2nd order linear differential equations:
\[ u'' + p(x) u' + q(x) u = 0 \quad \text{(homogeneous case for now)} \]
- Existence & Uniqueness:
  - wherever \( p, q \) are continuous \( \text{(with } u(x_0) = u_0, u'(x_0) = u'_0) \)

- If \( u_1, u_2 \) are solutions to \( Lu = 0 \), then
  - the general solution is \( u = c_1 u_1 + c_2 u_2 \) if and only if
    - \( u_1, u_2 \) are linearly independent

- Check Wronskian:
  \[ W(u_1, u_2) = u_1 u'_2 - u'_1 u_2 \neq 0 \text{ if and only if } u_1, u_2 \text{ are linearly independent.} \]

Also, two functions are linearly dependent on \([a, b]\) if
\[ u_1 = c u_2. \]

**Constant coefficient case:** \( a u'' + b u' + c u = 0 \)

**Guess:** \( u = e^{rt} \)

\[ \Rightarrow u \text{ is a solution if } (ar^2 + br + c) e^{rt} = 0 \]

\[ \Leftrightarrow \begin{cases} ar^2 + br + c = 0 \quad \text{characteristic equation} \\ \end{cases} \text{ solve using quadratic formula} \]

- If \( r_1 \neq r_2 \), then \( W(e^{r_1t}, e^{r_2t}) \neq 0 \).

- Two remaining cases: ① \( r_1, r_2 \) are complex
  - ② \( r_1 = r_2 \)
Complex-valued Roots

Often times, we will compute \( r_1, r_2 \) that are complex. This happens when \( 4ac > b^2 \).

In this case, the roots are always of the form \( r_i = \alpha + i\beta \).

And therefore
\[
\begin{align*}
  u_1 &= e^{r_1t} \\
  u_2 &= e^{r_2t} \\
  &= e^{(\alpha + i\beta)t} \\
  &= e^{\alpha t} e^{i\beta t} \\
  &= \overline{u_2} \\
  &= \overline{u_1} \quad \text{(complex conjugate)}
\end{align*}
\]

But what if we want a real solution?

Lemma: If \( u(t) = x(t) + iy(t) \), with \( x, y \) real-valued, and \( Lu = 0 \), then \( x, y \) are both real-valued solutions.

Proof. Since \( L \) is a linear operator.

Taking real/imag parts of \( u_1, u_2 \):
\[
\begin{align*}
  u_1 &= e^{\alpha t} (\cos \beta t + i \sin \beta t) \\
  u_2 &= e^{\alpha t} (\cos \beta t - i \sin \beta t)
\end{align*}
\]

Therefore, two real-valued linearly independent solutions are:
\[
y_1 = e^{\alpha t} \cos \beta t \quad \text{and} \quad y_2 = e^{\alpha t} \sin \beta t.
\]

Check: \( W[y_1, y_2] = ... \) (exercise for the reader).

Example:
\[
\begin{align*}
  u'' + 2u' + 3u &= 0 \\
  u'' + u' + 2u &= 0, \quad u(0) = 1, \quad u'(0) = -2
\end{align*}
\]
Repeating Roots

What if \( r_1 = r_2 \)?  
\[ \text{Ex: } u'' - 2u' + u = 0 \]
\[ \rho(r) = r^2 - 2r + 1 = 0 \]
\[ (r - 1)^2 = 0 \implies r_1 = r_2 = 1. \]

Idea: Given that \( u(x) \) solves \( Lu = 0 \), can we find another solution?  
Let \( u_2(t) = u(t) \frac{d}{dt} \)
Compute derivatives:
\[ u_2' = u' v + u v' \]
\[ u_2'' = u'' v + u' v' + u v'' + u'' v \]
note:
\[ = u v'' + 2u' v' + u'' v \]
The function \( u_2 \) is a solution to \( Lu_2 = 0 \) if \( Lu_2 = 0 \)
\[ = u v'' + 2u' v' + u'' v + p(u v' + u' v) + q u v \]
\[ = u v'' + (2u' + pu) v' + \underbrace{(u'' + pu' + qu)}_{Lu=0} v \]
\[ = u v'' + (2u' + pu) v' \]
This is just a first order equation in terms of the function \( v' \):
\[ w(v')' + (2u' + pu)(v') = 0 \]
The solution can be obtained by separation:

\[
\frac{(v')'}{(v')} = \frac{-2vv' + pv}{v}
\]

\[
\Rightarrow v' = C e^{-\int \frac{2v'}{v} - \int p}
\]

\[
\Rightarrow 2\int \frac{w'(t)}{w(t)} dt = 2\int \frac{d}{dt}(\log |w|) dt
\]

\[
= 2 \log |w| = \log |w|^2
\]

\[
= C e^{-\int p} d
\]

\[
= C e^{-\int p}
\]

We only need one \( v \), so we set \( C = 1 \) and then integrate:

\[
v' = \frac{1}{u^2} e^{-\int p}
\]

\[
\Rightarrow \int v' = \int \frac{1}{u^2} e^{-\int p} dt
\]

\[
\Rightarrow v = \int \frac{1}{u^2} e^{-\int \gamma(t) dt} + \int \frac{\gamma(t)}{u^2} dt
\]

This method is called the Method of Reduction of Order since the change of variable \( u_2 = uv \) required only the solution to a 1st order equation to obtain \( v \).
Application: Equal roots in the characteristic equation for \( au'' + bu' + cu = 0 \).

In this case, rewrite as:

\[
\frac{u'' + \frac{b}{a} u' + \frac{c}{a} u}{w'' + \frac{p}{q} w} = 0
\]

Then \( u_1 = e^{rb} \), and set \( u_2 = u, v \).

By the above calculation,

\[
v = \int \frac{1}{u^2} \int e^{p dt} dt
\]

\[
= \int \frac{1}{e^{rt}} \int e^{\frac{b}{a} dt} dt = \int e^{-2rt} e^{-\frac{b}{a} t} dt
\]

But if the root was repeated, then \( b^2 = 4ac \), and \( r = \frac{-b}{2a} \), so

\[
v = \int e^{\frac{2b}{2a} t} e^{-\frac{b}{a} t} dt
\]

\[
= \int 1 dt = t
\]

Therefore, a second linearly independent solution is \( u_2 = te^{rt} \).

Check: \( u_2' = e^{rt} + re^{rt} \)

\[
\frac{u_2'' = re^{rt} + re^{rt} + r^2 te^{rt}}{\text{Recall: } r = \frac{-b}{2a}}
\]

so \( Lu_2 = \left( a \left( \frac{-b}{a} + \frac{b^3}{4a^2} t \right) + b \left( 1 - \frac{b}{2a} t \right) + ct \right) \)

But \( c = \frac{b^2}{4a} \)
\[-b + \frac{b^2}{4a}t + b - \frac{b^2}{2a} t + \frac{b^2}{4a} \]

\[= 0.\]

**Application**

Other differential equations: (non-constant coefficients)

**Ex:** \((1-t^2) y'' + 2ty' - 2y = 0\)

Verify: \(y_1 = t\) is one solution.

Rewrite to apply the Method of Reduction of Order:

\[
y'' + \frac{2t}{1-t^2} y' - \frac{2}{1-t^2} y = 0.
\]

Other solution is \(y_2 = y_1 v = tv\)

\[v = \int \frac{1}{t^2} e^{-\int \frac{2t}{1-t^2} dt} dt \]

\[= \int \frac{1}{t^2} e^{\log(1-t^2)} dt \]

\[= \int \frac{1-t^2}{t^2} dt = -\frac{1}{t} - t \]

\(\Rightarrow y_2 = -t \left( \frac{1}{t} + t \right) = -(1 + t^2)\)