Sept 18, 2019

Last time:

- Exact differential equations
  - General form is $M(t,y) + N(t,y)\,y' = 0$

Exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$. In this case $M + Ny' = \frac{d}{dt}(\varphi(t,y)) = 0$

Goal: Find φ.

- Integrating factors for non-exact DE:

If $M + Ny' = 0$ is not exact, then

$\mu M + \mu N y' = 0$ is exact if and only if

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial t}(\mu N)$$

$$\Rightarrow \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial t} N + \mu \frac{\partial N}{\partial t}$$

In general, this is not solvable analytically for μ unless it turns out that $\mu = \mu(t)$ or $\mu = \mu(y)$.

Finally Existence & Uniqueness

For an initial value problem:

$$(*) \quad \begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

1. Does a solution exist near t_0?
2. Is there more than one solution?
3. Why do we care about 1 & 2 if we can't solve $(*)$ analytically?

Answer (3) first: The existence/uniqueness is <u>crucial</u> for designing numerical methods for solving ODEs, which are the only option most of the time.

Idea of existence proof:

- We need to prove the existence of a function <u>without</u> constructing it. Motivation for technique:

In calculus:

$$e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!}$$

$$= \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots}$$

it can be shown that this series converges for <u>any</u> <u>x</u>. $\Rightarrow$ this implies the existence of <u>some</u> function f s.t. $f(x) = \sum_{l=0}^{\infty} \frac{x^l}{l!}$

(even if we don't know that $f(x) = e^x$).

<u>Outline for Existence</u>

Construct sequence of functions $y_l, y_{l+1}, y_{l+2}, \dots$ that has a limit y on some interval $t_0 \le t \le t_0 + \alpha$, and show that $y' = f(t,y)$ on this interval.

Examine the exact solution:

$$y' = f(t,y) \quad \Rightarrow \quad \int_{t_0}^{t} y' \, dt = \int_{t_0}^{t} f(s,y) \, ds$$

$$y(t) - y(t_0) = \int_{t_0}^{t} f(s,y) \, ds$$

$$\boxed{y(t) = y(t_0) + \int_{t_0}^{t} f(s,y) \, ds}$$

Rewrite as

$$y(t) = L(t, y(t))$$

$$L(t, y(t)) = y(t_0) + \int_{t_0}^{t} f(s, y)\, ds$$

L is an <u>operator</u> or <u>transformation</u> or <u>map</u>

Therefore, a solution $y(t)$ to $\begin{matrix} y' = f(t,y) \\ y(t_0) = y_0 \end{matrix}$ is a <u>fixed point</u> of L. This means $L(y(t)) = y(t)$.

<u>Algorithm</u>

1. Guess a solution, $y_0(t) = y_0$
2. Compute $y_1(t) = y_0 + \int_{t_0}^{t} f(t, y_0(s))\, ds$
3. IF $L(y_1) = y_1$, stop, solution found
4. Otherwise, Let $y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))\, ds$

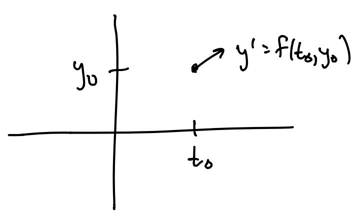
$\vdots$

(N) And so on.

The computation $y_{\ell+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_\ell(s))\, ds$ is called a **Picard Iteration.**

The sequence $y_1, y_2, y_3, \dots$ defined above via Picard Iterations can be shown to <u>always converge</u> for $t \in [t_0, t_0 + \alpha]$, for some suitably small α.

Idea:



- Integrate a little, get a better approximation to $y(t_0 + \alpha)$.
- Repeat

<u>Example</u>: $\begin{matrix} y' = y \\ y(0) = 1 \end{matrix}$ | solution: $y = e^t$).

$y_0 = 1$

$y_1 = y_0 + \int_0^t 1 \cdot ds = 1 + t$

$$y_2(t) = y_0 + \int_0^t y_1(s)\, ds = y_0 + \int_0^t (1+s)\, ds = 1 + t + \frac{1}{2} t^2$$

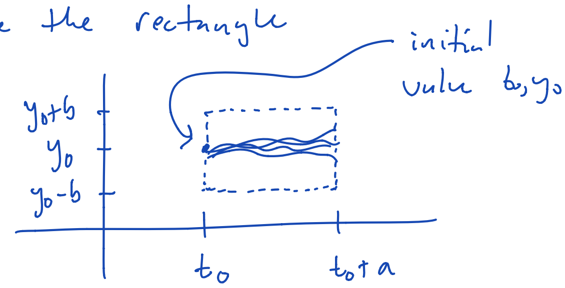
$y_3 = \dots$

(Think of Taylor series, in reverse).

Convergence of the Picard Iterations

Lemma: Choose any $a, b > 0$, let R be the rectangle

$$R = \left\{ \begin{matrix} t_0 \le t \le t_0 + a \\ |y - y_0| \le b \end{matrix} \right\}$$



Define $M = \max_{t,y \in R} |f(t,y)|$

$\alpha = \min(a, b/M)$

Then $|y_n(t) - y_0| \le M(t - t_0)$ for $t_0 \le t \le t_0 + \alpha$.

(Proof page 72 of textbook)

Thm If $\frac{\partial f}{\partial y}$ exists and is continuous in R, the Picard iterates $y_n(t)$ converge for any $t_0 \le t \le \alpha$.

(Proof page 72 in text)

Main Theorem Let f and $\partial f/\partial y$ be continuous in the rectangle $t_0 \le t \le a$, $|y - y_0| \le b$. Compute

$M = \max_R |f(t,y)|$ and set $\alpha = \min(a, b/M)$.

Then the IVP $y' = f$, $y(t_0) = y_0$ has a unique solution on $[t_0, t_0 + \alpha]$.

Example: $y' = t\sqrt{1 - y^2}$, $y(0) = 1$

$f(t,y) = t\sqrt{1-y^2}$ is continuous

$$\frac{\partial f}{\partial y} = t \, \frac{1}{2} \frac{1}{\sqrt{1-y^2}} (-2y) = \underbrace{\frac{-y}{\sqrt{1-y^2}}}_{\text{NOT continuous in } R \text{ (recall } y_0 = 1) \ldots}$$