Sept 11, 2019

Announcements: HW 1 due Monday, in class

Last time: * Inhomogeneous equations with integrating factors:

$$y' + ay = b \implies \mu y' + a\mu y = \mu b$$

choose μ such that $\mu y' + a\mu y = (\mu y)'$

- Separable equations:

$$\frac{dy}{dt} = \frac{g(t)}{f(y)} \implies f(y)\, dy/dt = g(t)$$

Rewrite as: $\frac{d}{dt}(F(y)) = g(t)$ where F is anti-derivative of f

$$\implies F(y) = \int g(t)\, dt + C$$

Add initial conditions via integration: (using $y(t_0) = y_0$)

$$\underbrace{\int_{t_0}^{t} \frac{d}{d\tau} F(y(\tau))\, d\tau}_{} = \int_{t_0}^{t} g(\tau)\, d\tau$$

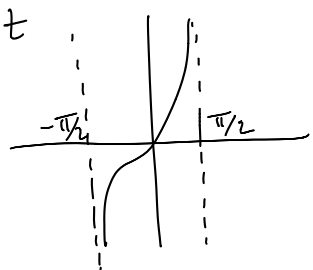
$$= F(y(t)) - F(y(t_0))$$
$$= F(y) - F(y_0)$$
$$= \int_{y_0}^{y} f(w)\, dw.$$

- Application: Radioactive dating:

$$N'(t) = N_0 \lambda \implies N(t) = N_0 e^{-\lambda(t-t_0)}$$

- Failure of solution to exist outside some interval:

$$y' = 1 + y^2, \quad y(0) = 1 \implies y = \tan t$$



Another method of failure:

Ex: $yy' + (1+y^2)\sin t = 0$

$$y(0) = 1$$

Separating we have: $\frac{dy}{dt}\frac{y}{1+y^2} = -\sin t$

$$\Rightarrow \int_1^y \frac{w}{1+w^2}\,dw = -\int_0^t \sin\tau \, d\tau$$

$$\Rightarrow \frac{1}{2}\log(1+w^2)\Big|_1^y = \cos\tau\Big|_0^t$$

$$\Rightarrow \frac{1}{2}\log(1+y^2) - \frac{1}{2}\log 2 = \cos t - 1$$

$$\Rightarrow \log\left(\frac{1+y^2}{2}\right) = 2(\cos t - 1)$$

$$y^2 = 2e^{2(\cos t - 1)} - 1$$

$$\Rightarrow y = \pm\sqrt{2e^{2(\cos t-1)} - 1}$$ which branch? $\pm$?

Since $y(0) = 1$, it must be the $+$ branch

$$= \sqrt{2e^{2(\cos t - 1)} - 1}$$

Furthermore, the solution is only real-valued when

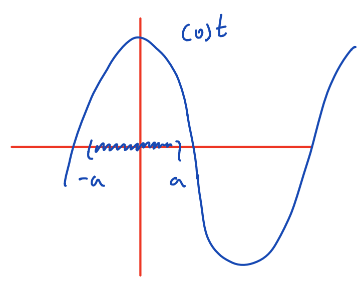
$$2e^{2(\cos t-1)} - 1 > 0$$

$$\Rightarrow e^{2(\cos t - 1)} > \frac{1}{2}$$

$$\Rightarrow 2(\cos t - 1) > -\log 2$$

$$\cos t - 1 > -\frac{1}{2}\log 2$$

$$\cos t > \underbrace{-\frac{1}{2}\log 2 + 1}_{\approx .65}$$



solution only exists in this interval

We can see this by rewriting the original DE as

$$y' = \frac{-(1+y^2)\sin t}{y}$$

y' is not defined when $y=0$, $y' \to \infty$

Two types of solution to separable equations:

Explicit and Implicit

Explicit → $y(t) = \text{RHS}$ (all previous examples)

Implicit → $F(y) = g(t)$, y cannot be solved for

<u>Ex:</u>

$$(1+e^y)\,y' = \cos t$$

$$y\left(\tfrac{\pi}{2}\right) = 3$$

Integrating:

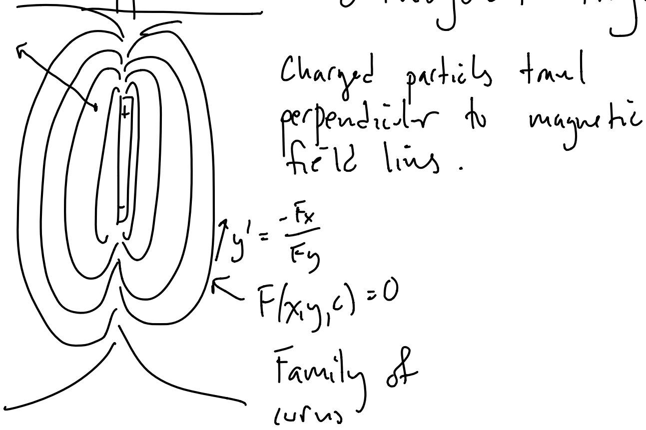
$$\int_3^y (1+e^w)\,dw = \int_{\pi/2}^t \cos\tau\,d\tau$$

$$\Rightarrow \quad w + e^w \Big|_3^y = \sin\tau \Big|_{\pi/2}^t$$

$$\Rightarrow \quad y + e^y = \sin t + 2 + e^3$$

] Implicit solution.

<u>Application</u> - orthogonal trajectories



Differentiate: $\frac{d}{dx} F(x,y,c) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

$$\Rightarrow y' = \frac{-F_x}{F_y}$$

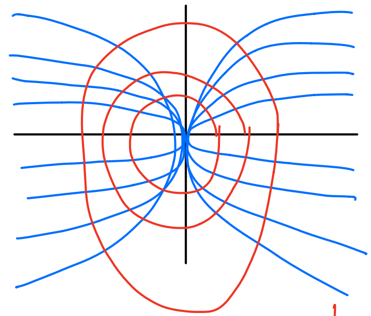
Therefore orthogonal trajectory is the solution to

$$y' = F_y / F_x$$

Ex: A family of parabolas:

$$F(x,y,c) = x - cy^2$$

$$\Rightarrow x = cy^2$$



Differentiate $\Rightarrow$ $1 = 2cyy'$

$$\Rightarrow y' = \frac{1}{2cy} = \frac{1}{2y}\frac{1}{x/y^2} = \frac{y}{2x}$$

Orthogonal trajectories are given by

$$y' = \frac{-2x}{y} \Rightarrow yy' = -2x$$

$$\int yy'\,dx = \int 2x\,dx + C \Rightarrow \frac{1}{2}y^2 + x^2 = C$$

$$\Rightarrow 2x^2 + y^2 = C$$ ← new constant

equation of an ellipse.

Exact equations §1.9

The form of differential equation we have studied so far is generally:

$$\frac{d}{dt}(\sim) = g(t).$$

↳ this "something" has generally been assumed to be only a function of y

E.g. ① $y' + ay = 0 \Rightarrow \frac{y'}{y} = -a \Rightarrow \frac{d}{dt}\log|y| = -a$

② $\frac{dy}{dt} = \frac{g(t)}{f(y)} \Rightarrow \frac{d}{dt}F(y) = g(t).$

The most general form of this problem is:

$$\frac{d}{dt}\varphi(t,y) = 0 \Rightarrow \varphi(t,y) = c,$$ then solve for y above.

Example:

$$\overbrace{2t\sin y + y^3e^t + (t^2\cos y + 3y^2e^t)}^{\Phi}\, dy/dt = 0$$

Note that: $\frac{d}{dt}\left(t^2\sin y + y^3e^t\right) = 2t\sin y + y^3e^t$

$\frac{d}{dy}\left(t^2\sin y + y^3e^t\right) = t^2\cos y + 3y^2e^t$

and therefore $\frac{d}{dt}\left(t^2\sin y + y^3e^t\right) = \Phi(t,y) = \frac{d}{dt}\varphi(t,y)$.

The existence of φ, given Φ is generally not obvious.

Recall: $\frac{d}{dt}\varphi(t,y) = \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt}$

(partial derivatives.)

Therefore:

The differential equation $M(t,y) + N(t,y)\,y' = 0$ can be written as $\frac{d}{dt}\varphi(t,y) = 0$ if and only if there is some φ such that $\frac{\partial\varphi}{\partial t} = M$ and $\frac{\partial\varphi}{\partial y} = N$