Oct 21, 2019

Last time:

Model DE:

$$\mathcal{L}u = P(t)u'' + Q(t)u' + R(t)u = 0$$

If P, Q, R are polynomials in t, then look for a solution u of the form $u(t) = \sum_{n=0}^{\infty} a_n t^n$.

$$\Rightarrow \left(\sum p_n t^n\right)\left(\sum n(n-1)a_n t^{n-2}\right) + \left(\sum q_n t^n\right)\left(\sum n a_n t^{n-1}\right)$$
$$+ \left(\sum r_n t^n\right)\left(\sum a_n t^n\right) = 0$$

Collect terms with t^n factor, determine recurrence relationship for the a_n's:

Note: Initial conditions determine a_0, a_1:

$$u(0) = \sum a_n (0)^n = a_0$$
$$u'(0) = \sum n \cdot a_n (0)^{n-1} = a_1$$

Next topic: Singular Points

Euler's Equation: $\underbrace{t^2 y''} + \underbrace{\alpha t y'} + \beta y = 0$

at $t=0$, these terms disappear. Can we still apply the series solution method?

If $y = t^r$, then both $ty' \sim t^r$ and $t^2 y'' \sim t^r$.

Ansatz: $y = t^r$.

$$\Rightarrow r(r-1)t^r + \alpha r t^r + \beta t^r = 0$$
$$(r^2 - r + \alpha r + \beta)\, t^r = 0$$
$$(r^2 + (\alpha - 1) r + \beta)\, t^r = 0$$

The solutions are:

$$r = -\frac{1}{2}\left((\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}\right)$$

Once again, there are three cases:

<u>Case 1</u> $\underline{(\alpha-1)^2 - 4\beta > 0}$

$\Rightarrow$ Two distinct real roots, solution is $y(t) = c_1 t^{r_1} + c_2 t^{r_2}$

<u>Case 2</u> $\underline{(\alpha-1)^2 - 4\beta = 0}$

$\Rightarrow$ Repeated roots, use Method of Reduction of Order to show that general solution is

$$y(t) = c_1 t^r + c_2 \underbrace{t^r \log t}$$

Alternative calculation to show that $t^r \log t$ is a solution, Since the roots are repeated:

$$\mathcal{L} t^r = (r - r_1)^2 t^r$$

and $\frac{\partial}{\partial r}\left(\mathcal{L} t^r\right) = \mathcal{L}\left(\frac{\partial}{\partial r} t^r\right) = \mathcal{L}\left(\log t \; t^r\right)$

$= 2(r-r_1)t^r + (r-r_1)^2 \log t \; t^r$

$= 0$ if $r = r_1$.

$\Rightarrow$ $\log t \; t^r$ is a solution.

If $y = t^r$, then

$\log y = r \log t$

$\frac{\partial}{\partial r}(\log y) = \frac{1}{y}\frac{\partial y}{\partial r}$

$\Rightarrow \frac{1}{y}\frac{\partial y}{\partial r} = \log t$

$\Rightarrow \frac{\partial y}{\partial r} = y \log t = t^r \log t$

<u>Case 3</u> $(\alpha-1)^2 - 4\beta < 0$

$\Rightarrow$ Two distinct complex roots $r = \lambda \pm i\mu$.

What is $t^{\lambda + i\mu}$?

$t^{i\mu} : \left(e^{\log t}\right)^{i\mu} = e^{i\mu \log t} = \cos(\mu \log t) + i \sin(\mu \log t)$

$\Rightarrow$ Real-valued general solution is $y(t) = c_1 t^\lambda \cos(\mu \log t) + c_2 t^\lambda \sin(\mu \log t)$.

Case of negative t $\left(t \in (-\infty, 0) \right)$.

$t^2 y'' + \alpha t y' + \beta y = 0$ seems to make sense for $t<0$, but often t^r does not stay real-valued.

Ex: $r = \frac{1}{2} \Rightarrow (-1)^{1/2} = i$, not real valued

$r = i\mu \Rightarrow \cos(\mu \log(-1))$ not defined (unless done very carefully).

These problems can be fixed with a change of variable:

Let $t = -x$, $x>0$

Then $\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = -\frac{du}{dx}$

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(-\frac{du}{dx}\right) = \frac{d}{dx}\left(-\frac{du}{dx}\right)\frac{dx}{dt} = \frac{d^2u}{dx^2}$$

Under this change of variable we have:

$t^2 u'' + \alpha t u' + \beta u \longrightarrow x^2 u'' + \alpha x u' + \beta u = 0$ Exactly the same equation!

$\Rightarrow$ Solutions $u(x)$ are the same.

Since $x = -t = |t|$ if $t<0$, we have that the solutions are of the form:

$(\alpha-1)^2 - 4\beta > 0$: $u = c_1 |t|^{r_1} + c_2 |t|^{r_2}$

$(\alpha-1)^2 - 4\beta = 0$ $u = c_1 |t|^r + c_2 \log|t| \, |t|^r$

$(\alpha-1)^2 - 4\beta < 0$ $u = c_1 |t|^\lambda \cos \mu \log|t| + c_2 |t|^\lambda \sin \log|t|$

Next The Frobenius Method

More general class of singular ODE's than the Euler equation

$$t^2 u'' + \tilde{p}(t) u' + \tilde{q}(t) u = 0$$

polynomials

Dividing by t^2 we have:

$$u'' + p(t)\,u' + q(t)\,u = 0 \qquad (*)$$

with p, q have expansions:

$$p(t) = \frac{p_0}{t} + p_1 + p_2 t + p_3 t^2 + \dots$$

$$q(t) = \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + q_3 t + \dots$$

If this is the case, $(*)$ is said to have a <u>Regular Singular Point</u> at $t=0$.

<u>Example</u>: Bessel's equation:

$$t^2 u'' + t u' + (t^2 - \nu) u = 0$$

$$\Rightarrow \quad u'' + \underbrace{\frac{1}{t}}_{p} u' + \underbrace{\left(1 - \nu/t^2\right)}_{q} u = 0 \qquad \Rightarrow \quad t=0 \text{ is a regular singular point.}$$

<u>Example</u> $t^2 u'' + u' + u = 0$

$$\Rightarrow \quad u'' + \underbrace{\frac{1}{t^2}}_{} u' + \frac{1}{t^2} u = 0$$

$\Rightarrow$ <u>irregular</u> singular point

Back to $u'' + p(t)\,u' + q(t)\,u = 0$, $\quad p(t) = \frac{p_{-1}}{t} + p_0 + p_1 t + p_2 t^2 \dots$

$$q(t) = \frac{q_{-1}}{t} + q_0 + q_1 t + \dots$$

<u>Ansatz</u>: Look for solution of the form $u(t) = t^r \sum_{n=0}^{\infty} a_n t^n$

t^r ← from Euler solutions

$\sum a_n t^n$ ← from general series solutions