Sept. 16, 2019

Last time:

- Mods of failure for the solution to an ODE to exist
  - Intervals of existence
  - Ex: $\cos t < a$ in order for solution to be real.

- Application: Orthogonal trajectories
  - If $F(x,y,c) = 0$ defines a family of smooth curves, then these curves satisfy $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$,

$\Rightarrow \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$ | Orthogonal (perpendicular) curve satisfies: $\frac{dy}{dx} = \frac{\partial F/\partial y}{\partial F/\partial x}$

- Exact differential Equations:

These have the form $\frac{d}{dt}\varphi(t,y) = 0$,

and therefore the solution is $\varphi(t,y) = C$.

Expand: $\frac{d}{dt}\varphi = \underbrace{\frac{\partial\varphi}{\partial t}}_{M} + \underbrace{\frac{\partial\varphi}{\partial y}}_{N}\frac{dy}{dt} = 0$

Theorem: Let M, N be continuously differentiable in $(a,b)\times(c,d)$. There exists a function φ s.t. $M = \partial\varphi/\partial t$ and $N = \partial\varphi/\partial y$ if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ in $(a,b)\times(c,d)$.

Proof: $M = \partial\varphi/\partial t$ if and only if

$$\varphi(t,y) = \int M(t,y)\,dt + h(y)$$

arbitrary function of y

so that $\frac{\partial\varphi}{\partial t} = M + \cancel{\frac{\partial h}{\partial t}}^{0}$

Then, we have that $\frac{\partial \varphi}{\partial y} = \int \frac{\partial M}{\partial y}\, dt + h'$

Therefore, $\frac{\partial \varphi}{\partial y} = N$ if and only if $N(t,y) = \int \frac{\partial M(t,y)}{\partial y}\, dt + h'(y)$,

or rather $\underbrace{h'(y)}_{\text{function of } y} = \underbrace{N(t,y) - \int \frac{\partial M}{\partial y}(t,y)\, dt}_{\text{function of } y \text{ and } t}$

This only makes sense if the RHS is a function of only y, so that

$$\frac{\partial}{\partial t}\left(N - \int \frac{\partial M}{\partial y}\, dt\right) = \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0$$

So, if $\frac{\partial N}{\partial t} \neq \frac{\partial M}{\partial y}$ then no φ exists.

If $\frac{\partial N}{\partial t} = \frac{\partial M}{\partial y}$, then $h(y) = \int N\, dy - \int\int \frac{\partial M}{\partial y}\, dt\, dy$

and $\varphi = \int M\, dt + \int N\, dy - \iint \frac{\partial M}{\partial y}\, dt\, dy$ (from above).

Definition The ODE $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$.

$\Rightarrow$ $M + N\frac{dy}{dt}$ is the exact derivative of some function $\varphi = \varphi(t,y)$.

Note: $\frac{dy}{dt} = f(t,y)$ can always be written in the above form by $\underbrace{-f(t,y)}_{M} + \underbrace{1}_{N} \cdot \frac{dy}{dt} = 0$.

How do we compute φ for exact ODEs?

Option 1: Start with $M = \partial\varphi/\partial t$. This determines φ up to an arbitrary function of y:

$$M = \frac{\partial \varphi}{\partial t} \Rightarrow \varphi = \int M \, dt + h(y).$$

Next solve $h'(y) = N - \int \frac{\partial M}{\partial y} \, dt$

Option 2: Vice versa, $N = \partial\varphi/\partial y$ implies that

$$\varphi = \int N \, dy + k(t)$$

then since $M = \partial\varphi/\partial t \Rightarrow M = \int \frac{\partial N}{\partial t} \, dy + k'$,

compute $k = \int M \, dy - \iint \partial N/\partial t \, dy \, dt + C$

Option 3: By inspection, note that

$$\varphi(t,y) = \int M(t,y) \, dt + h(y)$$
$$= \int N(t,y) \, dy + k(t)$$

Example Find general solution to

$$\underbrace{3y + e^t}_{M} + \underbrace{(3t + \cos y)}_{N} y' = 0$$

Check exactness: $\frac{\partial M}{\partial y} = 3$ ✓

$\frac{\partial N}{\partial t} = 3$

Option 1: We have $\varphi(t,y) = \int M \, dt + h(y)$

$= 3ty + e^t + h(y)$

so $h'(y) = \frac{\partial \varphi}{\partial y} - 3t$

$= N - 3t = \cos y \Rightarrow h(y) = \sin y$

$\Rightarrow \varphi = 3ty + e^t + \sin y + C.$

By inspection:

$$\varphi(t,y) = \int M\,dt + h(y)$$
$$= \int N\,dy + k(t)$$

$$\Rightarrow \quad \varphi(t,y) = 3ty + e^t + h(y)$$
$$= 3ty + \sin y + k(t)$$

relationship is obvious.

Choose whatever method is easiest based on ability to integrate M, N, etc.

<u>Example</u>:

$$3t^2 + 4ty + (2y + 2t^2)\,y' = 0$$
$$y(0) = 1$$

<u>Integrating Factors for Exact Diff. Eq.</u>

If an equation is <u>not</u> exact, can it be made exact?

Examine: $\quad M(t,y) + N(t,y)\,y' = 0$

Multiplying through by μ yields the condition:

$$\mu M + \mu N y' = 0$$

This is exact <u>if and only if</u>

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial t}(\mu N)$$

$$\Rightarrow \quad \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial t} N + \mu \frac{\partial N}{\partial t}$$

In general, we cannot find an explicit solution to this.

If, however, we assume $\mu = \mu(t)$ (i.e. doesn't depend on y) then

$$\Rightarrow \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial t} N + \mu \frac{\partial N}{\partial t}$$

$$\Rightarrow \underbrace{\frac{\partial \mu}{\partial t}}_{\text{only a function of } t} = \underbrace{\left(\frac{\partial M/\partial y - \partial N/\partial t}{N} \right)}_{\text{must also only be a function of } t} \mu . \qquad \Rightarrow \mu = e^{\int \frac{M_y - N_t}{N} dt} \text{ is an integrating factor.}$$

And vice versa when assuming $\mu = \mu(y)$.

Example: Find general solution to:

$$\underbrace{\frac{y^2}{2} + 2ye^t}_{M} + \underbrace{(y+e^t)}_{N} \, dy/dt = 0$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= y + 2e^t \\ \partial N/\partial t &= e^t \end{aligned} \right\} \Rightarrow \text{NOT EXACT}$$

But $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} \right) = \frac{1}{y+e^t}\left(y + 2e^t - e^t \right) = \frac{y+e^t}{y+e^t} = 1$

$\Rightarrow$ Integrating factor exists,

$$\mu(t) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} \right) dt} = e^t .$$

These examples are rather contrived, involving very special-case scenarios. Not very physically relevant.