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Last time:

Inhomogeneous equations:

$$y'' + p(x)\, y' + q(x) y = g(x) \quad = Ly$$

The general solution is of the form

$$= \underbrace{y_h}_{\text{solves } Ly = 0} + \underbrace{y_p}_{\text{"particular solution" solves } Ly = g.}$$

Given y_1, y_2, solutions to $Ly = 0$, a particular solution can be found using <u>Method of Variation of Parameters</u>:

Look for a particular solution of the form:

$$y_p = \Psi = u_1 y_1 + u_2 y_2$$

Insert into $Ly = g$ and <u>choose</u> extra condition

$$u_1' y_1 + u_2 y_2 = 0$$

so that things simplify. Then

$$u_1' = \frac{-g y_2}{W(y_1, y_2)} \qquad u_2' = \frac{g y_1}{W(y_1, y_2)} \leftarrow \text{Wronskian}$$

Solution can be obtained by integration.

<u>Next topic</u> <u>Series solutions</u>

Slightly different form of the homogeneous equation:

$$Ly = P(t)\, y'' + Q(t)\, y' + R(t)\, y = 0.$$

Assume P, Q, R <u>are polynomials in t</u>.

This suggests that y is <u>also a polynomial</u> in t.

Example:

$$y'' - 2t\,y' - 2y = 0$$

Ansatz: $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots$

$$= \sum_{n=0}^{\infty} a_n t^n$$

Compute derivatives: $y'(t) = \sum_{n=0}^{\infty} n \cdot a_n t^{n-1}$

$$y''(t) = \sum_{n=0}^{\infty} n \cdot (n-1) \cdot a_n t^{n-2}$$

Insert into equation:

$$\sum n(n-1)\, a_n t^{n-2} - 2t \sum n \cdot a_n t^{n-1} - 2 \sum a_n t^n = 0$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} \Big(n(n-1)\, a_n t^{n-2}}_{} - 2n a_n t^n - 2 a_n t^n \Big) = 0$$

Rewrite this so that term is of order t^n:

Note: $\sum_{n=0}^{\infty} n(n-1)\, a_n t^{n-2} = 2 \cdot 1 \cdot a_2 t^0 + 3 \cdot 2 \cdot a_3 t + \ldots$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, t^n$$

$$\Rightarrow \mathcal{L}y = \sum_{n=0}^{\infty} \Big(\underbrace{(n+2)(n+1)\, a_{n+2} - 2n a_n - 2a_n}_{} \Big) t^n = 0$$

Each coefficient must be zero in order for the equation to be satisfied:

$$\Rightarrow (n+2)(n+1)\, a_{n+2} - 2(n+1)\, a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{2\cancel{(n+1)}}{(n+2)\cancel{(n+1)}}\, a_n = \frac{2}{n+2}\, a_n .$$

This is a Recurrence Formula.

Given a_0, $a_2 \to a_4 \to a_6 \cdots$ are determined.

a_1, $a_3 \to a_5 \to a_7 \cdots$ are determined.

a_0, a_1 are determined from initial conditions.

To find two lin. indep. solutions, first specify $a_0 = 1, a_1 = 0$

$$\Rightarrow \quad a_0 = 1 \qquad\qquad a_1 = 0$$

$$a_2 = \frac{2}{0+2} a_0 = 1 \qquad\qquad a_3 = 0$$

$$a_4 = \frac{2}{2+2} a_2 = \frac{1}{2} \qquad\qquad a_5 = 0$$

$$a_6 = \frac{2}{4+2} a_4 = \frac{1}{3} \cdot \frac{1}{2} \qquad\qquad \vdots$$

$$\vdots \qquad\qquad a_{2n+1} = 0$$

$$a_{2n} = \frac{1}{n \cdot (n-1) \cdots 2} = \frac{1}{n!}$$

$$\Rightarrow \quad y_1(t) = \sum_{n=0} a_{2n} t^{2n}$$

$$= \sum_{n=0} \frac{1}{n!} t^{2n} = e^{t^2}$$

Another solution can be obtained by setting $a_0 = 0$, $a_1 = 1$

$$a_0 = 0 \qquad a_1 = 1$$

$$a_2 = 0 \qquad a_3 = \frac{2}{1+2} a_1 = \frac{2}{3}$$

$$a_4 = 0$$

$$a_5 = \frac{2}{3+2} a_3 = \frac{2}{5} \cdot \frac{2}{3}$$

$$a_7 = \frac{2}{5+2} a_5 = \frac{2}{7} \cdot \frac{2}{5} \cdot \frac{2}{3}$$

$$\vdots$$

$$a_{2n+1} = \frac{2^n}{3 \cdot 5 \cdot \cdots \cdot (2n+1)} \qquad \Rightarrow \quad y_2(t) = \sum_{n=0}^{\infty} a_{2n+1} t^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{2^n t^{2n+1}}{3 \cdot 5 \cdots (2n+1)}$$

Cannot be reduced any further.

<u>Note</u>: In the original equation

$$y'' - 2ty' - 2y = 0$$

$P(t) = 1$, $Q(t) = -2t$, $R(t) = -2$ <u>all of finite degree</u>

<u>But</u> the solution was an infinite series – a <u>power series</u>.

<u>Recall</u>

(1) Power series about t_0:

$$p(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$$

(2) Radius of convergence

there exists a $\rho > 0$ s.t. if $|t-t_0| < \rho$, $p(t)$ converges

$|t-t_0| > \rho$, $p(t)$ diverges.

(3) $p'(t)$, $\int p(t)$ have the same radius of convergence

(4) Cauchy ratio test:

If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$, then radius of convergence is $\frac{1}{\lambda} = \rho$.

(5) Both $(\sum a_n t^n)(\sum b_n t^n)$ and $\frac{\sum a_n t^n}{\sum b_n t^n}$ (with $b_0 \neq 0$) are power series

(6) If a function f can be written

$f(t) = \sum a_n (t-t_0)^n$ then it is analytic at t_0.

$\left(a_n = \frac{f^{(n)}(t_0)}{n!}, \text{ Taylor series} \right)$

(7) Radius of convergence can be obtained similarly for Taylor series.

The previous series solution method could be applied about a specific point t_0 as well. (<u>Ansatz</u>: $y(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$).

Why do we care about series solutions and analyticity?

Thm: Let $Q(t)/P(t)$ and $R(t)/P(t)$ have convergent Taylor series at t_0 for $|t-t_0|<\rho$. Then every solution to

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

is analytic at t_0 with radius of convergence $\rho' \geq \rho$.

Example: $y'' - t^3 y = 0$, $y(0)=0$, $y'(0)=-2$

If $y = \sum_{n=0}^{\infty} a_n t^n$, then

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - t^3 \sum_{n=0}^{\infty} a_n t^n = 0$$

$\Rightarrow$ adjust to collect terms:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=3}^{\infty} a_{n-3} t^n = 0$$

$$\Rightarrow \underbrace{2a_2}_{n=0} + \underbrace{6a_3 t}_{n=1} + \underbrace{12a_4 t^2}_{n=2} + \sum_{n=3}^{\infty} \left((n+2)(n+1)a_{n+2} - a_{n-3} \right) t^n = 0$$

$$\Rightarrow a_2 = a_3 = a_4 = 0$$

$$a_{n+2} = \frac{a_{n-3}}{(n+2)(n+1)} \quad \Rightarrow \quad n=3 \to a_5 = \frac{a_0}{(5)(4)} = \frac{1}{20}a_0$$

$$n=4 \to a_6 = \frac{a_1}{(6)(5)} = \frac{1}{30}a_1$$

$$n=5 \to a_7 = \frac{a_2}{(7)(6)} = 0$$

$$n=6 \to a_8 = \frac{a_3}{8 \cdot 7} = 0$$

$$n=7 \to a_9 = \frac{a_4}{9 \cdot 8} = 0$$

Continue...

$$n=8 \to \quad a_{10} = \frac{a_5}{10\cdot 9} = \frac{1}{20}\frac{1}{10\cdot 9} a_0$$

$$n=9 \to \quad a_{11} = \frac{a_6}{11\cdot 10} = \frac{1}{30}\frac{1}{11\cdot 10} a_1$$

So solution is

$$y(t) = a_0\left(1 + \frac{1}{20}t^5 + \frac{1}{20}\frac{1}{10\cdot 9}t^{10} + \dots\right)$$
$$+ a_1\left(t + \frac{1}{30}t^6 + \frac{1}{30}\frac{1}{11\cdot 10}t^{11} + \dots\right)$$

$$y(0) = 0 \implies a_0 = 0$$

$$y'(0) = -2 \implies a_1 = -2$$

$$\implies y(t) = -2\left(t + \frac{1}{5\cdot 6}t^6 + \frac{1}{5\cdot 6}\frac{1}{10\cdot 11}t^{11} + \frac{1}{5\cdot 6}\frac{1}{10\cdot 11}\frac{1}{15\cdot 16}t^{16} + \dots\right)$$

Slightly messy, but very systematic.