11/15/2022

Births, deaths, and equilibrium

Relaxing some of the assumptions we made last week

When would the no births/deaths assumption be reasonable?

Including births and deaths

To maintain population, any births must = deaths.

Denote rate of birth/death by μ, with $1/\mu$ = lifespan

$$\frac{dS}{dt} = \mu(S+I+R) - \beta SI - \mu S.$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

Now, $S+I+R=1$, so this can be rewritten as

$$\frac{dS}{dt} = \mu - \beta SI - \mu S.$$

$$\frac{dI}{dt} = \beta SI - (\gamma+\mu) I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

Calculate R_0 in the same way,

$$I'_{t=0} = (\beta S(0) - \gamma - \mu) I(0) < 0.$$

$$\longrightarrow \frac{\beta S(0)}{\gamma+\mu} < 1.$$

So $R_0 = \dfrac{\beta}{\gamma+\mu}$.

Compare this with previous β/γ, it is smaller.

the natural deaths and births help you (here it's deaths of infectious individuals)

Equilibrium points

$$\frac{dS}{dt} = 0 \quad \frac{dI}{dt} = 0. \quad \frac{dR}{dt} = 0.$$

Exercise: solve for them!

1) $\mu - \beta SI - \mu S = 0$

2) $\beta SI - (\gamma+\mu) I = 0.$

3) $\gamma I - \mu R = 0$

2) $\longrightarrow \beta S = \gamma + \mu.$ [If $I \neq 0$?]

$$S = \frac{\gamma+\mu}{\beta} = \frac{1}{R_0}$$

1) → $I = \frac{\mu(1-S)}{\beta S} = \frac{\mu\left(1-\frac{1}{R_0}\right)}{\gamma+\mu} = \frac{\mu}{\beta} R_0 \left(1-\frac{1}{R_0}\right)$

$$I = \frac{\mu}{\beta}(R_0 - 1)$$

$$R = 1 - \frac{1}{R_0} - \frac{\mu}{\beta}(R_0 - 1).$$

Called the <u>endemic equilibrium</u> in contrast to Disease free equilibrium $(0,0,0)$

<u>How</u> can we tell if these points are stable? i.e., which one of them will the solution approach?

<u>Linear stability analysis</u>

The idea is as follows: suppose you are solving the ODE

(*) $\frac{dy}{dt} = \lambda y$. The solution is $y(t) = y(0)e^{\lambda t}$.

If $\lambda > 0$, this solution <u>decays</u> to 0, while if $\lambda > 0$, the solution grows to ∞.

Now, every ODE look like (*) in the neighborhood of the fixed point.

Let's consider the ODE

$$\frac{dy}{dt} = y^2 - 7y + 10.$$

This ODE has 2 steady states: $y = 5$, $y = 2$. Which, if any is stable?

If $\frac{dy}{dt} = f(y) = \cancel{f(y_0)} + (y-y_0) f'(y_0) + O((y-y_0)^2)$.

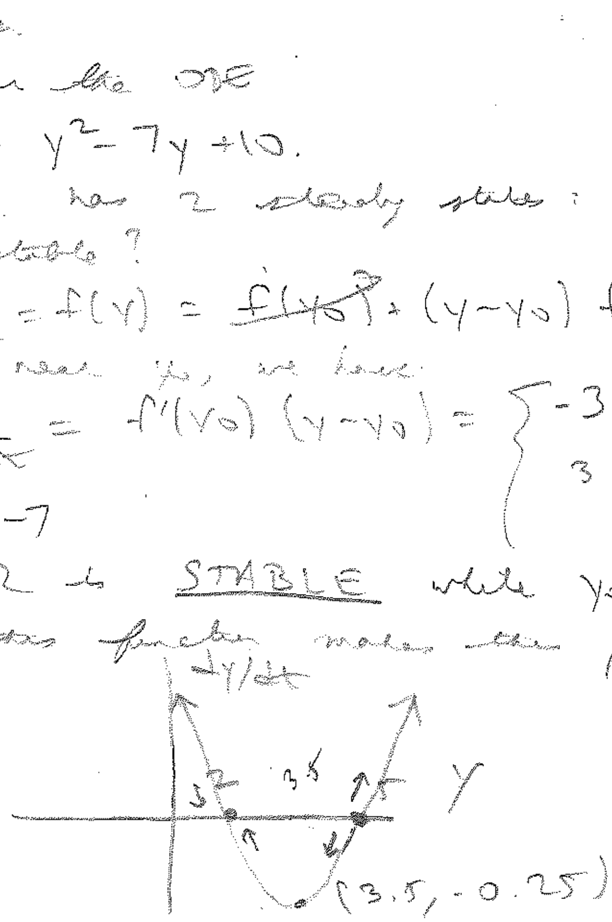
For points near y_0, we have

$$\frac{dy}{dt} = f'(y_0)(y-y_0) = \begin{cases} -3(y-y_0) & y_0 = 2 \\ 3(y-y_0) & y_0 = 5 \end{cases}$$

$f' = 2y - 7$

So $y_0 = 2$ is <u>STABLE</u> while $y_0 = 5$ is UNSTABLE.

Plotting this function makes this possible to see



If we apply this analysis to a multivariable system

$$\frac{d\begin{pmatrix} S \\ I \\ R \end{pmatrix}}{dt} = f\begin{pmatrix} S \\ I \\ R \end{pmatrix}$$

where f is a 3 vector, then the analysis has to be conducted in terms of the eigenvalues of the 3×3 matrix.

$$A = \frac{\partial f}{\partial p} = \begin{pmatrix} \partial f_1/\partial S & \frac{\partial f_1}{\partial I} & \partial f_1/\partial R \\ \partial f_2/\partial S & & \\ \partial f_3/\partial S & \cdots & \end{pmatrix}$$

Then we can rewrite the problem as

$$\frac{dx}{dt} = Ax \quad \text{near the fixed point,}$$

and the eigenvalues of A determine stability.
You can show in this case that endemic eq is stable when $R_0 > 1$ and unstable when $R_0 < 1$. while DFE is the opposite.