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Lotka-Volterra eqns.

Describe predator-prey interaction

U = # of prey V = # of predators

Rate of change of U = growth rate w/o predation − loss due to predation

change of V = growth rate due to predation − loss w/o prey

Assumptions

1) Prey limited only by predators, grows exponentially in their absence
2) Predation term is linear in U (growth rate $\propto$ # of prey)
3) No interference among predators in finding prey (predation term linear in V)
4) In absence of prey, predators die off exponentially
5) Every prey death contributes the same to the predator population

The equations which describe these assumptions are

$$\frac{dU}{dt} = \alpha U - \gamma UV$$

$$\frac{dV}{dt} = e\gamma UV - \beta V. \quad , \text{ where } \alpha, \gamma, e, \beta > 0$$

Easy to check (1) and (4): if $\gamma = 0$ get

$$U' = \alpha U \to U = U_0 e^{\alpha t}$$

$$V' = -\beta V \to V = V_0 e^{-\beta t}$$

Exercise Find the steady states of the LV eqns

$$\alpha \cancel{U} = \gamma \cancel{U} V \to V = \alpha/\gamma.$$

$$\beta \cancel{V} = e\gamma U \cancel{V} \to U = \beta/e\gamma$$

Remember that the stability of the steady state is determined by the eigenvalues of the Jacobian matrix

$$J = \begin{pmatrix} \partial F/\partial U & \partial F/\partial V \\ \partial G/\partial U & \partial G/\partial V \end{pmatrix}\Bigg|_{U^*, V^*} \quad \text{where} \quad \frac{dU}{dt} = F \quad \frac{dV}{dt} = G$$

$$= \begin{pmatrix} \alpha - \gamma V^* & -\gamma U^* \\ e\gamma V^* & e\gamma U^* - \beta \end{pmatrix} = \begin{pmatrix} 0 & -\beta/e \\ e\alpha & 0 \end{pmatrix}$$

The trace of this matrix (sum of the eigenvalues) is zero, while the determinant is positive ($\alpha\beta$)

This means the eigenvalues are pure imaginary

Thus the linearized eqns. have periodic solutions, but we can't say if the dynamics will approach or move away from the fixed point based on this linearized analysis.
We will have to do some nonlinear analysis. To do this, it will help to rescale the variables.

Recall $\frac{dU}{dt} = \alpha U - \gamma U V$

$\frac{dV}{dt} = e\gamma U V - \beta V.$

Let $u = U/U^*$ $v = V/V^*$ and set $\tau = \alpha t$.

$U = (U^* u)$ $V = (v V^*)$ $t = \tau/\alpha$

$$\frac{d(\cancel{U^*} u)}{d(\tau/\alpha)} = \alpha \cancel{U^*} u - \gamma \cancel{U^*} V^* uv \implies \frac{du}{d\tau} = u - \frac{\gamma V^*}{\alpha} uv.$$

$$\frac{d(\cancel{V^*} v)}{d(\tau/\alpha)} = e\gamma U^* \cancel{V^*} uv - \beta \cancel{V^*} v \qquad \frac{dv}{d\tau} = \frac{e\gamma}{\alpha} U^* uv - \frac{\beta}{\alpha} v.$$

But $U^* = \beta/e\gamma$, $V^* = \alpha/\gamma$. get

$$\frac{du}{d\tau} = u(1-v)$$

$$\frac{dv}{d\tau} = \frac{\beta}{\alpha} uv - \beta v$$

$$= \frac{\beta v}{\alpha}(u-1)$$

Letting $a = \beta/\alpha$, we have the system

$$\frac{du}{dt} = u(1-v)$$

$$\frac{dv}{dt} = av(u-1).$$

Dividing these by each other, we get

$$\frac{dv}{du} = \frac{av(u-1)}{u(1-v)}$$

this equation in the (u,v) plane is called the <u>phase plane</u> equation. It expresses the changes in (dimensionless) predator population per changes in the prey population.

<u>Exercise</u> Solve this equation implicitly in (u,v) by separation of variables

$$a \frac{u-1}{u} du = dv \frac{1-v}{v}$$

$$= a\left(1 - \frac{1}{u}\right) du = dv\left(\frac{1}{v} - 1\right).$$

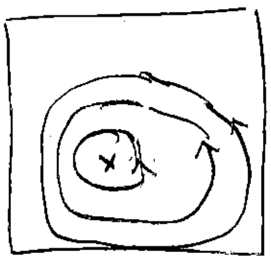
$$a(u - \ln u) = \ln v - v + C$$

Can write this as

$$c = a(u - \ln u) + v - \ln v.$$

which is an equation of a family of curves which look like

Predators v.



Prey u

Arrows denote trajectory in time

the system undergoes oscillations in time
Possible to show that these are oscillations around the
steady state u^*, v^* (1)
Reason for these oscillations make physical sense:
Scarce prey → scarce predators → abundant prey
→ abundant predators → ...