HW 6 Solutions

(a) Attachment

$$\frac{\#\text{ attach}}{\text{time}} = (1-U)n_0 \int_{x_0}^{A} \alpha_0 \, dx$$

$$= (1-U)n_0 \alpha_0 (A - x_0)$$

(b) Detachment

$$\frac{\#\text{ detach}}{\text{time}} = \beta n_0 \int_{x_0}^{A} e^{x/A} u(x) \, dx$$

(c) Sliding speed v:

$$\frac{\#\text{ sliding}}{\text{time}} = v n_0 u(x_0)$$

(d) integral eqn.

Attach = Detach + Sliding

$$(1-U)n_0\alpha_0(A-x_0) = \beta n_0 \int_{x_0}^{A} e^{x/A} u(x)\,dx + v n_0 u(x_0)$$

Differentiate to get ODE

$$-(1-U)n_0\alpha_0 = -\beta n_0 e^{x_0/A} u(x_0) + v n_0 u'(x_0)$$

(e) Special case $v=0$:

$$-(1-U)n_0\alpha_0 = -\beta n_0 e^{x/A} u(x)$$

$$\boxed{u(x) = \frac{(1-U)\alpha_0}{\beta} e^{-x/A}}$$

NOT done yet! Still need to get $U = \int_{-A}^{A} u(x)\,dx$

$$\underbrace{\int_{-A}^{A} u(x)\,dx}_{U} = \frac{(1-U)\alpha_0}{\beta} \int_{-A}^{A} e^{-x/A}\,dx$$

$$= \frac{(1-U)\alpha_0}{\beta}\left[-A e^{-x/A}\Big|_{-A}^{A}\right] = \frac{(1-U)\alpha_0 A (e - e^{-1})}{\beta}$$

Solve for U: $U\left(1 + \frac{\alpha_0 A}{\beta}(e - e^{-1})\right) = \frac{\alpha_0 A (e - e^{-1})}{\beta}$

$$\boxed{U = \frac{\alpha_0 A (e - e^{-1})}{\beta} \left(1 + \frac{\alpha_0 A}{\beta}(e - e^{-1})\right)^{-1}}$$

(f) $P = n_0 \int_{-A}^{A} u(x)\, p(x)\, dx$

$$= \int_{-A}^{A} \frac{(1-u)\alpha_0 p_0 n_0}{\beta} e^{-x/A}\left(e^{kx/A} - 1\right) dx$$

$$= \frac{(1-u)\alpha_0 p_0 n_0}{\beta} \int_{-A}^{A} e^{(k-1)x/A} - e^{-x/A}\, dx$$

$$= \frac{(1-u)\alpha_0 p_0 n_0}{\beta} \left(\frac{A}{k-1} e^{(k-1)x/A} + A e^{-x/A} \right) \Big|_{-A}^{A}$$

$$= \frac{(1-u)\alpha_0 p_0 n_0}{\beta} \left(\frac{A}{k-1}\left(e^{k-1} - e^{1-k}\right) + A\left(e^{-1} - e\right) \right)$$

(g) If $k = 2$, we get

$$P = \frac{(1-u)\alpha_0 p_0 A n_0}{\beta} \left[e^{1} - e^{-1} + (e^{-1} - e) \right] = 0$$

while if $k = 3$

$$P = \frac{(1-u)\alpha_0 p_0 A n_0}{\beta} \left(\frac{1}{2}\left(e^{2} - e^{-2}\right) + \left(e^{-1} + e\right) \right)$$

$$= \frac{(1-u)\alpha_0 p_0 A n_0}{\beta} (1.27) > 0.$$

So $k = 3$ is minimum positive integer.