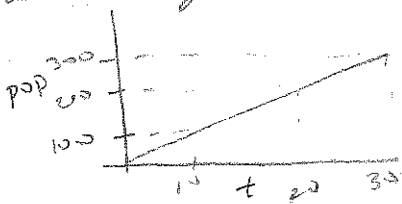
12/01/2022

Single species population dynamics

types of population growth

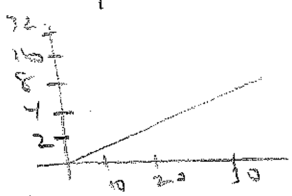
1) Linear growth (increasing by 100 every 10 years)



$$Y = ax$$

2) Exponential growth (doubling every 10 years).

$$Y = 2^{(x/10)}$$



Straight line when plotted on a semilog plot.

Populations cannot grow forever - they are limited by competition for resources. For simplicity let's ignore that to start.

Let b = per capita production or reproduction rate per time (probability that an individual produces a new one in $b \Delta t$).

d = per capita death rate per time

$$N(t+\Delta t) = N(t) + bN(t)\Delta t - dN(t)\Delta t$$

$$\xrightarrow{\Delta t \to 0} \quad \frac{dN}{dt} = (b-d)N := rN.$$

with initial condition $N(0) = N_0$.

Exercise: solve this ODE

$$N(t) = N_0 e^{rt}$$

This kind of population growth is called <u>Malthusian growth</u>.

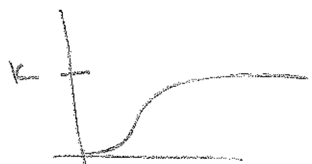
It can be understood in terms of the reproductive ratio $R_0 = b/d$, $\quad r = d(R_0 - 1)$

If $R_0 > 1$, $r > 0$ and $N \to \infty$ as $t \to \infty$.

if $R_0 < 1$, $r < 0$ and $N \to 0$ as $t \to \infty$

Notice here that the birth and death rates are independent of population size.

If we want to include density-dependent effects, assume that the population reaches a carrying capacity K that slows down its growth



If $\frac{dN}{dt} = N F(N)$

where $f(N)$ is the net per capita growth rate, what should be true about $F(0)$ and $F(K)$?

Simple model:

$$\boxed{\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)} \qquad (*)$$

Called the logistic eqn.

For small N, get Malthusian growth.

Exercise Find eq point of $(*)$ and analyze their stability

$N^* = 0, K$

$f(N) = rN - \frac{rN^2}{K}$

$f'(N) = r - \frac{2rN}{K}$

$f'(0) = r > 0$ Unstable.

$f'(K) = -r < 0$. Stable

Can solve this eqn

$$\int \frac{dN}{N(K-N)} = \int \frac{r}{K}$$

partial fraction

$$\frac{1}{N(K-N)} = \frac{A}{N} + \frac{B}{K-N} \rightarrow A(K-N) + BN = 1.$$

$N = K$, $B = 1/K$. $N = 0$, $A = 1/K$.

$$\int \frac{1}{KN} + \frac{1}{K(K-N)}\, dN = \int \frac{r}{K}\, dt$$

$$\cancel{\frac{1}{K}}\ \ln N - \ln(K-N) = \frac{rt}{\cancel{K}} + c$$

$$\ln\left(\frac{N}{K-N}\right) = rt$$

$$\frac{N}{K-N} = ce^{rt}$$

Solve for N: $N = cKe^{rt} - Nce^{rt}$.

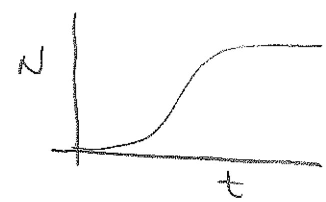
$$N(1+ce^{rt}) = cKe^{rt}$$

$$N = \frac{cKe^{rt}}{1+ce^{rt}}$$

$$N(0) = \frac{cK}{1+c} = N_0 \rightarrow cK = N_0 + cN_0$$

$$c = \frac{N_0}{K-N_0}$$

$$N = \frac{\frac{N_0K}{K-N_0}e^{rt}}{1+\frac{N_0}{K-N_0}e^{rt}} = \boxed{\frac{N_0Ke^{rt}}{K-N_0+N_0e^{rt}}}$$



Logistic growth doesn't work indefinitely because the carrying capacity K changes over time due to technical or medical advances.

Important feature: logistic model is compensatory, since F(N) decreases as N increases $[f'(N) = r(1 - \frac{2N}{K})]$ is negative when $N > K/2$.

Some growth models are depensatory, where F(N) increases with N over some range (higher density helps breeding).

Extreme case: critical depensation - growth rate negative for small values of N, leading to minimum pop. size (HW).