Sept 25, 2019

Last time:

$$y'(t) = f(t,y)$$
$$y(t_0) = y_0$$

- Numerical Methods for Initial Value problems

Explicit Euler's Method

Since $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$, the solution can be approximated as:

$$y(t) \approx y_0 + (t-t_0)\, f(t_0, y_0)$$

the error is $\mathcal{O}((t-t_0)^2)$ (on one step).

$\Rightarrow$ If $t_\ell = t_0 + h\ell$, then

$$y_{\ell+1} = y_\ell + h\, f(t_\ell, y_\ell) \qquad \underline{\text{EXPLICIT EULER}}$$

More generally, after many timesteps, if $h \to h/2$, then the error decreases by a factor of $\frac{1}{2}$ as well $\Rightarrow$ first order convergent

Implicit Euler's Method

The integral above could have been approximated by

$$y(t) \approx y_0 + (t-t_0)\, f(t, y(t))$$ now solve for $y(t)$.

$$\Rightarrow \quad y_{\ell+1} = y_\ell + h\, f(t_{\ell+1}, y_{\ell+1}).$$

- Error is the same as explicit Euler, stability is better.

A better approximation

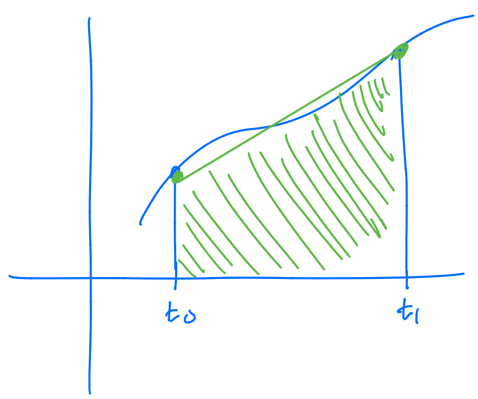
$$y(t_1) = y_0 + \int_{t_0}^{t_1} f(s, y(s))\, ds$$

$$\approx y_0 + \underbrace{\frac{h}{2}\Big(f(t_0, y_0) + f(t_1, y(t_1))\Big)}$$

Trapezoidal Rule

error is $\underline{\mathcal{O}(h^3)}$.

Also an implicit Method



$\Rightarrow$ Global Error is $\mathcal{O}(h^2)$.

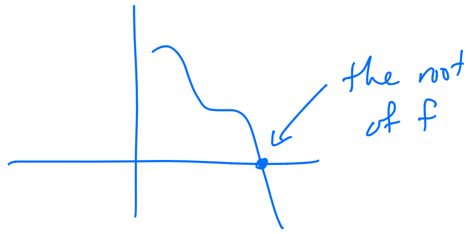
How do we solve these equations for $y_{\ell+1}$ in implicit methods for IVPs?

Newton's Method

In general, can be used to solve equations of the form:

$f(x) = 0$

"Nonlinear root finding"



To derive, Taylor expand f around some point near the root, x':

$$f(x) \approx f(x') + f'(x')(x - x')$$

Now, if x is the root of f, then $f(x) = 0$, solve for x:

$$0 \approx f(x') + f'(x')(x - x')$$

$$\frac{-f(x')}{f'(x')} \approx x - x' \quad \Rightarrow \quad \boxed{x \approx x' - \frac{f(x')}{f'(x')}}.$$

Starting with initial guess x_0, Newton's Method is then

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

It can be shown that if x_0 is close enough to the root, x^* and $f'' \neq 0$, then

$$\underbrace{|x_{k+1} - x^*| \leq C\,|x_k - x^*|^2}_{\text{Quadratic Convergence.}}$$

This is very fast: If $|x_0 - x^*| \sim 10^{-1}$

$|x_1 - x^*| \sim 10^{-2}$

$|x_2 - x^*| \sim 10^{-4}$

$|x_3 - x^*| \sim 10^{-8}$

$|x_4 - x^*| \sim 10^{-16}$ ← "machine precision"

Computers only do arithmetic to about 16 digits of precision

Quick sketch of Proof

Taylor expand (use Taylor's Thm)

$$f(x) = f(x') + f'(x')(x-x') + \frac{f''(\xi)}{2}(x-x')^2.$$

Solve for x^* if x^* is root $(f(x^*)=0)$:

$$x^* = x' - \frac{f(x')}{f'(x')} - \frac{f''(\xi)}{2f'(x')}(x^*-x')^2$$

Set $x' = x_\ell$.

$$x^* = x_\ell - \frac{f(x_\ell)}{f'(x_\ell)} - \frac{f''(\xi)}{2f'(x_\ell)}(x^*-x_\ell)^2$$

Subtract

$$x_{\ell+1} - x^* = \underbrace{x_\ell - \frac{f(x_\ell)}{f'(x_\ell)}}_{\text{Newton}} - \underbrace{\left(x_\ell - \frac{f(x_\ell)}{f'(x_\ell)} - \frac{f''(\xi)}{2f'(x_\ell)}(x^*-x_\ell)^2\right)}_{\text{Taylor}}$$

$$= \frac{f''(\xi)}{2f'(x_\ell)}(x^*-x_\ell)^2$$

$$\Rightarrow |x_{\ell+1} - x^*| \leq \frac{1}{2}\left|\frac{f''(\xi)}{f'(x_\ell)}\right| |x^*-x_\ell^*|^2 \qquad \square$$

Next topic 2nd order linear differential equations

General form $y'' = f(t,y,y')$

Linear implies that the form of the equation is:

$$y'' + p(t)y' + q(t)y = g(t).$$

This is a much richer and more interesting class of equations.

Unsurprisingly, two extra conditions need to be enforced to uniquely specify the solution.

Ex: $y(t_0) = y_0$, $y'(t_0) = y'_0$ | STILL AN IVP

Contrast this with the other type of 2^{nd} order Linear Differential equation that we will study: two-point boundary value problems:

Solve $y'' + py' + qy = g$ on $[a,b]$ with $y(a) = y_0$
$y(b) = y_1$

The theory and methods is very different, in fact.

Ex: IVP 2^{nd} order equation from physics: $F = ma$

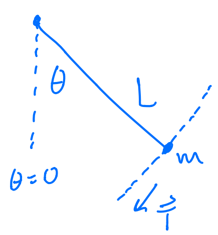
If $x(t)$ = location of particle at time t, then

$x'(t)$ = velocity

$x''(t)$ = acceleration

So if a force function $F(t)$ is applied, the equation of motion is $F(t) = ma(t) = mx''(t)$.

Ex2: Motion of pendulum



Force on the mass in the tangential direction $\vec{T}$ is given by $F = -mg\sin\theta$ ← with respect to xyz-coordinates.

The change in arclength s is related to θ by

$s = L\theta$, so that

$$\frac{ds}{dt} = L\frac{d\theta}{dt}, \quad \frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2}$$

non-linear equation.

$$\Rightarrow F = ma \Rightarrow -mg\sin\theta = mL\frac{d^2\theta}{dt^2} \Rightarrow \frac{d^2\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

Specifying $\theta(0) = \theta_0$
$\theta'(0) = \theta_0'$ uniquely determines the motion of the pendulum.