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Introduction to compartment models for infectious disease

Infectious diseases – ~~pathogen~~ passed between individuals

Pathogens can be divided into microparasites and macroparasites

Each of these broken down again: direct transmission (close contact – covid!) and indirect transmission (pathogen survives outside of its host – vector borne diseases)

We will focus on microparasitic directly transmitted infectious diseases

Compartment models

Take a population and divide into compartments based on an individual's status. Simplest model?

1) Population is closed (no deaths or births)
2) Only infection & recovery occur

$$\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$$

Let λS = rate of new infections. λ = force of infection, per capita rate at which susceptibles acquire infection

Infection must be related to I, but how?

1) Density dependent $\lambda = \beta I$, where β = transmission rate. ?
Composite: contact rate × transmission probability

I = # infected

$\lambda S = \frac{\text{new infection}}{\text{time}}$ $\qquad \beta = \frac{\text{new infection}}{\text{susceptible} \times \text{time} \times \text{(infected people)}}$

$\lambda = \frac{\text{new infection}}{\text{susc} \times \text{time}}$ $\qquad \frac{\text{contacts}}{\text{infected} \times \text{susceptible} \times \text{time}} \cdot \frac{\text{infection}}{\text{contact}}$

2) Frequency dependent transmission. $\lambda = \beta \frac{I}{N}$. λ is independent of pop size, depends on fraction of infectious

$\beta = \frac{\text{new infection}}{\text{susceptible} \times \text{time} \times \text{person}}$ (fraction of people infected)

β = rate when everyone is infected

$\lambda = \beta \frac{I}{N}$ (we will use this)

Recovery – let γ be recovery rate. $1/\gamma$ = average time spent infected (e.g. 14 days rate is 1/14 people/day)

With these assumptions, the ODE model is given by

$$\frac{dS}{dt} = -\beta \frac{I}{N} S$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

$N = S + I + R =$ constant

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0.$$

Often useful to nondimensionalize state variables

$$\bar{S} = S/N \quad \bar{I} = I/N, \quad \bar{R} = R/N.$$

gives simple model

$$\frac{d\bar{S}}{dt} = -\beta \bar{I} \bar{S}$$

$$\frac{d\bar{I}}{dt} = \beta \bar{S} \bar{I} - \gamma \bar{I}$$

$$\frac{d\bar{R}}{dt} = \gamma \bar{I}$$

Now $S + I + R = 1$ and $S' + I' + R' = 0$.

Questions about this model. (dropping bars)

1) Under what conditions will disease die out?

$$\left.\frac{dI}{dt}\right|_{t=0} < 0.$$

$$I' = \beta S I - \gamma I = I(\beta S - \gamma).$$

$$I'(t=0) = I(0)(\beta S(0) - \gamma) < 0.$$

If $I(0) \geq 0$, need

$$\beta S(0) - \gamma < 0.$$

$$S(0) < \frac{\gamma}{\beta} \quad \text{or} \quad \frac{\beta S(0)}{\gamma} < 1.$$

fraction of individuals susceptible to infection in the absence spread...

< 1. Small enough so epidemic can't take off

At the onset $S(0) \approx 1$, so

$\boxed{\frac{\beta}{\gamma} < 1.}$ disease dies out

Otherwise it will [illegible]. Thus is a threshold phenomenon and β/γ is called R_0 a basic reproductive number #.
R_0 = average # of cases arising from 1 person in an entirely susceptible pop

$$\beta = \text{infection rate} = \frac{\text{new infections}}{\text{time} \cdot \text{person}}$$

$\frac{1}{\gamma}$ = time in infectious class. $\beta \cdot \frac{1}{\gamma}$ = new infections / person

R_0 is the most important quantity in disease modeling!

2) Is it possible for susceptible individuals to escape infection
i.e. is $\lim_{t\to\infty} S(t) > 0$?

$$\frac{dS}{dt} \quad \frac{dI}{dt} \quad \frac{dR}{dt}$$

$$\frac{dS}{dR} = \frac{dS}{dt} \Big/ \frac{dR}{dt} = \frac{-\beta S \cancel{I}}{\gamma \cancel{I}} = -\frac{\beta S}{\gamma} = -R_0 S$$

<u>Exercise</u> Solve the ODE $\frac{dS}{dR} = -R_0 S$ for S

$$S = C e^{-R_0 R(t)}$$

Answer [illegible] $S = S(0) \exp(-R_0 R(t))$.

$$\lim_{t\to\infty} S(t) = S(0) \exp(-R_0 R(\infty)) ..$$

Thus [illegible] positive number. So there is still a fraction of susceptible individuals remaining. Not enough infectives to sustain the epidemic. Called <u>epidemic burnout</u>.