1 Deriving the forward Kolmogorov equation

1.1 The transition probability density

Associated with a one-dimensional diffusion X_s is the transition probability density $p(z,T|x,t)$ such that

$$\int_A p(z,T|x,t)\,dz = \mathbb{P}\left[X_T \in A | X_t = x\right] \quad \text{for} \quad T \geq t. \tag{1}$$

So, the z-integral of p over the set A gives the probability of finding X_T in the set A under the condition that it started as $X_t = x$. It is helpful to visualize X_s as the random trajectory of a particle that started at x at time t and now might be at many different locations z at time T; in this view p is the spatial probability density of the particle to be found at any particular z. Note that p depends on two sets of space–time coordinates: the initial set (x,t) and the current set (z,T). Several properties of p are easily deduced and important to us.

1. Normalization. The particle has to be somewhere at time T, so

$$\int_{-\infty}^{+\infty} p(z,T|x,t)\,dz = 1. \tag{2}$$

   Note that this involves an integral of p over its first argument.

2. Initial conditions. If $T = t$ then we know that the particle is at $z = x$, therefore

$$p(z,T|x,T) = \delta(z-x) \tag{3}$$

   must hold for the density p at every T.

3. Chapman–Kolmogorov equation. If we introduce an intermediate time s such that $T \geq s \geq t$ then a continuous process must pass through some location y at time s on its way from the initial x to the final z. The transition probability must then satisfy an obvious consistency property in the form of the Chapman–Kolmogorov equation

$$p(z,T|x,t) = \int_{-\infty}^{+\infty} p(z,T|y,s)p(y,s|x,t)\,dy. \tag{4}$$

   Here the integral over y accumulates the probabilities of the process visiting any particular such intermediate location at time s.

4. Backward Kolmogorov equation. As usual, the probability in (1) can be written as an expectation, i.e.,

$$\mathbb{P}\left[X_T \in A | X_t = x\right] = \mathbb{E}\left[\mathbf{1}_A(X_T) | X_t = x\right] \tag{5}$$

   where $\mathbf{1}_A$ is the indicator function of the set A. This makes obvious that the integral in (1) satisfies the backwards Kolmogorov equation with respect to the initial variable pair (x,t). As this holds for any set A we can let A shrink to any point z and therefore we conclude that the same is true for p, i.e.,

$$p_t + L_{x,t}\,p = 0. \tag{6}$$

Here the explicit notation $L_{x,t}$ highlights that the generator is formed from the drift and diffusion functions evaluated at (x, t). Specifically, if

$$dX_s = a(X_s, s)ds + b(X_s, s)dW_s \tag{7}$$

then

$$L_{x,t} = a(x,t)\frac{\partial}{\partial x} + \frac{b(x,t)^2}{2}\frac{\partial^2}{\partial x^2}. \tag{8}$$

1.2 The forward Kolmogorov equation

Now, the claim is that p satisfies the following equation with respect to the current variable pair (z, T):

$$p_T = L^\dagger_{z,T}\, p. \tag{9}$$

This is called the *forward Kolmogorov equation* in mathematics and the *Fokker–Planck equation* in physics. The operator $L^\dagger$ is the adjoint of the operator L with respect to the quadratic inner product, i.e.,

$$\int_{-\infty}^{+\infty} f(x)Lg(x)\,dx = \int_{-\infty}^{+\infty} g(x)L^\dagger f(x)\,dx \tag{10}$$

must hold for all suitable functions f and g. In our case these functions are twice differentiable and vanish at infinity. It then follows from (8) and integration by parts that

$$L^\dagger f = -\frac{\partial(af)}{\partial x} + \frac{1}{2}\frac{\partial^2(b^2 f)}{\partial x^2}. \tag{11}$$

Crucially, the sign of the first-order term has switched but not the sign of the second-order term. In (9) the adjoint operator appears with a and b evaluated at (z, T) and also the spatial derivatives are with respect to z. Explicitly,

$$L^\dagger_{z,T} f = -\frac{\partial(a(z,T)f)}{\partial z} + \frac{1}{2}\frac{\partial^2(b(z,T)^2 f)}{\partial z^2}. \tag{12}$$

It is noteworthy that the forward equation involves derivatives of a and b; this was *not* true for the backward equation. It is easy to check that (9) and (12) maintain the normalization (2) provided that ap and $\partial(b^2 p)/\partial z$ both vanish at infinity in z. Note that this is not guaranteed simply by the vanishing of p there! Indeed, it is possible to lose or gain total probability from spatial infinity with the forward equation (for example, think of a particle whose drift a increases greatly with z such that ap goes to a finite limit at infinity). No such problem arose with the backward equation. Historically, the forward equation was derived first and the subsequent derivation of the backward equation marked a considerable improvement of the theoretical situation because it involved neither the local nor the global behaviour of a and b, i.e., it involved neither derivatives nor decay conditions.

1.3 Formal derivation of the forward equation

We derive (9) by using the Chapman–Kolmogorov equation (4) at an intermediate time $s = T - \epsilon$ that is close to the final time T, i.e., we are interested in the limit $\epsilon \to 0$. This allows us to Taylor-expand the relevant functions to first order in ϵ; the formal nature of this derivation lies in not worrying about the smoothness of the functions that are so expanded. So we obtain

$$\begin{aligned} p(z,T|x,t) &= \int_{-\infty}^{+\infty} p(z,T|y,T-\epsilon)p(y,T-\epsilon|x,t)\,dy \qquad (13)\\ &= \int_{-\infty}^{+\infty} [p(z,T|y,T)-\epsilon p_t(z,T|y,T)][p(y,T|x,t)-\epsilon p_T(y,T|x,t)]\,dy + O(\epsilon^2)\\ &= \int_{-\infty}^{+\infty} [\delta(z-y)+\epsilon L_{y,T}\,p(z,T|y,T)][p(y,T|x,t)-\epsilon p_T(y,T|x,t)]\,dy + O(\epsilon^2)\\ &= p(z,T|x,t)-\epsilon p_T(z,T|x,t)+\epsilon\int_{-\infty}^{+\infty} p(y,T|x,t)L_{y,T}\,p(z,T|y,T)\,dy + O(\epsilon^2) \end{aligned}$$

where we used (3) and (6) and discarded terms $O(\epsilon^2)$. At $O(\epsilon)$ we obtain

$$\begin{aligned} p_T(z,T|x,t) &= \int_{-\infty}^{+\infty} p(y,T|x,t)L_{y,T}\,p(z,T|y,T)\,dy \qquad (14)\\ &= \int_{-\infty}^{+\infty} p(z,T|y,T)L^{\dagger}_{y,T}\,p(y,T|x,t)\,dy\\ &= \int_{-\infty}^{+\infty} \delta(z-y)L^{\dagger}_{y,T}\,p(y,T|x,t)\,dy = L^{\dagger}_{z,T}\,p(z,T|x,t), \end{aligned}$$

which is the forward equation (9).