11/17/2022

Modifying SIR for disease features.

The goal of this lecture: learn confidently developing a compartment model based on a set of examples.

Fatal infection

In none discussed, all infections resulted in death (or 99%, which is large enough to assume all)

Let's also relax assumptions. What happens if deaths, since the Ebola deaths will wipe out some of the population anyway

Let μ = per capita death rate ν = per capita birth rate

$$S' = \nu - \beta SI - \mu S$$
$$I' = \beta SI - (\gamma + \mu) I$$

Here $1/\gamma$ = time to die for disease

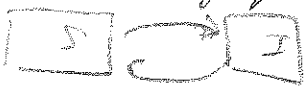
Exercise: Calculate R_0.

$$I'|_{t=0} = \beta S\cancel{I} - (\gamma+\mu)\cancel{I} < 0$$
$$\frac{\gamma+\mu}{\beta} > S(0). \quad \Rightarrow \quad 1 > S(0)\frac{\beta}{\gamma+\mu}.$$

$R_0 = \frac{\beta}{\gamma+\mu}$. Same as before! Why? In the old model, recovered individuals had no interaction with the rest of the population. Neither do dead people.

No immunity, multiple infections

Consider a case where there is no immunity from infection.



SIS model

$$\frac{dS}{dt} = -\beta SI + \gamma I$$
$$\frac{dI}{dt} = \beta SI - \gamma I.$$

Assuming no birth and death, so $S+I=1$ for all time, then

$$\frac{dI}{dt} = \beta(1-I) - \gamma I$$
$$= I(\beta - \gamma - \beta I)$$
$$= (\beta-\gamma) I \left(1 - \frac{\beta}{\beta-\gamma} I\right)$$

the equilibrium points are at $I=0$ and

$$I = \frac{\beta-\gamma}{\beta} = 1 - \frac{\gamma}{\beta} = 1 - \frac{1}{R_0}.$$

If $R_0 > 1$, the endemic equilibrium exists. Is it stable?

$$\frac{dI}{dt} = f(I) = (\beta - \gamma) I \left(1 - \frac{\beta}{\beta - \gamma} I\right).$$

$$f'(I) = (\beta - \gamma)\left(1 - \frac{\beta}{\beta - \gamma} I\right) + (\beta - \gamma) I \left(-\frac{\beta}{\beta - \gamma}\right).$$

$$= \beta - \gamma - \beta I - \beta I$$

$$= \beta - \gamma - 2\beta I.$$

$$I^* = 1 - \frac{\gamma}{\beta}; \rightarrow f'(I^*) = \beta - \gamma - 2\beta\left(1 - \frac{\gamma}{\beta}\right).$$

$$= \beta - \gamma - 2\beta + 2\gamma.$$

$$= \gamma - \beta.$$

If $R_0 > 1$, then $\frac{\beta}{\gamma} > 1$, then $\beta > \gamma$ and $f'(I^*) < 0$.

So endemic equilibrium is stable.

The SEIR model

Let's account for diseases which have an incubation period, then an infectious period, after which individuals have lifelong immunity. Modify model to include an (exposed) class. This is known as the SEIR model.

Let $1/\sigma$ be the average amount of time spent in exposed class. Example: write the ODEs.

$$S' = \mu - \beta S I - \mu S.$$

$$E' = \beta S I - (\sigma + \mu) E$$

$$I' = \sigma E - (\mu + \gamma) I$$

$$R' = \gamma I - \mu R$$

The rates have some details on finding R_0 which I feel are beyond scope of this class.

The issue is that if we try to write $I' = \sigma E - (\mu + \gamma) I < 0$, we can no longer factor out an I. So we need a new method for finding the R_0.

Next generation matrix

Create a "new" model where m of compartments are diseased individuals (E and I). Write this model in the form

$$\frac{dx_i}{dt} = \mathcal{F}_i(x) - \mathcal{V}_i(x)$$

where $\mathcal{F}_i(x)$ = rate of new infections entering each compartment,
$\mathcal{V}_i$ = rate with other inputs + outputs. For this model,

$$\frac{dx_i}{dt} = \begin{pmatrix} \beta S I \\ 0 \end{pmatrix} - \begin{pmatrix} (\sigma+\mu)E \\ -\sigma E + (\mu+\gamma)I \end{pmatrix}.$$

Can rewrite this in matrix form as

$$\begin{pmatrix} E \\ I \end{pmatrix}' = \begin{pmatrix} 0 & \beta S \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \mu+\sigma & 0 \\ -\sigma & \mu+\gamma \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix}$$

$$= (\mathcal{F} - \mathcal{V}) \begin{pmatrix} E \\ I \end{pmatrix}.$$

We can see from this that $R_0 < 1$ if the (real part of) the eigenvalues of $\mathcal{F}-\mathcal{V}$ is less than zero. An equivalent condition to this is that the largest eigenvalue of $\mathcal{F}\mathcal{V}^{-1}$ is less than 1. The largest eigenvalue of the next generation matrix defines R_0. If matrices are 1×1 it makes sense.

$$\mathcal{F}-\mathcal{V} < 0 \iff \mathcal{F}\mathcal{V}^{-1} < 1.$$