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Multiple species

How interaction between small number of species affect their dynamics

3 categories of interaction:

1) Competition $(-,-)$

2) Symbiosis $(+,+)$

3) Predation or parasitism $(+,-)$

We will concentrate this week on 1) and 3)

Host parasite interactions

Define

H_n = # of host at generation n

P_n = # of parasites at generation n

R_0 = host reproductive ratio - per capita production of host in absence of parasites

c = average # of eggs laid by adult parasite in host

$f(H,P)$ = fraction of host NOT infected w/ parasite

$$H_{n+1} = R_0 H_n f(H_n, P_n) \quad \text{[Only non-infected hosts survive]}$$

$$P_{n+1} = c H_n (1 - f(H_n, P_n)) \quad \text{[Dead hosts become new parasites]}$$

Assumed here no density dependent effects

The first model of this class called the Nicholson-Bailey model, assumed that within a generation

$$\frac{dH}{dt} = -\alpha P H$$

encounters between parasite and host take place at rate α.

Solving this with P_n constant, get

$$H(n+\tau) = H(n)\, e^{-\alpha P_n \tau} = H(n)\, e^{-a P_n}$$

where $a = \alpha\tau$ thus in $(*)$ $f = e^{-\alpha P_n}$ and

$$H_{n+1} = R_0 H_n \exp(-aP_n)$$

$$P_{n+1} = c H_n (1 - \exp(-aP_n))$$

Exercise find constant population steady state $H_{n+1} = H_n$

$$H^* = R_0 H^* \exp(-aP_n)$$

$$\rightarrow P_n = \log(R_0)/a$$

$$H_n = \frac{P^*}{c(1-\exp(-aP_n))} = \frac{P^*}{c(1-\exp(-\log(R_0)))}$$

$$= \frac{P^*}{c(1-1/R_0)} = \frac{P^* R_0}{c(R_0-1)}$$

So we have the steady states

$$P^* = \frac{1}{a}\log(R_0)$$

$$H^* = \frac{1}{a}\log(R_0)\left(\frac{R_0}{c(R_0-1)}\right)$$

How do we analyze its stability?

Linearize system about steady state

$$\begin{pmatrix}\Delta H_{n+1}\\ \Delta P_{n+1}\end{pmatrix} = A\begin{pmatrix}\Delta H_n\\ \Delta P_n\end{pmatrix} = \begin{pmatrix}a_{11}H_n + a_{12}H_n\\ a_{21}P_n + a_{22}P_n\end{pmatrix} \qquad \Delta H = H - H^*, \text{ etc.}$$

If we have eqns

$$H_{n+1} = R_0 H_n \exp(-aP_n)$$

$$P_{n+1} = cH_n(1-\exp(-aP_n))$$

Remember $f(H,P) = f(H^*,P^*) + f_H(H^*,P^*)(H-H^*) + f_P(H^*,P^*)(P-P^*)$

In this case:

$$H_{n+1} = \underbrace{R_0 H^* \exp(-aP^*)}$$

$$H_{n+1} = H^* + R_0\exp(-aP^*)(H-H^*) - aR_0H^*\exp(-aP^*)(P-P^*)$$

$$\exp(-aP^*) = 1/R_0$$

$$P_{n+1} = P^* + c(1-\exp(-aP^*))(H-H^*) + acH^*\exp(-aP^*)(P-P^*)$$

So

$$\begin{pmatrix}\Delta H_{n+1}\\ \Delta P_{n+1}\end{pmatrix} = \underbrace{\begin{pmatrix}1 & -aH^*\\ c(1-\frac{1}{R_0}) & \frac{acH^*}{R_0}\end{pmatrix}}_{J}\begin{pmatrix}\Delta H_n\\ \Delta P_n\end{pmatrix}$$

Determinant of this matrix is

$$\det J = \frac{acH^*}{R_0} = \left(\frac{\log R_0}{R_0-1}\right) + acH^*\left(1-\frac{1}{R_0}\right)$$

$$= \frac{\log R_0}{R_0-1} + \frac{(\log R_0)R_0}{R_0-1}\left(1-\frac{1}{R_0}\right)$$

$$= \frac{\log R_0}{R_0-1}(R_0-1+1)$$

$$= \frac{R_0\log(R_0)}{R_0-1} > 1.$$

Since the trace of the matrix is also positive, both of the eigenvalues are positive and we see that the equilibrium point is <u>unstable</u>.
It predicts that the host-parasite relationship exhibits oscillations whose amplitude grows without limit. What errors have we made in the model - knowing that in real life dynamics are stable?

1. Host is not limited entirely by parasite (there is intra-host competition / logistic growth)
2. For large H, the rate of contact between H and P should not be $\propto HP$, but something smaller.
3. Parasites could compete with each other for same host.