Sept 30, 2019

Last time:

Finished Numerical Methods for IVPs:

- Explicit Euler
- Implicit Euler
- Newton's Method

used in combination with Implicit methods

Started: 2^{nd} order linear differential equations:

$$u'' + p(t)\, u + q(t)\, u = g(t)$$

IVP version

$u(t_0) = u_0$

$u'(t_0) = u_0'$

Boundary value version

$u(a) = u_a$

$u(b) = u_b$

Solve on entire interval $[a,b]$.

Today:

- Some more on 2^{nd} order linear DE's
- Review for Prelim Exam 1.

Get right to the point: More or less, we will only be concerned with the equation $y'' + p(t)\,y' + q(t)\,y = g(t)$. If $g=0$, we have the following E&U theorem:

Thm If p,q are continuous in $\alpha < t < \beta$, then the equation

$$(*) \qquad \begin{aligned} &y'' + p(t)\,y' + q(t)\,y = 0 \\ &y(t_0) = y_0 \qquad t_0 \in (\alpha,\beta) \\ &y'(t_0) = y_0' \end{aligned}$$

has exactly one solution on (α,β). In particular, if $y_0 = y_0' = 0$, then $y=0$ on (α,β).

To begin studying this equation, start by examining the operator $\mathcal{L}$: functions $\rightarrow$ functions:

$$\mathcal{L}f = f'' + pf' + qf$$

$\mathcal{L}$ is a linear operator / transformation / map:

$$\begin{aligned} \mathcal{L}(cf+dg) &= (cf''+dg'') + p(cf'+dg') + q(f+g) \\ &= c(f''+pf'+qf) + d(g''+pg'+qg) \\ &= c\mathcal{L}f + d\mathcal{L}g \end{aligned}$$

$\Rightarrow$ solutions of $(*)$ satisfy $\mathcal{L}y = 0$.

Ex: $\dfrac{d^2y}{dt^2} + y = 0 \Rightarrow \mathcal{L}y = \dfrac{d^2y}{dt^2} + y = 0 \qquad (**)$

Trivially, solutions are $y_1 = \cos t$
$y_2 = \sin t$

and therefore any linear combination of y_1, y_2 are solutions.

$$\mathcal{L}(c_1 \cos t + c_2 \sin t) = 0.$$

Adding conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$ determine c_1, c_2.

The obvious question is: are <u>all</u> solutions to (**) of the form $c_1y_1 + c_2y_2$? <u>Yes!</u>

<u>Thm:</u> Let y_1, y_2 be solutions to $\mathcal{L}y = 0$ on (α,β). If

$$y_1y_2' - y_1'y_2 \not\equiv 0 \text{ anywhere in } (\alpha,\beta), \text{ then } y = c_1y_1 + c_2y_2 \text{ is the}$$

general solution to $\mathcal{L}y = 0$.

<u>Proof:</u> Let y be <u>any</u> solution to $Ly = 0$. Compute $y_0 = y(t_0)$, $y_0' = y'(t_0)$. Then we must solve for c_1, c_2 via the system:

$$y_0 = y(t_0) = c_1y_1(t_0) + c_2y_2(t_0)$$

$$y_0' = y'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0)$$

$$\Rightarrow \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Solution exists and is unique if the determinant is non-zero:

$$y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0) \neq 0.$$

Therefore, for any $t_0 \in (\alpha,\beta)$, we can compute unique c_1, c_2. ∎

Definition: The quantity $y_1y_2' - y_1'y_2$ is called the Wronskian of y_1, y_2, denoted by $W(y_1,y_2)$.

<u>Thm:</u> Let p, q be continuous on (α,β) and let y_1, y_2 be two solutions to $\mathcal{L}y = 0$. Then $W(y_1,y_2) = 0$ identically, or is <u>never</u> equal to 0 on (α,β).

<u>Proof:</u> First, note that W satisfies the ODE:

$$W' + p(t)\, W = 0.$$

Just compute W':

$$W(t) = y_1(t)\, y_2'(t) - y_1'(t)\, y_2(t)$$

$$\Rightarrow \quad W' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$

$$= y_1 y_2'' - y_1'' y_2$$

Also, $y_1'' = -p(t)\, y_1' - q(t)\, y_1$

$y_2'' = -p(t)\, y_2' - q(t)\, y_2$

And therefore:

$$W' = y_1\left(-p y_2' - q y_2\right) - \left(-p y_1' - q y_1\right) y_2$$

$$= -p y_1 y_2' - \cancel{q y_1 y_2} + p y_1' y_2 + \cancel{q y_1 y_2}$$

$$= -p\left(y_1 y_2' - y_1' y_2\right)$$

$$= -p W$$

Given that $W' + pW = 0$, then

$$W(t) = C \underbrace{e^{-\int p(t)\, dt}}_{\text{never equal to zero.}}$$

therefore, either $C = 0$ and $W = 0$ everywhere, or $W \neq 0$ for any t. ∎

<u>Definition</u>: Two functions f, g are linearly dependent on an interval $[a,b]$ if $f(t) = c\, g(t)$ for $t \in [a,b]$. Otherwise they are linearly independent.

<u>Thm</u> Two solutions y_1, y_2 to (*) on $[a,b]$ are linearly independent if and only if $W[y_1, y_2] \neq 0$ on $[a,b]$. Therefore, y_1 & y_2 form a fundamental solution set iff they are linearly independent.

Review for Prelim Exam 1

- Oct 2nd in class, 9:30 - 10:45
- Closed book

Topics

- First order equations (§1.2)
  - linear vs. nonlinear
  - general solutions
  - solution to $y' + a(t)\,y = b(t)$ (Application: Carbon dating §1.3)
    - integrating factors.
- Separable equations (§1.4), orthogonal trajectories (§1.8)
- Exact equations (§1.9)
  - conditions for exactness
  - solution methods
  - integrating factors
- Existence & Uniqueness to IVP (§1.10)
  - Conditions required to prove existence/uniqueness (Thm 2')
  - Picard iterations
- Newton's Method (§1.11.1)
  - rate of convergence
- Euler's Method (§1.13)
  - Explicit vs. Implicit
  - Rate of convergence.