} + (1 - \frac{u}{t}) \right) u = 0
\]

\[
\Rightarrow p = \frac{1}{t}, \quad q = 1 - \frac{u}{t} \quad \Rightarrow t=0 \text{ is a regular singular point.}
\]

**Example** \( t^2 u'' + u' + u = 0 \)

\[
\Rightarrow u'' + \frac{1}{t} u' + \frac{1}{t} u = 0
\]

\[
\Rightarrow \text{irregular singular point}
\]

Back to \( u'' + p(t) \frac{u'}{t} + q(t) u = 0 \), \( p(t) = \frac{p_0 + p_1 t + p_2 t^2 + \cdots}{t} \)

\[
q(t) = \frac{q_0 + q_1 t + q_2 + q_3 t + \cdots}{t}
\]

**Ansatz:** Look for solution of the form \( u(t) = t^r \sum_{n=0}^{\infty} a_n t^n \)

From Euler solutions from general series solutions