Last time:
- Modes of failure for the solution to an ODE to exist
  - Intervals of existence
  - Existence cost & a in order for solution to be real.

- Application: Orthogonal Trajectories
  - If \( F(x,y,c) = 0 \) defines a family of smooth curves, then these curves satisfy
    \[
    \frac{dy}{dx} = -\frac{\partial F}{\partial x} \quad \text{Orthogonal (perpendicular)}
    \]
    
- Exact differential Equation:
  - These have the form \( \frac{1}{dt} \frac{dy}{dt} (by + y) = 0 \), and therefore the solution is \( y(by) = C \).
  - Expand: \( \frac{1}{dt} \frac{dy}{dt} = \frac{dy}{dt} + \frac{dy}{dy} \frac{dt}{dt} = 0 \)

Theorem: Let \( M, N \) be continuously differentiable in \((a,b) \times (c,d)\).
There exists a function \( y \) s.t. \( M = \frac{\partial y}{\partial t} \) and \( N = \frac{\partial y}{\partial y} \)
if and only if \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \) in \((a,b) \times (c,d)\).

Proof: \( M = \frac{dy}{dt} \) if and only if:
\[
\frac{dy}{dt} = \int M(t,y) \, dt + h(y)
\]
so that \( \frac{dy}{dt} = M + \frac{\partial h}{\partial y} \).
Then, we have that \( \frac{dq}{dy} = \int \frac{dM}{dy} \, dt + h' \).

Therefore, \( \frac{dq}{dy} = N \) if and only if \( N(t,y) = \int \frac{dM}{dy} \, dt \, + \, h'(y) \),
or rather, \( h'(y) = N(t,y) - \int \frac{dM}{dy} \, (t,y) \, dt \).

This only makes sense if the RHS is a function of only \( y \), so that
\[
\frac{d}{dt} \left( N - \int \frac{dM}{dy} \, dt \right) = \frac{dN}{dt} - \frac{dM}{dy} = 0
\]

So, if \( \frac{dN}{dt} + \frac{dM}{dy} \), then no \( q \) exists.

If \( \frac{dN}{dt} = \frac{dM}{dy} \), then
\[
h(y) = \int N \, dy - \int \int \frac{dM}{dy} \, dt \, dy
\]

and \( q = \int M \, dt + \int N \, dy - \int \int \frac{dM}{dy} \, dt \, dy \) (from above).

**Definition.** The ODE \( M(t,y) + N(t,y) \frac{dy}{dt} = 0 \) is exact if \( \frac{dM}{dy} = \frac{dN}{dt} \).

\[\rightarrow \quad M + N \frac{dy}{dt} \text{ is the exact derivative of some function } q = q(t,y).\]

Note: \( \frac{dy}{dt} = f(t,y) \) can always be written in the above form by \( -\frac{f(t,y)}{N} \, dt = 0 \).
How do we compute $q$ for exact ODEs?

**Option 1:** Start with $M = dy/dt$. This determines $q$ up to an arbitrary function of $y$:

$$M = \frac{dy}{dt} \Rightarrow q = \int M \, dt + h(y).$$

Next solve $h'(y) = N - \int \frac{dN}{dy} \, dt$

**Option 2:** Vice versa, $N = \frac{dx}{dy}$ implies that

$$q = \int N \, dy + k(t)$$

then since $M = \frac{dy}{dt} = \frac{dx}{dt}$ implies $M = \int \frac{dN}{dy} \, dy + k'$, compute

$$k = \int M \, dy - \int \frac{dN}{dt} \, dy \, dt + C$$

**Option 3:** By inspection, note that

$$q(ty) = \int M(ty) \, dt + h(y)$$

$$= \int N(ty) \, dy + k(t)$$

**Example:** Find general solution to

$$3y + e^t + (3t + \cos y) \, y' = 0$$

Check exactness: $\frac{dM}{dy} = 3$

$$\frac{dN}{dt} = 3$$

**Option 1:** We have $q(ty) = \int M \, dt + h(y)$

$$= 3ty + e^t + h(y)$$

so $h'(y) = \frac{dh}{dy} = 3t$

$$= N - 3t = \cos y \Rightarrow h(y) = \sin y$$

$$\Rightarrow q = 3ty + e^t + \sin y + C.$$
By inspection:

\[
q(1, y) = \int M \, dt + h(y)
\]

\[
= \int N \, dy + k(t)
\]

\[
\Rightarrow q(1, y) = 3ty + e^t + h(y)
\]

\[
= 3ty + \sin y + k(t)
\]

relationship is obvious.

Choose whatever method is easiest based on ability to integrate \(M, N, \) etc.

Example:

\[
3t^2e + 4ty + (2y + 2e^2) \, y' = 0
\]

\[
y(0) = 1
\]

**Integrating Factors for Exact Diff. Eq.**

If an equation is not exact, can it be made exact?

Example: \(M(1, y) + N(1, y) \, y' = 0\)

Multiplying through by \(\mu\) yields the condition:

\[
\mu M + \mu N \, y' = 0
\]

This is exact if and only if

\[
\frac{1}{M} \left( \frac{\partial}{\partial y} (\mu M) \right) = \frac{1}{N} \left( \frac{\partial}{\partial t} (\mu N) \right)
\]

\[
\Rightarrow \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial t} N + \mu \frac{\partial N}{\partial t}
\]

In general, we cannot find an explicit solution to this.

If, however, we assume \(\mu = \mu(t)\) (i.e. does not depend on \(y\)) then
\[
\begin{align*}
\mu \frac{dM}{dy} &= \frac{dM}{dt} + \nu \frac{dN}{dt} \\
\mu \frac{dN}{dt} &= \left( \frac{dM}{dy} - \frac{dN}{dt} \right) M.
\end{align*}
\]

\[\Rightarrow \mu = \frac{\int M \, dy - N \, dt}{\int N \, dt} \text{ is an integrating factor.}\]

And vice versa when assuming \( \mu = \mu(y) \).

**Example:** Find general solution to:

\[
\frac{\mu^2}{2} + 2y\mu + (y+\mu) \frac{d\mu}{dt} = 0
\]

\[
\text{Only a function of } \mu \text{ must also only be a function of } t.
\]

These examples are rather contrived, involving very special-case scenarios. Not very physically relevant.