Wave–mean interaction theory

Oliver Bühler
Courant Institute of Mathematical Sciences
New York University, New York, NY 10012, U.S.A.
obuhler@cims.nyu.edu

Abstract.

This is an informal account of the fluid-dynamical theory describing nonlinear interactions between small-amplitude waves and mean flows. This kind of theory receives little attention in mainstream fluid dynamics, but it has been developed greatly in atmosphere and ocean fluid dynamics. This is because of the pressing need in numerical atmosphere–ocean models to approximate the effects of unresolved small-scale waves acting on the resolved large-scale flow, which can have very important dynamical implications. Several atmosphere ocean example are discussed in these notes (in particular, see §5), but generic wave–mean interaction theory should be useful in other areas of fluid dynamics as well.

We will look at a number of examples relating to the basic problem of classical wave–mean interaction theory: finding the nonlinear $O(a^2)$ mean-flow response to $O(a)$ waves with *small amplitude* $a \ll 1$ in *simple geometry*. Small wave amplitude $a \ll 1$ means that the use of linear theory for $O(a)$ waves propagating on an $O(1)$ background flow is allowed. Simple geometry means that the flow is periodic in one spatial coordinate and that the $O(1)$ background flow does not depend on this coordinate. This allows the use of averaging over the periodic coordinate, which greatly simplifies the problem.

1 Two-dimensional incompressible homogeneous flow

This is our basic starting point. We first develop the mathematical equations for this kind of flow and then we consider waves and mean flows in it.

1.1 Mathematical equations

We work in a flat, two-dimensional domain with Cartesian coordinates $\boldsymbol{x} = (x, y)$ and velocity field $\boldsymbol{u} = (u, v)$. In the y-direction the domain is bounded at $y = 0$ and $y = D$ by solid impermeable walls such that $v = 0$ there. In the x-direction there are periodic boundary conditions such that $\boldsymbol{u}(x + L, y, t) = \boldsymbol{u}(x, y, t)$. In an atmospheric context we can think of x as the "zonal" (i.e. east–west) coordinate and of y as the "meridional" (i.e. south–north) coordinate. The period length L is then the Earth's circumference.

The flow is incompressible, which means that the velocity field is area-preserving and hence has zero divergence:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \Leftrightarrow \quad u_x + v_y = 0. \tag{1.1}$$

The velocity field induces a time derivative following the fluid flow, which is called the *material* derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla}) \quad . \tag{1.2}$$

Evaluating the material derivative of any flow variable at location $\boldsymbol{x}$ and time t gives the rate of change of this variable as experienced by the fluid particle that is at location $\boldsymbol{x}$ at the time t. The quadratic nonlinearity of the material derivative when applied to $\boldsymbol{u}$ itself gives fluid dynamics its peculiar mathematical flavour.

The momentum equations for inviscid ideal flow is provided by Newton's law as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \frac{1}{\rho}\boldsymbol{\nabla} p = 0, \tag{1.3}$$

where ρ is the fluid density per unit area and p is the pressure. We assume that the flow is homogeneous (i.e. $\boldsymbol{\nabla}\rho = 0$) and hence ρ can be absorbed in the definition of p so that we can set $\rho = 1$ throughout. The mathematical problem is completed by specifying the boundary conditions mentioned before:

$$\boldsymbol{u}(x+L,y,t) = \boldsymbol{u}(x,y,t), \qquad v = 0,\, p_y = 0 \tag{1.4}$$
$$p(x+L,y,t) = p(x,y,t) \qquad \text{at } y = \text{ and } y = D. \tag{1.5}$$

The boundary condition for the pressure at the wall follows from evaluating the y-component of (1.3) at the wall, where $v = 0$:

$$v_t + uv_x + vv_y + p_y = 0 \quad \Rightarrow \quad p_y = 0. \tag{1.6}$$

Together with (1.1) we now have three equations for the three variables u, v, p.

However, the pressure is not really an independent variable. This is a peculiarity of incompressible flow and can be seen as follows. Taking the divergence of (1.3) results in

$$\nabla^2 p = -\boldsymbol{\nabla}\cdot\boldsymbol{u}_t - \boldsymbol{\nabla}\cdot[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}] = -\boldsymbol{\nabla}\cdot[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}] \tag{1.7}$$

due to (1.1). Hence (1.7) is a Poisson equation for p in terms of the velocities, which can be solved for p. This means that $\boldsymbol{u}$ *determines* the pressure p instantaneously at any given moment in time. In other words, we can only specify initial conditions for $\boldsymbol{u}$ but not for p.

It turns out that we can eliminate p at the outset, which reduces the number of variables that need to be considered. To do this we take the curl of (1.3), which eliminates the pressure gradient term. This brings in the *vorticity* vector $\boldsymbol{\nabla}\times\boldsymbol{u}$, which is

$$\boldsymbol{\nabla}\times\boldsymbol{u} = (0, 0, v_x - u_y) \tag{1.8}$$

in two dimensions. There is only one nonzero component, which we will denote by

$$q \equiv v_x - u_y. \tag{1.9}$$

So, subtracting the y-derivative of the x-component of (1.3) from the x-derivative of its y-component leads to

$$\frac{\mathrm{D}q}{\mathrm{D}t} + q\boldsymbol{\nabla}\cdot\boldsymbol{u} + p_{yx} - p_{xy} = 0 \quad \Rightarrow \quad \frac{\mathrm{D}q}{\mathrm{D}t} = 0. \tag{1.10}$$

This means that q is advected by the flow. We say that q is a *material invariant*.

We can satisfy (1.1) exactly by introducing a *stream function* ψ such that

$$u = -\psi_y, \quad v = +\psi_x. \tag{1.11}$$

Clearly, ψ is determined only up to an arbitrary constant. The relationship between q and ψ is

$$q = v_x - u_y = \psi_{xx} + \psi_{yy} = \nabla^2\psi. \tag{1.12}$$

For given q this equation can be inverted to find ψ, though some care is needed because our channel domain is doubly connected. This means we require boundary conditions on ψ at both walls (in addition to requiring ψ to be x-periodic with period L). At the walls $\psi_x = 0$ and hence ψ is a constant there. We can set $\psi = 0$ at $y = 0$ without loss of generality due to the arbitrary constant in ψ. Then we have

$$\psi|_{y=D} = -\int_0^D u\mathrm{d}y = A(t). \tag{1.13}$$

The evolution of $A(t)$ has to be determined from (1.3). This gives A =const. and so A is determined once and for all from the initial conditions. Physically, $A \neq 0$ corresponds to a uniform, vorticity-free flow along the channel. Together, q and A determine ψ uniquely in our channel domain. With this understood, we will not consider A explicitly from now on.

So, in summary, if we take q, ψ as our basic two variables then we have the closed system of two equations

$$\nabla^2\psi = q \tag{1.14}$$

and

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \quad \Leftrightarrow \quad q_t + uq_x + vq_y = 0, \quad \Leftrightarrow \quad q_t - \psi_y q_x + \psi_x q_y = 0. \tag{1.15}$$

This is called the *vorticity–stream function* formulation of two-dimensional fluid dynamics. The material invariance of q allows many analytical simplifications, as we will see. Initial conditions can be specified either in q or in ψ; one can compute one from the other via (1.14).

1.2 Waves on shear flows

What is the linear dynamics of the system (1.14-1.15) relative to a state of rest? The first equation is already linear and the linear part of the second is simply

$$q_t = 0. \tag{1.16}$$

This means that *any* vorticity distribution is a steady solution of the linearized equations. So the linear state is infinitely degenerate and has no dynamics: all dynamics is necessarily nonlinear in the present case.[1] However, this changes when we consider the linear dynamics relative to a shear flow along the channel. Specifically, we consider an $O(1)$ shear flow

$$\boldsymbol{U}(y) = (U(y), 0) \tag{1.17}$$

with vorticity

$$Q(y) = -U_y. \tag{1.18}$$

It is easy to check that this gives a trivial steady state for all profiles $U(y)$.

We now consider linear waves on top of this shear flow. The wave amplitude is denoted by a suitable non-dimensional positive number $a \ll 1$ such that

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{U} + \boldsymbol{u}' + O(a^2) && (1.19)\\ q &= Q + q' + O(a^2) && (1.20)\\ \psi &= \Psi + \psi' + O(a^2) && (1.21) \end{aligned}$$

where $\{\boldsymbol{u}', q', \psi'\}$ are all understood to be $O(a)$ and Ψ is the stream function belonging to $\boldsymbol{U}$. So we have expanded the flow into an $O(1)$ background flow, $O(a)$ waves, and as yet unspecified further $O(a^2)$ terms. It is straightforward to show that (1.14-1.15) yield

$$\begin{aligned} \nabla^2\psi' &= q' \quad \text{and} && (1.22)\\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) q' + v' Q_y &= 0 && (1.23) \end{aligned}$$

at $O(a)$. The operator acting on q' in the second equation gives the time derivative along $O(1)$ material trajectories. We will use the short-hand

$$\mathrm{D}_t \equiv \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) \tag{1.24}$$

for it. The equation itself expresses that q' along these trajectories changes due to advection of particles in the y-direction in the presence of an $O(1)$ vorticity gradient Q_y.

The system (1.22-1.23) (and its viscous counterpart) has been studied for a long time (e.g. Drazin and Reid (1981)), mainly in order to find unstable growing modes, i.e. complex-valued modes of the form $\psi' \propto \exp(i(kx - ct))\hat{\psi}(y)$ with a nonzero imaginary part of the phase speed c. The modal approach has many practical advantages, but clarity and physical insight are not among them. We will take a different approach to this system, which lends itself to far-reaching generalizations. As a bonus, we will derive Rayleigh's theorem (one of the main results of modal theory) *en passant*.

First, we introduce the helpful linear particle displacement η' in the y-direction via

$$\mathrm{D}_t \eta' = v'. \tag{1.25}$$

[1] This peculiar fact gives rise to the popular quip that even linear fluid dynamics is more complicated than quantum mechanics! This always raises a laugh, especially among fluid dynamicists.

So the rate of change of η' along $O(1)$ material trajectories is given by v'. Of course, a complete specification of η' requires initial conditions as well. Now, combining (1.23) and (1.25) gives

$$\mathrm{D}_t\{q' + \eta' Q_y\} = 0, \tag{1.26}$$

where we have used that $\mathrm{D}_t Q_y = 0$. This means that if η' is initialized to be

$$\eta' = -\frac{q'}{Q_y} \quad \text{at } t = 0 \tag{1.27}$$

then this relation will hold at all later times as well.

We now introduce the important concept of *zonal averaging* along the channel: for any field $A(x, y, t)$ we define

$$\overline{A} \equiv \frac{1}{L}\int_{-L/2}^{+L/2} A(x+s, y, t)\mathrm{d}s \tag{1.28}$$

to be the *mean* part of A. For x-periodic A, the mean $\overline{A}$ is simply its x-average at fixed y and t, and then $\overline{A}$ does not depend on x. However, this particular definition of $\overline{A}$ has the advantage that $\overline{(x, y, t)} = (x, y, t)$, which is a useful property. Averaging is a linear operation and hence

$$\overline{A + B} = \overline{A} + \overline{B} \tag{1.29}$$

holds for all A, B. Furthermore, averaging commutes with taking partial derivatives in space and time (it also commutes with D_t though it does not in general commute with the full material derivative). Most importantly, this implies that

$$\overline{A_x} = (\overline{A})_x = 0 \tag{1.30}$$

for *all* x-periodic functions A. Clearly, averaging introduces an x-symmetry in $\overline{A}$ that need not have been present in A. We can now define the *disturbance* part of A to be

$$A' \equiv A - \overline{A} \quad \text{such that} \quad \overline{A'} = 0. \tag{1.31}$$

This is the *exact* definition of the disturbance A', i.e. this definition holds without restriction to small-amplitude disturbances.

Combining (1.28) and (1.31) we note that averaging nonlinear terms results in a mixture of mean and disturbance parts. Specifically, the mean of a quadratic term is easily shown to be

$$\overline{AB} = \overline{AB} + \overline{A'B'} \tag{1.32}$$

after using $\overline{A'\overline{B}} = \overline{A'}\,\overline{B} = 0$ etc. This is the most important nonlinear average in fluid dynamics and it is exact, i.e. not restricted to small wave amplitudes.

Let us pause for a second to consider the two distinct mathematical tools we are using: zonal averaging and small-amplitude expansions. For instance, consider the explicit vorticity small-amplitude expansion

$$q = Q + q_1 + q_2 + \ldots + q_n + O(a^{n+1}), \tag{1.33}$$

where the expansion subscripts mean that $q_n = O(a^n)$ (except for the $O(1)$ background term). Each of these terms can be decomposed into a mean and a disturbance part relative to the averaging operation:

$$q_n = \overline{q_n} + q_n'. \tag{1.34}$$

Now, the background vorticity has no disturbance part (i.e. $Q' = 0$, or $Q = \overline{Q}$) whilst the linear, first-order vorticity has no mean part: $\overline{q_1} = 0$, or $q_1 = q_1'$. Starting with q_2 all terms usually have both mean and disturbance parts. We can see now that strictly speaking q' and the other disturbance variables in (1.19) should be denoted by q_1' etc. However, we will usually suppress the cumbersome expansion subscripts when the meaning is clear from the context. From time to time we will use the expansion subscripts to highlight the nature of a particular approximation.

Returning to (1.23) now, we perform a crucial operation: multiply (1.23) by η' and then take the zonal average of that equation. This yields

$$\overline{\eta' \mathrm{D}_t q'} + \overline{\eta' v'} Q_y = 0 \quad \Rightarrow \quad \mathrm{D}_t \left(\frac{\overline{\eta' q'}}{2} \right) - \overline{q' v'} = 0 \tag{1.35}$$

after using (1.27). The first term is the time derivative of a new important variable, the zonal **pseudomomentum** per unit mass

$$\boxed{\mathsf{p}(y,t) \equiv \frac{\overline{\eta' q'}}{2} = -\frac{\overline{\eta'^2}}{2} Q_y = -\frac{\overline{q'^2}}{2Q_y}}\,. \tag{1.36}$$

The dimensions of p are that of a velocity and the second equality in (1.36) is useful because it makes clear that the sign of p is opposite to that of Q_y. The second term in (1.35) can be manipulated as follows:

$$\begin{aligned} -\overline{q'v'} &= -\overline{(v_x' - u_y')v'} = -\overline{v_x' v'} + \overline{u_y' v'} &(1.37)\\ &= -\frac{1}{2}(\overline{v'^2})_x + (\overline{u'v'})_y - \overline{u' v_y'} &(1.38)\\ &= 0 + (\overline{u'v'})_y + \overline{u' u_x'} = (\overline{u'v'})_y. &(1.39) \end{aligned}$$

This made use of the continuity equation $u_x' + v_y' = 0$, integration by parts, and the key symmetry (1.30). Noting that $\mathrm{D}_t \mathsf{p} = \mathsf{p}_t$ finally gives the pseudomomentum evolution equation

$$(\mathsf{p}_2)_t + (\overline{u_1' v_1'})_y = 0. \tag{1.40}$$

The pseudomomentum is an $O(a^2)$ quantity, but it is completely determined by the linear, $O(a)$ wave fields, i.e.

$$\mathsf{p}_2 = \frac{\overline{\eta_1' q_1'}}{2} \tag{1.41}$$

if expansion subscripts are used. Such $O(a^2)$ quantities are called *wave properties* to distinguish them from other $O(a^2)$ quantities (such as q_2) that depend on more than just the linear equations. Now, (1.40) yields a conservation law for the total, channel-integrated pseudomomentum, which is

$$\mathcal{P}(t) \equiv \int_0^L \int_0^D \mathsf{p}\, \mathrm{d}x\mathrm{d}y = L \int_0^D \mathsf{p}\, \mathrm{d}y. \tag{1.42}$$

The time derivative of $\mathcal{P}$ is

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = L \int_0^D \mathsf{p}_t\, \mathrm{d}y = -L\, \overline{u'v'}|_{y=0}^{y=D} = 0, \tag{1.43}$$

because $v' = 0$ at the channel walls. Therefore the total pseudomomentum $\mathcal{P}$ is constant, which is a new conservation law.

This conservation law leads directly to Rayleigh's famous instability criterion, namely that the existence of an unstable normal mode for a given $U(y)$ implies that the vorticity gradient $Q_y = -U_{yy}$ must change sign somewhere in the domain. This follows from

$$\mathcal{P} = L \int_0^D -\frac{\overline{\eta'^2}}{2} Q_y\, \mathrm{d}y = \text{const.} \tag{1.44}$$

and the fact that for a growing normal mode (which has constant shape in η' but grows in amplitude) the profile

$$\overline{\eta'^2}(y,t) = \exp(2\alpha t)\, \overline{\eta'^2}(y,0) \tag{1.45}$$

for some growth rate $\alpha > 0$. (For a normal mode α is proportional to the imaginary part of c.) This means that

$$\mathcal{P}(t) = \exp(2\alpha t)\, \mathcal{P}(0), \tag{1.46}$$

which is compatible with $\mathcal{P} = \text{const.}$ only if $\mathcal{P} = 0$. Therefore for a growing normal mode $\mathcal{P} = 0$ and (1.44) then implies that Q_y must change its sign somewhere in the domain. This is Rayleigh's famous theorem.

It can be noted that although a sign-definite Q_y therefore implies stability of normal modes, it does not preclude the localized transient growth of non-normal modes (e.g. Haynes (1987)).

1.3 Mean-flow response

We now consider the leading-order mean-flow response to the waves. This response occurs due to the quadratic nonlinearity of the equations, which produces a leading-order response at $O(a^2)$. However, as was first shown by Reynolds, in simple geometry it is often possible to write down a mean-flow equation that holds at finite amplitude. We will do this first and then specialize to $O(a^2)$. Substituting the exact decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}'$ into the continuity equation and averaging yields

$$\overline{u_x} + \overline{v_y} = 0 \quad \Rightarrow \quad \overline{v} = \text{const.} = 0 \tag{1.47}$$

The last equality comes from the fact that at the walls $v = 0$. So we see that $\overline{v} = 0$ everywhere. Now, substituting the decomposition in the x-component of the momentum equation (1.3) and averaging yields

$$\overline{u}_t = -(\overline{u'v'})_y \tag{1.48}$$

after some manipulations using $\overline{v} = 0$ and $u'_x + v'_y = 0$. This equation is exact, i.e. it does not depend on small wave amplitudes. It expresses the fact that the zonal mean flow accelerates in response to the convergence of the meridional flux of zonal momentum $\overline{u'v'}$. In regions of constant (but not necessarily zero) flux there is no acceleration.

How does this fit together with our wave solution at $O(a)$? Clearly, the momentum flux $\overline{u'v'}$ is a wave property in that at $O(a^2)$ it can be evaluated from the linear, $O(a)$ wave solution as $\overline{u'_1 v'_1}$. This means we can combine (1.40), which is valid only at $O(a^2)$, with (1.48) to obtain

$$(\overline{u}_2)_t = (\mathsf{p}_2)_t \;\Rightarrow\; \boxed{\overline{u}_t = \mathsf{p}_t + O(a^3)}\,. \tag{1.49}$$

This innocuous-looking equation is our main result: *to $O(a^2)$ the zonal mean flow acceleration equals the pseudomomentum growth.*

Together with the sign of p that can be read off from (1.36), we see that a growing wave leads to *positive* zonal acceleration where Q_y is negative, i.e. where $U_{yy} > 0$. Indeed, we can re-write (1.49) as

$$\overline{u}_t = \left(\frac{\overline{\eta'^2}}{2}\right)_t \overline{u}_{yy} + O(a^3), \tag{1.50}$$

from which it is easy to see qualitatively that for a growing mode the mean shear is eroded as the mode grows. This is a basis for the sometimes-observed nonlinear growth saturation of marginally unstable modes: the mode grows until the induced mean-flow response shuts off the instability mechanism. At this point the growth ceases and the mode saturates.

2 The beta plane

The waves on shear flows considered above gave us a first example of wave–mean interaction theory. However, it is hard to be more specific without writing down actual wave solutions and those are complicated because $U(y)$ must be non-constant to get waves in the first place. Also, most profiles $U(y)$ have unstable modes, and these will quickly render invalid our linear, $O(a)$ theory for the waves.

For this reason we now turn to fluid systems with a simpler background state that is still sufficient to support waves. The particular example we are going to study is the so-called mid-latitude β-plane, which is a local tangent-plane approximation to our rotating gravitating planet Earth (Pedlosky (1987)).

2.1 The beta-effect

You will know that the Earth spins with frequency Ω around its pole-to-pole axis. If we denote the rotation vector along this axis by $\boldsymbol{\Omega}$ then we know that the momentum equation relative to the spinning Earth must be augmented by suitable Coriolis and centrifugal forces based on $\boldsymbol{\Omega}$. The latter can be absorbed in the gravitational potential and need not concern us any further. The former means that a term $\boldsymbol{f} \times \boldsymbol{u}$ must be added to the material derivative, where the Coriolis vector

$$\boldsymbol{f} \equiv 2\boldsymbol{\Omega}. \tag{2.1}$$

However, it turns out that because of the strong gravitational field of our planet the large-scale motion is mostly "horizontal", i.e. along two-dimensional, nearly spherical stratification surfaces. What is relevant to this horizontal flow is not the full Coriolis vector but only its projection onto the local "upward" direction denoted by the unit vector $\widehat{\boldsymbol{z}}$. That is, if we introduce the usual latitude θ, then we can define the Coriolis *parameter*

$$f \equiv \boldsymbol{f} \cdot \widehat{\boldsymbol{z}} = 2\Omega \sin(\theta). \tag{2.2}$$

This parameter increases monotonically with latitude θ, is zero at the equator, negative in the southern hemisphere, and positive in the northern hemisphere. If we look at a local tangent plane around latitude θ_0 we can introduce the Cartesian coordinates

$$x \qquad \text{Zonal: west-to-east} \tag{2.3}$$
$$y \qquad \text{Meridional: south-to-north} \tag{2.4}$$
$$z \qquad \text{Vertical: low-to-high,} \tag{2.5}$$

and the governing two-dimensional equations are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + \frac{1}{\rho}p_x = 0 \tag{2.6}$$
$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu + \frac{1}{\rho}p_y = 0 \tag{2.7}$$
$$u_x + v_y = 0 \tag{2.8}$$

where all fields depend on (x, y, t), as before. The origin of y has been chosen at θ_0 such that $y = R(\theta - \theta_0)$, where $R \approx 6300$km is the Earth's radius. Strictly speaking, the tangent-plane approximation is only valid in a range of x and y that is small compared to R, so that the spherical geometry can be neglected. This is often relaxed for the zonal coordinate x, which is usually allowed to go once around the globe such that periodic boundary conditions in x make sense. We continue having solid walls at $y = 0$ and $y = D$. Now, the parameter f is given by a local Taylor expansion as

$$f = 2\Omega \sin(\theta_0) + 2\Omega \cos(\theta_0)(\theta - \theta_0) \equiv f_0 + \beta y, \tag{2.9}$$

which introduces the important constant

$$\beta = \frac{2\Omega}{R} \cos(\theta_0). \tag{2.10}$$

So $\beta > 0$ is the rate of change of f per unit northward distance. This will have a profound dynamical effect, as we shall see.

Can we find a vorticity stream function formulation of (2.6-2.8)? The answer is yes, provided we use the *absolute* vorticity

$$q = v_x - u_y + f_0 + \beta y. \tag{2.11}$$

This is the normal fluid vorticity as seen by a non-rotating, inertial observer: it is the relative vorticity $v_x - u_y$ plus the vorticity due to the rotating frame. It is easy to show that we again get

$$\frac{\mathrm{D}q}{\mathrm{D}t} + q\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \Rightarrow \quad \frac{\mathrm{D}q}{\mathrm{D}t} = 0. \tag{2.12}$$

This means that q is a material invariant, just as before. However, the stream function now satisfies

$$\nabla^2\psi + f_0 + \beta y = q, \tag{2.13}$$

which is the new equation. We can note as an aside that if $\beta = 0$ then the constant f_0 can be absorbed in the definition of q, because $q - f_0$ still satisfies (2.12). This means that the $\beta = 0$ dynamics is *exactly* the same as in the non-rotating system studied before! This peculiar fact implies that frame rotation in a two-dimensional incompressible flow is not noticeable.[2]

Let us now consider $O(a)$ waves again. The $O(1)$ background vorticity now satisfies

$$Q = f_0 + \beta y - U_y \quad \Rightarrow \quad Q_y = \beta - U_{yy} \tag{2.14}$$

and this is the only change to the $O(a)$ equations (1.22-1.23). Rayleigh's theorem now states that $\beta - U_{yy}$ must change sign in order to have unstable normal modes. For instance, this means that $|U_{yy}| \leq \beta$ implies stability, i.e. $\beta \neq 0$ has a stabilizing influence.

2.2 Rossby waves

We will now set U =const. such that $Q_y = \beta$. Searching for plane-wave solutions $\psi' = \hat{\psi}\exp(i(kx+ly-\omega t))$ to (1.22-1.23) then yields the Rossby-wave dispersion relation

$$\omega = Uk - \frac{\beta k}{k^2 + l^2} = Uk - \frac{\beta k}{\kappa^2}, \tag{2.15}$$

where κ is the magnitude of the wavenumber vector $\boldsymbol{k} = (k, l)$. The absolute frequency ω is the sum of the Doppler-shifting term Uk and the *intrinsic* frequency

$$\hat{\omega} = -\frac{\beta k}{\kappa^2} \quad \text{such that} \quad \omega = Uk + \hat{\omega}. \tag{2.16}$$

The intrinsic frequency captures the wave dynamics relative to the background flow U whereas the absolute frequency ω gives the frequency as seen by an observer fixed on the ground.

We can see by inspection that large-scale Rossby waves (i.e. small κ) have higher intrinsic frequencies than small-scale Rossby waves. The absolute speed of phase propagation for a plane wave at *fixed* y is

$$c \equiv \frac{\omega}{k} = U + \frac{\hat{\omega}}{k} = U - \frac{\beta}{\kappa^2}. \tag{2.17}$$

We can introduce the intrinsic phase speed $\hat{c}$ at fixed y such that $c = U + \hat{c}$:

$$\hat{c} = \frac{\hat{\omega}}{k} = -\frac{\beta}{\kappa^2}. \tag{2.18}$$

[2] This is not exactly true in the case of unbounded flow: with background rotation there can be pressure-less uniform inertial oscillations $(u, v) = (\cos(f_0 t), -\sin(f_0 t))$, which depend on f_0. You may want to ponder for a second where the loophole in the mathematical argument is that allows this to happen! However, the statement is exact for the bounded channel geometry.

This speed is always negative, i.e. the phase always travels westward relative to the background flow.

The continuity equation for plane waves is $ku' + lv' = 0$, i.e. $\boldsymbol{u}'$ and $\boldsymbol{k}$ are at right angles to each other. This means that Rossby waves are transverse waves, with velocities parallel to lines of constant phase.[3] The absolute Rossby-wave *group velocity* is

$$u_g \equiv \frac{\partial \omega}{\partial k} \quad = \quad U + \beta \frac{k^2 - l^2}{\kappa^4} \tag{2.19}$$

$$v_g \equiv \frac{\partial \omega}{\partial l} \quad = \quad \beta \frac{2kl}{\kappa^4}, \tag{2.20}$$

where the partial derivatives are understood to take (k, l) as independent variables. We recall that, in general, the group velocity gives the speed of propagation for a slowly varying *wavepacket* containing many wave crests and troughs. For example, let the initial condition be

$$\psi'(x, y, 0) = a \exp(-(\boldsymbol{x}\kappa\mu)^2) \, \exp(i(\boldsymbol{k} \cdot \boldsymbol{x})) \tag{2.21}$$

where $\mu \ll 1$ is a small parameter measuring the scale separation between the wavelength $2\pi/\kappa$ and the scale $1/(\mu\kappa)$ of the slowly varying Gaussian envelope of the wavepacket (the Gaussian is not essential; other smooth envelope functions work as well). Then linear theory predicts that the solution for small enough later times $t \leq O(\mu^{-1})$ is

$$\psi'(x, y, t) = a \exp(-((\boldsymbol{x} - \boldsymbol{c}_g t)\kappa\mu)^2) \, \exp(i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)) \tag{2.22}$$

where $\boldsymbol{c}_g = (u_g, v_g)$ is given by (2.19-2.20). In other words, for $t \leq O(\mu^{-1})$ the wavepacket simply moves with the group velocity. This is long enough to move the envelope a distance comparable to its size. Over longer times, e.g. $t \leq O(\mu^{-2})$, dispersion effects for the wavepacket envelope need to be taken into account, as is done in the (nonlinear) Schrödinger equation for modulated wavepackets, for instance.

We note in passing that a *vectorial* phase velocity $\boldsymbol{c}_p$ can be defined that gives the speed of propagation of individual wave crests or troughs in the xy-plane. It is natural to let $\boldsymbol{c}_p$ be parallel to $\boldsymbol{k}$, which means that the vector

$$\boldsymbol{c}_p = (u_p, v_p) \equiv \frac{\omega}{\kappa^2} \, \boldsymbol{k} \tag{2.23}$$

is the desired definition. In general, phase and group velocity can have different magnitude as well as direction, which is important when interpreting observations. Finally, we note that the phase velocity c in (2.17) agrees with u_p only if $l = 0$. This is because c is defined as the phase speed at constant y, which as a direction is only parallel to $\boldsymbol{k}$ if $l = 0$.

Indeed, for the special case of $l = 0$ and $U = 0$, the lines of constant phase are $y =$const., the particle velocity satisfies $u' = 0$, the meridional group velocity $v_g = 0$,

[3] It is easy to show that this implies that a *single* plane Rossby wave is also an exact solution of the nonlinear equations. This is because the only nonlinearity enters through the nonlinear part of the material derivative, which turns out to be exactly zero for transverse waves.

and the zonal group velocity is

$$u_g = \frac{\beta}{k^2} = -c = -u_p. \tag{2.24}$$

So the group velocity for $l = 0$ is always *eastward* and is opposite to the zonal phase velocity! This enjoyable fact is one of the reasons why atmospheric weather systems usually travel from west to east.

Finally, the *intrinsic* group velocities are defined by replacing ω by $\hat{\omega}$ in (2.19) and (2.23), for example:

$$\hat{u}_g \equiv \frac{\partial \hat{\omega}}{\partial k} = \beta \frac{k^2 - l^2}{\kappa^4} \tag{2.25}$$

$$\hat{v}_g \equiv \frac{\partial \hat{\omega}}{\partial l} = \beta \frac{2kl}{\kappa^4} = v_g. \tag{2.26}$$

We have $(u_g, v_g) = (U, 0) + (\hat{u}_g, \hat{v}_g)$ overall.

2.3 Momentum flux and pseudomomentum

Consider now the wave-induced momentum flux $\overline{u'v'}$, i.e. the meridional flux of x-momentum per unit length across lines of constant y. Using $u' = -v'l/k$ from continuity we get

$$\overline{u'v'} = -\frac{l}{k}\,\overline{v'^2} = -\frac{kl}{k^2}\,\overline{v'^2}, \tag{2.27}$$

which makes it obvious that

$$-\mathrm{sgn}(\overline{u'v'}) = \mathrm{sgn}(kl) = \mathrm{sgn}(v_g), \tag{2.28}$$

provided that $\beta > 0$. This means that a northward-moving wavepacket (i.e. $v_g > 0$) has *negative* momentum flux, and vice versa. Also, (2.28) implies that the lines of constant phase (at fixed time) for a northward moving Rossby wave are slanted from south–east to north–west, and vice versa for a southward moving wave. This can be used to diagnose wave direction from single-time snapshots.

The Rossby-wave pseudomomentum is given by

$$\mathsf{p} = \frac{\overline{\eta' q'}}{2} = -\frac{Q_y}{2}\,\overline{\eta'^2} = -\frac{\beta}{2}\,\overline{\eta'^2} \leq 0. \tag{2.29}$$

It is always negative for Rossby waves, which is why in meteorology one sometimes uses the opposite sign convention for p. If we define the wave disturbance energy as

$$E \equiv \frac{1}{2}\,(\overline{u'^2} + \overline{v'^2}) \tag{2.30}$$

and make use of

$$\hat{\omega}^2\,\overline{\eta'^2} = \overline{v'^2}, \tag{2.31}$$

which follows directly from (1.25) and $\mathrm{D}_t = -i\hat{\omega}$ for plane waves, then it is straightforward to show that

$$\boxed{\mathsf{p} = \frac{k}{\hat{\omega}}\,E}\,. \tag{2.32}$$

This link between p and E for propagating plane waves will be seen to hold much more generally. Among other things, it implies that the sign of p is equal to the sign of the intrinsic zonal phase speed $\hat{c} = \hat{\omega}/k$. It is also easy to show that for a plane wave the momentum flux

$$\overline{u'v'} = \mathsf{p} v_g \tag{2.33}$$

and therefore

$$\mathsf{p}_t + (\mathsf{p} v_g)_y = 0 \tag{2.34}$$

holds for slowly varying wavetrains.

Let us now consider the Rossby waves generated by uniform flow U over a small-amplitude undulating sidewall at $y = 0$. That is, the southern channel boundary is undulated at $O(a)$ according to

$$h(x) = h_0 \cos(kx) \tag{2.35}$$

where $h_0 = O(a)$ and k is the wavenumber of the undulations. (Other shapes of $h(x)$ can be built up by linear superposition of such Fourier modes.) The linear kinematic boundary condition at $y = 0$ becomes

$$v'(x, 0, t) = U h_x = U k h_0 \sin(kx) \quad \text{or more simply} \quad \eta'(x, 0, t) = h_0 \cos(kx) \tag{2.36}$$

after using (1.25). Now, k is fixed by the undulations and we see from (2.36) that the disturbance is time-independent when observed from the ground. This means that the forced Rossby waves have *absolute* frequency $\omega = 0$ and hence

$$\omega = Uk + \hat{\omega} = 0 \;\Rightarrow\; \hat{\omega} = -Uk \;\Rightarrow\; U = \frac{\beta}{k^2 + l^2} \;\Rightarrow\; l^2 = \frac{\beta}{U} - k^2. \tag{2.37}$$

This fixes l^2 in terms of k and U. To have propagating waves requires $l^2 > 0$ and we see that this is only possible if U satisfies the so-called Charney–Drazin conditions

$$0 < U < \frac{\beta}{k^2}. \tag{2.38}$$

If U falls outside this window then $l^2 < 0$ and the waves are trapped, or evanescent, in the y-direction. In other words, in order to excite propagating Rossby waves the background wind must be eastward and not too fast.

It remains to pick the sign of l for propagating waves. If $k > 0$ then we must pick $l > 0$ in order to satisfy the radiation condition $v_g > 0$, i.e. in order to have waves propagating away from the wave maker at $y = 0$. If $k < 0$ the same argument gives $l < 0$. Therefore we have

$$l = \mathrm{sgn}(k)\sqrt{\frac{\beta}{U} - k^2} \tag{2.39}$$

and the wave field is $\eta' = h_0 \cos(kx + ly)$. The corresponding pseudomomentum from (2.29) is

$$\mathsf{p} = -\frac{\beta}{4}\, h_0^2. \tag{2.40}$$

Interestingly, p does not depend explicitly on either U or k. However, if (2.38) is not satisfied then $\mathsf{p} = 0$. So there is an implicit dependence on Uk^2 here.

Now, we imagine that the wall undulations are growing smoothly from zero to their final amplitude h_0 over some time interval that is long compared to the intrinsic period of the wave. Then there will be a smooth transition zone in space that separates a far-field region without waves from a region with waves. This transition zone, or wave front, will travel with speed $v_g > 0$, which can be computed from (2.20). (For simplicity, we will not consider what happens when the waves reach the other channel wall at $y = D$.) So, below the wave front (2.40) will be valid whilst above it $\mathsf{p} = 0$.

To get to the mean-flow response we note that (1.48) is still valid on the β-plane because $\overline{fu} = f\overline{v} = 0$ and hence we still have

$$\overline{u}_t = -(\overline{u'v'})_y = \mathsf{p}_t + O(a^3). \tag{2.41}$$

Integrating in time we get

$$\overline{u} = U + \mathsf{p} + O(a^3). \tag{2.42}$$

We see that mean-flow acceleration is confined to the wave front, where p changes from zero to $-\beta h_0^2/4$. In other words, the mean flow is *decelerated* as the Rossby waves arrive. Once the wave field is steady there is no further mean-flow change. Furthermore, if we imagine the wall undulations to return to zero again then we see that the mean flow is now *accelerated* back to its original value U. So the waves did not create a lasting, irreversible change in the mean flow. We see from this example that

- only transient waves (i.e. $\mathsf{p}_t \neq 0$) can accelerate the mean flow;
- the mean-flow changes are uniformly bounded (in time) at $O(a^2)$;
- and the mean-flow changes are reversible.

The first point (and its dissipative generalization considered below) is often called a "non-acceleration theorem" in meteorology. The second and third points show that these wave-induced mean-flow changes can be ignored if the amplitude a is small enough. To create lasting mean-flow changes in simple geometry we must allow for wave dissipation.

2.4 Forcing and dissipation

We can include forcing and dissipation by adding a body force $\boldsymbol{F} = (F, G)$ to the right-hand side of the momentum equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\widehat{\boldsymbol{z}} \times \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{\nabla} p = \boldsymbol{F}. \tag{2.43}$$

For instance, the usual Navier–Stokes equations correspond to $\boldsymbol{F} = \nu\nabla^2\boldsymbol{u}$, where $\nu > 0$ is the kinematic viscosity. The forced vorticity equation is

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \widehat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} = G_x - F_y \tag{2.44}$$

and q is not a material invariant any more. We will assume that there is no $O(1)$ part of $\boldsymbol{F}$, i.e. the background flow is unforced. The expansion of F in terms of wave amplitude a therefore has the form

$$\boldsymbol{F} = \boldsymbol{F}' + O(a^2) \tag{2.45}$$

where $\boldsymbol{F}' = O(a)$. The $O(a)$ vorticity equation is

$$\mathrm{D}_t q' + v' Q_y = \widehat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F}' \tag{2.46}$$

but the relation $q' = -Q_y \eta'$ does not hold any more, because of the friction. The most robust definition of pseudomomentum that survives introducing friction is

$$\mathsf{p} \equiv -\frac{\overline{q'^2}}{2Q_y}, \tag{2.47}$$

which equals (1.36) in the absence of friction. The pseudomomentum equation is

$$\mathsf{p}_t + (\overline{u'v'})_y = -\frac{1}{Q_y}\overline{q' \boldsymbol{\nabla} \times \boldsymbol{F}'} \equiv \mathcal{F} \tag{2.48}$$

after introducing the useful short-hand $\mathcal{F}$. Clearly, the total pseudomomentum in the channel is not conserved any more:

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = L \int_0^D \mathcal{F}\,\mathrm{d}y. \tag{2.49}$$

For Rossby waves with constant U the pseudomomentum is always negative, so we see that $\mathcal{F} < 0$ corresponds to wave forcing whereas $\mathcal{F} > 0$ corresponds to wave dissipation.

The forced zonal mean-flow equation at $O(a^2)$ is

$$\overline{u}_t = -(\overline{u'v'})_y + \overline{F} = \mathsf{p}_t - \mathcal{F} + \overline{F} + O(a^3), \tag{2.50}$$

where $\overline{F}$ is evaluated correct to $O(a^2)$.

We now make an important assumption: we assume that $\overline{F}$ can be neglected in (2.50). We will see later that this assumption is linked to the momentum-conservation character of $\boldsymbol{F}$: if the body force does not add any mean zonal momentum to the system then $\overline{F}$ will indeed be negligible in (2.50). (This is physically reasonable in many circumstances; however, it can be difficult to satisfy in numerical simulations. For instance, simple Rayleigh damping $\boldsymbol{F} \propto -(\boldsymbol{u} - \boldsymbol{U})$ will violate this assumption because then $\overline{F}$ turns out to be comparable to $\mathcal{F}$ (Bühler (2000)). With this assumption we obtain the simple result

$$\boxed{\overline{u}_t = \mathsf{p}_t - \mathcal{F} + O(a^3)}, \tag{2.51}$$

which shows that we now have two ways to force the mean flow: either through wave transience or through wave forcing/dissipation.

Let us reconsider the Rossby-wave problem with the undulating wall, but this time with dissipation. We shall assume that

$$\boldsymbol{\nabla} \times \boldsymbol{F}' = -\frac{\alpha}{2} q' \tag{2.52}$$

to $O(a)$ with a constant damping rate $\alpha > 0$. This means that

$$\mathcal{F} = -\frac{\overline{q' \boldsymbol{\nabla} \times \boldsymbol{F}'}}{Q_y} = -\alpha\, \mathsf{p}. \tag{2.53}$$

If the damping rate α is small enough then we are dealing with a slowly varying wavetrain whose structure can be computed by using the plane-wave result (2.33) in (2.48):

$$(\overline{u'v'})_y = (\mathsf{p} v_g)_y = -\alpha \mathsf{p}. \tag{2.54}$$

This is the weak dissipation approximation, which can be formally justified as a first-order approximation in $|\alpha/\hat{\omega}|$. The group velocity v_g is constant in (2.54) and hence we obtain

$$\mathsf{p}(y) = \mathsf{p}(0) \exp\left(-\frac{\alpha}{v_g} y\right) \quad \text{with} \quad \mathsf{p}(0) = -\frac{\beta}{4} h_0^2. \tag{2.55}$$

So the wavetrain amplitude decays exponentially in y with spatial decay rate α/v_g. The mean-flow response to the steady wavetrain is given explicitly by

$$\overline{u}_t = -\mathcal{F} = +\alpha \mathsf{p} = -\alpha \frac{\beta}{4} h_0^2 \exp\left(-\frac{\alpha}{v_g} y\right). \tag{2.56}$$

This shows that $\overline{u}_t < 0$ everywhere so the flow decelerates, just as in the spin-up phase of the transient waves. However, unlike the mean-flow response to the transient waves, the dissipative mean-flow changes *persist* after the waves are switched off: dissipative mean-flow changes are irreversible. Connected to this fact is another, even more important fact: the mean-flow change grows without bound in time. Indeed, integrating (2.56) in time gives approximately

$$\overline{u} = U + (1 + \alpha t)\mathsf{p}, \tag{2.57}$$

where the transient non-dissipative acceleration stemming from $\mathsf{p}_t \neq 0$ has been included. The last term gives secular growth, which will invalidate the assumed scaling when $t = O(a^{-2})$. So even small-amplitude waves can give rise to $O(1)$ mean-flow changes after a long time.

Of course, following the solution for such a long time requires a singular perturbation analysis, which is more work than we have done here (our expansion is valid for times $t = O(1)$). However, the conclusion remains valid: wave dissipation can lead to persistent, irreversible mean-flow changes that can grow to $O(1)$ after times $t = O(a^{-2})$.

Let us now consider a variant of this problem with interior wave forcing due to $\boldsymbol{\nabla} \times \boldsymbol{F}'$, say in a region centred around the middle of the channel. The waves propagate away northward and southward and are then subject to dissipation say at a certain distance away from the forcing region. For simplicity, we neglect the influence of the channel walls and we set $U = 0$. By construction we have $\mathcal{F} < 0$ in the wave forcing region near $y = D/2$ and $\mathcal{F} > 0$ is the dissipation regions above and below. If we assume that the wave field is steady on average (i.e. $\mathsf{p}_t = 0$) then the mean-flow response is $\overline{u}_t < 0$ in the dissipation regions, as before. However, we also obtain that $\overline{u}_t > 0$ in the forcing region! This means the mean flow is accelerated eastward in the forcing region, even though we have assumed that $\overline{F} = 0$ and hence there is no net momentum input by the force. How this is possible becomes clear once we consider the wave-induced momentum flux (2.33) above and below the forcing region. Integrate the first equality in (2.50) (with $\overline{F} = 0$) in y to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{y_1}^{y_2} \overline{u} \, \mathrm{d}y = -(\overline{u'v'}|_{y_2} - \overline{u'v'}|_{y_1}), \tag{2.58}$$

where $y1$ and $y2$ are southward and northward of the forcing region, respectively. It is easiest to think of these locations to lie in between the forcing region and the dissipation regions. To the north $v_g > 0$ and hence $\overline{u'v'}|_{y_2} < 0$. To the south $v_g < 0$ and hence $\overline{u'v'}|_{y_1} > 0$. This means that there is a net influx of zonal momentum into the region between y_1 and y_2, and (2.58) quantifies how this leads to positive, eastward acceleration.

By momentum conservation the eastward momentum gained by the forcing region is exactly compensated by the westward momentum gained by the dissipation regions. This means that the Rossby waves mediated a **wave-induced non-local momentum transfer** between these regions. The stirring by the body force allowed this momentum transfer to happen, but the forcing itself did not change the momentum in the channel.

It is important to note the signs here: the *arrival and/or dissipation* of Rossby waves accelerates the mean flow westwards, whereas the *departure and/or generation* of Rossby waves leads to eastward acceleration! Such wave-induced momentum transfer between wave forcing and dissipation regions is a key concept in wave–mean interaction theory and atmosphere ocean fluid dynamics. For instance, the Rossby-wave generation caused by large-scale baroclinic instability at mid-latitudes is well known to contribute to the eastward acceleration in the upper troposphere (e.g. Held (2000)).

2.5 Critical layers

We now return to the case of variable $U(y)$. Among other things, understanding this more general case will allow us to understand feedback cycles between wave-induced mean-flow changes and the waves themselves.

To fix ideas we consider again non-dissipative Rossby waves forced by flow past an undulating wall at $y = 0$ (cf. (2.35)), but this time allow for $U(y)$. Specifically, we want to consider the case where $U = 0$ at some $y = y_c$, which is called the *critical line*. The standard approach is to use a normal-mode Ansatz for the steady disturbance stream function

$$\psi' = \hat{\psi}(y) \exp(ikx) \quad \text{such that} \quad q' = \left(\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}y^2} - k^2\hat{\psi} \right) \exp(ikx) \tag{2.59}$$

with a modal structure $\hat{\psi}(y)$ to be determined from the relevant $O(a)$ vorticity equation

$$\begin{aligned} U(y)q'_x + \psi'_x(\beta - U_{yy}) &= 0, \text{ or} && (2.60) \\ U(y)\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}y^2} + (\beta - U_{yy} - k^2U)\hat{\psi} &= 0. && (2.61) \end{aligned}$$

This reduces to (2.37) if $U_y = 0$. We see that at y_c the coefficient of the highest derivative in this ODE vanishes, i.e. y_c is a singular point of the equation. Singular points are standard for many linear equations arising in physics, such as Bessel's equation for instance. The standard way to continue is to investigate the power series expansion of U near the singular point in order to find the local behaviour of $\hat{\psi}$ there, which usually involves a logarithmic term centred at y_c. However, we know that the $O(a)$ equations are only a small-amplitude approximation to the nonlinear fluids equations. Linear critical layer theory might be a poor guide to real fluid dynamics! Indeed, most linear (or weakly nonlinear) critical layer theory is wholly unrealistic, as we shall see that the flow at a critical layer is almost always *strongly* nonlinear. This is an important (and not very widely appreciated) fact for practical purposes, which puts a lot of theoretical critical layer research into perspective.

To see the breakdown of linear theory at the critical layer we shall use a ray-tracing (or JWKB) approximation valid for slowly varying waves on a slowly varying shear flow $U(y)$ (e.g. Lighthill (1978)). This is a very powerful general tool in wave theory. It amounts to approximating the wave structure everywhere by a *local* plane wave determined by the local value of $U(y)$. This means making the ray-tracing Ansatz

$$\psi' = \tilde{\psi}(y)\exp(ikx)\exp\left(i\int_0^y l(\bar{y})\mathrm{d}\bar{y}\right) \quad \text{with} \tag{2.62}$$

$$l(y) = +\sqrt{\frac{\beta}{U(y)} - k^2} \tag{2.63}$$

for the steady stream function. So the local meridional wavenumber $l(y)$ itself varies with y in according to the plane-wave formula (2.39). The zonal wavenumber $k > 0$ does not change. The amplitude $\tilde{\psi}$ is undetermined at this stage. For this ray-tracing approach to make sense we need $\beta \gg |U_{yy}|$ and also a condition such as

$$\left(\frac{U_y}{U}\right)^2 \ll l^2 \tag{2.64}$$

that quantifies that $U(y)$ is slowly varying over a meridional wavelength $2\pi/l$. (The exact asymptotic conditions for ray-tracing to be accurate are more complicated than (2.64), but (2.64) is a good guide in practice.)

Now, we see immediately from (2.63) that at the critical line y_c the local wavenumber $l \to \infty$, so the wave structure becomes non-smooth near y_c. Specifically, if U is proportional to $y_c - y$ just below the critical line then

$$l \propto (y_c - y)^{-1/2} \tag{2.65}$$

there. We can consider the motion of a wave front moving with v_g in order to find out how long it takes for a wave front to reach the critical line y_c. Using ray-tracing we know that v_g will depend on y through $l(y)$ such that

$$v_g = \beta\frac{2kl}{(k^2+l^2)^2} \propto \frac{1}{l^3} \propto (y_c - y)^{3/2} \tag{2.66}$$

just below y_c. This means that $v_g \to 0$ at y_c and so the wave front slows down. The travel time near the critical line can be easily estimated from

$$\int \mathrm{d}t = \int \frac{\mathrm{d}y}{v_g} \propto \int \frac{\mathrm{d}y}{(y_c - y)^{3/2}} \propto (y_c - y)^{-1/2} + \text{const.}, \tag{2.67}$$

which shows that it takes infinite time for the wave front to reach the critical line $y = y_c$.

This strongly suggests that the wavetrain accumulates just below y_c, in a region that has been named the critical *layer*. Now, the wave amplitude (i.e. $|\tilde{\psi}|$) for a slowly varying wavetrain is determined by the steady pseudomomentum conservation law

$$(\overline{u'v'})_y = (\mathsf{p} v_g)_y = 0 \Rightarrow \mathsf{p}(y) = \mathsf{p}(0)\frac{v_g(0)}{v_g(y)}, \tag{2.68}$$

which shows that p also becomes unbounded. A good physical definition for the local wave amplitude is the non-dimensional overturning amplitude

$$a \equiv \max |\eta'_y|, \tag{2.69}$$

where the maximum is taking over the x variable. This is because the undulating vorticity contours are given by

$$\begin{aligned} q &= Q + q' + O(a^2) && (2.70)\\ &= f_0 + \beta y - \beta \eta' + O(a^2) && (2.71)\\ q_y &= \beta(1 - \eta'_y) + O(a^2), && (2.72) \end{aligned}$$

which shows that overturning (i.e. $q_y = 0$) occurs first where η'_y exceeds unity. Such overturning means that $a \approx 1$ and linear theory must fail. With this amplitude definition we have the ray-tracing relations

$$a^2 = 2\,\overline{\eta'^2_y} = 2l^2\,\overline{\eta'^2} = -\frac{4l^2}{\beta}\,\mathsf{p}. \tag{2.73}$$

In the critical layer this implies

$$a^2 \propto \frac{l^2}{v_g} \propto l^5 \propto U^{-5/2} \Rightarrow a \propto U^{-5/4} \propto (y_c - y)^{-5/4}, \tag{2.74}$$

and hence waves must *break nonlinearly* in the critical layer just below y_c.

In summary, we have shown that linear theory must break down in the critical layer just below the critical line y_c. Ray-tracing itself also breaks down there, as can be checked from (2.64), which is violated in the critical layer. Furthermore, even the assumption of a steady state has broken down, although we could only formulate this assumption within linear theory. So, linear theory has predicted its own comprehensive breakdown in the critical layer. This is actually a good, strong scientific result, much preferable to the alternative of having a theory that continues to work fine even though it has ceased to be valid!

What happens at a true critical layer can be elucidated by more sophisticated theory as well as by nonlinear numerical simulations. The picture that emerges is roughly as follows (e.g Haynes (2003), Haynes (1985)).

- As the wave gets close to the critical line the overturning amplitude grows and eventually closed stream lines are formed in the critical layer, giving the flow a characteristic "cat eyes" appearance;
- the flow becomes unstable and breaks down into strongly nonlinear two-dimensional turbulence;
- the turbulence mixes the fluid and the materially advected vorticity distribution becomes approximately uniform, or *homogenized* in the critical layer (eventually, this is due to viscous diffusion acting on small-scale vorticity generated by the turbulence).

Surprisingly, we can compute the mean-flow changes in the critical layer based only on the homogenization of vorticity. If $\boldsymbol{\nabla} q = 0$ then

$$\overline{q_y} = 0 = \beta - \overline{u}_{yy} \Rightarrow \overline{u}(y) = A + By + \frac{\beta y^2}{2} \tag{2.75}$$

must hold in the critical layer for some A, B. If $\overline{u}$ should match $U(y)$ outside the critical layer then it is not hard to see that the mean-flow change $\overline{u} - U$ is a quadratic centred at the middle of the critical layer and zero at the edges of this layer. The sign of $\overline{u} - U$ is always negative and hence the net mean-flow acceleration has been westward, the same as in the case of wave dissipation. So we can see that wave breaking is similar to wave dissipation in some respects: it destroys the wave and it leads to westward mean-flow acceleration. However, once q has been homogenized in the critical layer then there is no further mean-flow acceleration possible inside it. This is different from straightforward wave dissipation and this fact goes hand-in-hand with the finding that mature critical layers tend to reflect rather than absorb further incoming Rossby waves: they cannot absorb any more momentum flux and hence must reflect the incoming waves (Killworth and McIntyre (1985)).

It is noteworthy that linear theory breaks down in the critical layer but can still be used to compute the momentum flux $\overline{u'v'}$ outside from the critical layer. Momentum conservation and the associated momentum fluxes are fully nonlinear concepts that hold without restriction to small wave amplitudes and this is another way of seeing that the mean-flow acceleration had to be westward.

In summary, we have seen that variable $U(y)$ can lead to critical lines where $U = 0$ and that wave propagation is not possible past such lines. Instead, nonlinear wave breaking occurs in a critical layer just below the critical line. This involves overturning of the vorticity contours and eventual homogenization of the vorticity field, concomitant with westward acceleration of the mean flow. It is straightforward to show that all of the above applies equally well for Rossby waves with $c \neq 0$, which encounter critical lines where $U = c$. In general, critical lines occur where the *intrinsic* phase speed $\hat{c} = c - U$ is zero.

2.6 Reflection

Variable $U(y)$ can lead to a second strong effect: wave reflection. This occurs when $U(y)$ reaches the upper speed limit β/k^2 and hence $l = 0$ there, from (2.63). This implies that $v_g = 0$ there as well and this means that simple ray-tracing predicts unbounded amplitudes again, just as in the critical layer case. However, this time

$$v_g^{-1} \propto l^{-1} \propto (\beta/k^2 - U)^{-1/2} \tag{2.76}$$

near the reflection line where $U = \beta/k^2$. This implies that the wave front reaches the reflection line in finite time. What actually happens is that the wave is reflected at this location, which is an effect that is not captured by simple ray-tracing. The wave field then consists of an two waves, one going northward and the reflected one going southward. Eventually, a steady state establishes itself that has nonzero pseudomomentum $\mathsf{p}(y)$ between the southern wall and the reflection line and zero $\mathsf{p}(y)$ to the north of the reflection line. The pseudomomentum or momentum flux $\overline{u'v'} = 0$ everywhere.

Wave reflection means that simple ray tracing breaks down (cf. (2.64)), but linear theory remains valid. Indeed, the modal equation (2.60) has no singularity at the reflection line, in contrast to the situation at the critical line, and it can straightforwardly be solved locally in terms of Airy functions. (As an exercise, you may consider which parts of the above remain true if it should happen that $U_y = 0$ at the reflection line.)

2.7 Another view on mean-flow acceleration

Recall the exact mean-flow equation (without forcing)

$$\overline{u}_t + (\overline{u'v'})_y = 0. \tag{2.77}$$

Using the equally exact definition of $q' = v'_x - u'_y$ and that $u'_x + v'_y = 0$ we can easily prove the *Taylor identity*

$$(\overline{u'v'})_y = -\overline{q'v'}. \tag{2.78}$$

Doing this mirrors the steps below (1.37), which made no use of small-amplitude approximations. Now, this means that (2.77) can be re-written as

$$\overline{u}_t = \overline{q'v'}. \tag{2.79}$$

In this form the exact zonal mean-flow acceleration equals the northward flux of vorticity.

Now, using the small-amplitude results $q' = -Q_y\eta'$ and $\mathrm{D}_t\eta' = v'$ we see that

$$\overline{u}_t = -Q_y\,\overline{\eta'\mathrm{D}_t\eta'} = \frac{\partial}{\partial t}\left(-Q_y\frac{\overline{\eta'^2}}{2}\right) = \mathsf{p}_t \tag{2.80}$$

holds to $O(a^2)$. This slick short derivation of our main result is worth knowing.

The exact law (2.79) also allows understanding why growing Rossby waves must be accompanied by westward acceleration, as follows. The absolute vorticity q is materially advected and increases with y in the undisturbed reference configuration. Consider one particular contour of q, say the one at $y = 5$ in the reference configuration, on which $q = Q(5)$. Now, growing Rossby waves undulate the contours of constant q such that the area between our contour and the line $y = 5$ is filled with $q < Q(5)$ where our contour is *northward* of $y = 5$ or with $q > Q(5)$ where our contour is *southward* of it. This means that there has been a net negative transport of vorticity across the line $y = 5$, i.e. $\overline{q'v'} < 0$

and hence $\overline{u}_t < 0$ follows. This qualitative picture remains valid nonlinearly, i.e. it does not depend on small wave amplitudes: the westward acceleration due to the arrival of Rossby waves is a robust feature.

3 Internal gravity waves

We now turn to a new physical system, with a new kind of waves (e.g. Staquet and Sommeria (2002)). Doing this will get us one step closer towards a robust theory of wave–mean interactions, i.e. a theory that is informed by as many different physical applications as possible. Internal gravity waves are important in their own right as contributing significantly to the global circulation of the atmosphere and oceans. They are usually too small in spatial scale (especially vertical scale) to be resolved in global computer models and this makes their theoretical study especially important, as global models depend crucially on the theory.

3.1 Boussinesq equations

The two-dimensional ideal Boussinesq equations are

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{1}{\rho_0} p_x = 0 \tag{3.1}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + \frac{1}{\rho_0} p_z = b \tag{3.2}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + N^2 w = 0 \tag{3.3}$$

$$u_x + w_z = 0. \tag{3.4}$$

Here we work in the xz-plane, with x west–east as before and z being altitude. The two-dimensional velocity $\boldsymbol{u} = (u, w)$ is non-divergent and the density ρ_0 is a uniform constant, as before. The new variable is the *buoyancy* $b(x, z, t) = -g(\rho - \rho_0)/\rho_0$, which arises due to joint effect of gravity and density contrasts in the vertical (e.g. Pedlosky (1987)). This leads to a force b in the vertical, as indicated. This force is upward if there is positive buoyancy $b > 0$ and vice versa.

The evolution equation for b expresses the fact that there are material *stratification surfaces* (or lines in two dimensions)

$$\theta = b + N^2 z \quad \text{such that} \quad \frac{\mathrm{D}\theta}{\mathrm{D}t} = 0 \tag{3.5}$$

and at undisturbed rest we have $b = 0$ and linear stratification $\theta = N^2 z$. Here the constant N is the *buoyancy frequency*, for reasons to become clear below. In the ocean, these stratification surfaces are surfaces of constant density (called “isopycnals”), whereas in the atmosphere they are surfaces of constant entropy (“isentropes”). The Boussinesq equations are a useful approximation in both cases, although in the atmospheric case the vertical extent of the domain must be small compared to a density scale height (approx. 7km), so that we can neglect the density decay in ρ_0.

We can eliminate the pressure p as before by focusing on the vorticity equation, which is

$$\frac{\mathrm{D}}{\mathrm{D}t}(u_z - w_x) = -b_x. \tag{3.6}$$

The nonzero right-hand side shows how y-vorticity can be generated by sloping stratification surfaces: this is called *baroclinic generation of vorticity*. You can convince yourself that the sign of the vorticity generation is such that sloping stratification surfaces tend to always rotate back towards the horizontal. Due to inertia, they overshoot the horizontal equilibrium position and this is the basic oscillation mechanism for internal gravity waves.

3.2 Linear gravity waves

Consider small-amplitude waves relative to a steady background state with $b = 0$ and $\boldsymbol{u} = (U(z), 0)$. The $O(a)$ equations are

$$\mathrm{D}_t(u'_z - w'_x) + b'_x = 0 \tag{3.7}$$
$$\mathrm{D}_t b' + N^2 w' = 0 \tag{3.8}$$
$$u'_x + w'_z = 0. \tag{3.9}$$

Introducing a stream function ψ' such that $u' = +\psi'_z$ and $w' = -\psi_x$ gives

$$\mathrm{D}_t(\psi'_{xx} + \psi'_{zz}) + b'_x = 0 \tag{3.10}$$

and after cross-differentiation we can eliminate b' to obtain the single equation

$$\boxed{\mathrm{D}_t\mathrm{D}_t(\psi'_{xx} + \psi'_{zz}) + N^2\psi'_{xx} = 0}. \tag{3.11}$$

This is called the *Taylor–Goldstein* equation. If $U =$ const. we can look for plane-wave solutions $\psi' = \hat{\psi}\exp(i(kx + mz - \omega t))$ in terms of absolute frequency ω and wavenumber vector $\boldsymbol{k} = (k, m)$. As before, the intrinsic frequency $\hat{\omega} = \omega - Uk$ and we have the usual plane-wave relations

$$\frac{\partial}{\partial t} = -i\omega, \quad \frac{\partial}{\partial x} = ik, \quad \frac{\partial}{\partial z} = im, \quad \Rightarrow \quad \mathrm{D}_t = -i\hat{\omega}. \tag{3.12}$$

Substituting in (3.11) gives

$$-\hat{\omega}^2(-k^2 - m^2) - k^2N^2 = 0 \quad \Rightarrow \quad \boxed{\hat{\omega}^2 = N^2\,\frac{k^2}{k^2 + m^2}}, \tag{3.13}$$

which is the dispersion relation for internal gravity waves. It is one of the most peculiar dispersion relations found in the natural world. We note a number of important facts about gravity waves.

1. There are two roots $\hat{\omega}$ for each $\boldsymbol{k}$:

$$\hat{\omega} = \pm N\frac{k}{\kappa}, \tag{3.14}$$

where $\kappa = |\boldsymbol{k}|$. This means there are two independent wave modes for each $\boldsymbol{k}$, which is because we need to specify initial conditions for *two* fields in the linear Boussinesq equations, say for the stream function ψ' and for the buoyancy b'. This is unlike the Rossby-wave case, where there was only one mode and only one field, say q'.

2. The continuity equation implies $\boldsymbol{u}' \cdot \boldsymbol{k} = 0$ and hence gravity wave are transverse waves, with fluid velocities perpendicular to $\boldsymbol{k}$ and hence tangent to planes of constant phase. The same was true for Rossby waves, and single plane gravity waves are again exact solutions of the nonlinear equations.

3. There is a finite frequency bandwidth

$$0 \leq \hat{\omega}^2 \leq N^2. \tag{3.15}$$

The lower limit is attained when $k = 0$ and the flow is entirely horizontal. The upper limit is attained when $m = 0$ and the flow entirely vertical, with bands of fluid moving up and down with horizontal spacing $2\pi/k$. These are called buoyancy oscillations, which explains the name of N. This most rapid gravity wave has a frequency of about 7 minutes in the atmosphere and about 1 hour in the ocean.

4. The frequency does not depend on spatial scale, i.e. $\hat{\omega}$ is a function only of k/m. Specifically, if polar coordinates are used for the wavenumber vector such that $k = \kappa \cos \alpha$ and $m = \kappa \sin \alpha$ then we have

$$\hat{\omega}^2 = N^2 \cos^2 \alpha. \tag{3.16}$$

So the frequency depends only on the angle of $\boldsymbol{k}$ but not on its magnitude.

5. The intrinsic group velocities are

$$\boxed{\hat{u}_g = \frac{\partial \hat{\omega}}{\partial k} = \pm N \, \frac{m^2}{\kappa^3}, \quad \hat{w}_g = \frac{\partial \hat{\omega}}{\partial m} = \mp N \, \frac{km}{\kappa^3}}, \tag{3.17}$$

where the sign cases correspond to the two wave modes. This means that $k\hat{u}_g + m\hat{w}_g = 0$, which is a direct consequence of (3.16). Therefore, just like the particle velocity, the group velocity is also perpendicular to $\boldsymbol{k}$ and hence the intrinsic group velocity *makes a right angle* with the intrinsic phase velocity (cf. 2.23 with ω replaced by $\hat{\omega}$). Furthermore, it is easy to see that for *both* wave modes

$$\operatorname{sgn}(\hat{\omega}/k) = \operatorname{sgn}(\hat{u}_g), \quad \text{but} \quad \operatorname{sgn}(\hat{\omega}/m) = -\operatorname{sgn}(\hat{w}_g). \tag{3.18}$$

This means that a wave with eastward intrinsic phase speed (i.e. $\hat{c} = \hat{\omega}/k > 0$) also has an eastward intrinsic group velocity. On the other hand, *downward* phase propagation corresponds to *upward* group velocity! Historically, this has been very important for the correct interpretation of observations (Hines (1989)).

6. The wave-induced vertical flux of zonal momentum is

$$\overline{u'w'} = -\frac{k}{m}\,\overline{u'^2} = -\frac{m}{k}\,\overline{w'^2} \tag{3.19}$$

after using the continuity equation, and this means that its sign is given by

$$\operatorname{sgn}(\overline{u'w'}) = -\operatorname{sgn}(km) = \operatorname{sgn}(\hat{u}_g)\operatorname{sgn}(\hat{w}_g) \tag{3.20}$$

for both wave modes. So, an upward–eastward-moving gravity wave has a positive vertical flux of zonal momentum and an upward–westward-moving gravity wave has a negative flux.

3.3 Gravity wave pseudomomentum

Now we would like to find the pseudomomentum of gravity waves. We work in analogy with the Rossby-wave case. First, we define the vertical particle displacement ζ' to $O(a)$ by

$$\mathrm{D}_t\zeta' = w' \quad \Rightarrow \quad b' = -\zeta' N^2 \tag{3.21}$$

follows from (3.8). We now multiply (3.7) by ζ', average over x, and manipulate the terms:

$$\begin{aligned} \overline{\zeta' \mathrm{D}_t(u'_z - w'_x)} = -N^2 \overline{\zeta' b'_x} &= \overline{b' b'_x} = \frac{1}{2}(\overline{b'^2})_x = 0 &(3.22)\\ \Rightarrow \mathrm{D}_t \overline{\zeta'(u'_z - w'_x)} - \overline{w'(u'_z - w'_x)} &= 0 &(3.23)\\ \Rightarrow \mathrm{D}_t \overline{\zeta'(u'_z - w'_x)} = \overline{w' u'_z} &= (\overline{u'w'})_z - \overline{u' w'_z} = (\overline{u'w'})_z. &(3.24) \end{aligned}$$

We have made repeated use of $u'_x + w'_z = 0$ here. We now define the gravity-wave pseudomomentum

$$\boxed{\mathsf{p} = -\overline{\zeta'(u'_z - w'_x)} = \frac{1}{N^2}\,\overline{b'(u'_z - w'_x)} \quad \Rightarrow \quad \mathsf{p}_t + (\overline{u'w'})_z = 0}\,. \tag{3.25}$$

This can be compared to the Rossby-wave pseudomomentum (2.29). The most conspicuous difference is a factor of two, which is related to the fact that the wave energy for gravity waves is

$$E = \frac{1}{2}\left(\overline{u'^2} + \overline{w'^2} + \frac{\overline{b'^2}}{N^2}\right), \tag{3.26}$$

which contains a second, potential energy term. Plane gravity waves (in the absence of background rotation) obey energy equipartition and hence $E = \overline{b'^2}/N^2$ holds, i.e. the total wave energy is twice the kinetic energy. This means that the generic expression $\mathsf{p} = kE/\hat{\omega} = E/\hat{c}$ holds for plane gravity waves as well. This is an easy way to see that eastward waves have positive pseudomomentum and vice versa. In addition, for plane waves the pseudomomentum flux $\overline{u'w'} = \mathsf{p} w_g$.

3.4 Another view on mean-flow acceleration, again

Just as in the Rossby-wave case there is a short-cut to the main result for gravity waves. Recall that the exact mean-flow equation (without forcing) can be rewritten

$$\overline{u}_t = -(\overline{u'w'})_z = -\overline{(u'_z - w'_x)w'} \tag{3.27}$$

by using the Taylor identity. Now, using the small-amplitude results $\mathrm{D}_t(u'_z - w'_x) = -b'_x$, $\mathrm{D}_t\zeta' = w'$, and $\zeta' = -b'/N^2$ we find that

$$\overline{u}_t = \mathrm{D}_t\left(-\overline{(u'_z - w'_x)\zeta'}\right) - \overline{\zeta' b'_x} = \mathsf{p}_t - 0 \tag{3.28}$$

holds to $O(a^2)$. Thus we again have a quick route to the pseudomomentum definition as well as to the mean-flow acceleration equation.

3.5 Rossby versus gravity waves

Here is a brief summary of some aspects of two-dimensional Rossby and non-rotating gravity waves. There are similarities but the two are certainly not identical.

	Rossby waves	Gravity waves
Domain	(x, y)	(x, z)
Material invariant	absolute vorticity	stratification
	$q = v_x - u_y + f_0 + \beta y$	$\theta = b + N^2 z$
Pseudomomentum	$\mathsf{p} = -\overline{q'^2}/(2\beta)$	$\mathsf{p} = \overline{b'(u'_z - w'_x)}/N^2$
	sign-definite	not sign-definite
Momentum flux	$\overline{u'v'}$	$\overline{u'w'}$
Dispersion relation	$\hat{\omega} = -\beta k/(k^2 + l^2)$	$\hat{\omega} = \pm Nk/\sqrt{k^2 + l^2}$

Both waves can be viewed as undulations of the contours marking constant values of the respective material invariant. There has to be a background gradient in this invariant (i.e. nonzero β and $N^2 > 0$) for there to be a restoring mechanism and hence waves. Single plane waves always satisfy $\mathsf{p} = kE/\hat{\omega}$ and the corresponding pseudomomentum fluxes are $\overline{u'v'} = \mathsf{p}v_g$ and $\overline{u'w'} = \mathsf{p}w_g$, respectively.

3.6 Mountain waves

Consider a uniform background flow with velocity $U > 0$ over a mountain described by a small-amplitude surface undulation

$$h(x) = h_0 \cos(kx), \quad h_0 = O(a) \tag{3.29}$$

such that the vertical particle displacement at the lower boundary $z = 0$ is given by $\zeta'(x, 0) = h(x)$. The wave is steady with respect to the ground and hence the absolute frequency $\omega = 0$. This implies

$$0 = \omega = \hat{\omega} + Uk \quad \Rightarrow \quad \frac{\hat{\omega}}{k} = \hat{c} = -U \quad \Rightarrow \quad U = \frac{N}{\kappa}, \tag{3.30}$$

where the last expression uses (3.14) and the mode selection is dictated by the sign of U. For $U > 0$ the *lower* wave mode branch had to be selected to satisfy $\omega = 0$ and if $U < 0$ then the upper branch would have been selected. We have assumed $U > 0$ and hence we will use the lower sign in all expressions for the group velocities etc. The horizontal wavenumber k is given and we solve the third expression in (3.30) for the vertical wavenumber m to get $m^2 = N^2/U^2 - k^2$. The vertical group velocity $\hat{w}_g = w_g$ must be positive for waves propagating away from the lower boundary and hence $km > 0$. A convenient sign convention consistent with this and the mode selection is

$$k > 0, \quad m > 0, \quad \hat{\omega} < 0. \tag{3.31}$$

Flipping *all* of these signs would work equally well as a sign convention.

This means we have

$$m = +\sqrt{\frac{N^2}{U^2} - k^2}, \tag{3.32}$$

and the window for propagating waves is $0 < U < N/k$. (Repeating the argument for $U < 0$ yields the window $0 < U^2 < N^2/k^2$, i.e. positive and negative wind speeds are allowed, unlike in the Rossby-wave case.) The vertical group velocity is

$$w_g = \frac{kmN}{\kappa^3} = \frac{kU^2}{N}\sqrt{1 - \frac{k^2U^2}{N^2}}. \tag{3.33}$$

Now, the linear fields inside the established wavetrain are easily found to be

$$\begin{aligned}
\zeta'(x,z) = -b'(x,z)/N^2 &= h_0 \cos(kx+mz) && (3.34)\\
u'(x,z) &= Umh_0 \sin(kx+mz) && (3.35)\\
w'(x,z) &= -Ukh_0 \sin(kx+mz) && (3.36)\\
\mathsf{p} = \overline{b'(u'_z - w'_x)}/N^2 &= -Uh_0^2\kappa^2\,\overline{\cos^2(kx+mz)} && (3.37)\\
\Rightarrow\ \mathsf{p} &= -h_0^2\frac{U\kappa^2}{2} = -\frac{h_0^2N^2}{2U}, && (3.38)
\end{aligned}$$

where m is given by (3.32) and the last line made use of $U = N/\kappa$. The expression for p is easily checked to be consistent with $\mathsf{p} = E/\hat{c}$. It is interesting to note that p diverges as $U \to 0+$, even though $\mathsf{p} = 0$ for $U = 0$.

The drag force D exerted on the mountain range is equal to minus the wave-induced momentum flux $\overline{u'w'}$. From the linear solution this yields

$$\boxed{D = -\overline{u'w'} = -\mathsf{p}w_g = +\frac{h_0^2k}{2}U\sqrt{N^2 - k^2U^2}}. \tag{3.39}$$

So the mountain range feels a net force in the x-direction of this magnitude. The drag is linear in U for small U and, unlike p, goes to zero at the speed limits $U = 0$ and $U = N/k$. It is maximal for $U_* = N/(k\sqrt{2})$, with value $D_* = h_0^2N^2/4$.

The exact zonal mean-flow acceleration equation for the Boussinesq system is

$$\overline{u}_t + (\overline{u'w'})_z = 0, \tag{3.40}$$

which yields

$$\overline{u}_t = \mathsf{p}_t + O(a^3) \tag{3.41}$$

as in the Rossby-wave case. Therefore the mean flow inside the wavetrain is changed to

$$\overline{u} = U - \frac{h_0^2N^2}{2U} \tag{3.42}$$

as the wavetrain arrives. This nonlocal momentum transfer precisely balances the equal and opposite drag on the mountain below.

3.7 Forcing and dissipation

To effect lasting mean-flow changes we again have to allow for wave dissipation. The simplest way to do this is to add a body force $\boldsymbol{F} = (F, G)$ and a heating term H to the right-hand sides of the Boussinesq equations. Assuming that these terms have no $O(1)$, background part, we obtain for the linear equations

$$\begin{aligned} \mathrm{D}_t(u'_z - w'_x) + b'_x &= F'_z - G'_x && (3.43)\\ \mathrm{D}_t b' + N^2 w' &= H' && (3.44)\\ \Rightarrow \quad \mathsf{p}_t + (\overline{u'w'})_z &= \frac{1}{N^2}\overline{H'(u'_z - w'_x)} + \frac{1}{N^2}\overline{b'(F'_z - G'_x)} \equiv \mathcal{F} && (3.45) \end{aligned}$$

where $\mathsf{p} = \overline{b'(u'_z - w'_x)}/N^2$. If we assume again that $\overline{F} = 0$ at $O(a^2)$, then for a steady wavetrain we obtain the mean-flow acceleration equation

$$\overline{u}_t = -(\overline{u'w'})_z = -\mathcal{F} + O(a^3). \tag{3.46}$$

The simplest case has no body force but includes radiative damping given by the *Newtonian cooling* approximation

$$H' = -\alpha\, b', \quad \Rightarrow \quad \mathcal{F} = -\alpha \mathsf{p}, \quad \Rightarrow \quad \boxed{\overline{u}_t = +\alpha \mathsf{p}}, \tag{3.47}$$

where $\alpha > 0$ is a constant. In the atmosphere, this kind of damping arises due to radiative energy transfers within the atmosphere, which tend to dampen temperature disturbances. As the sign of p is the sign of $\hat{c}$, we see that the dissipation of left-going intrinsic wave speeds leads to negative $\overline{u}_t$ and vice versa.

The wave structure is found from the steady pseudomomentum equation (3.45) and $\mathsf{p} w_g = \overline{u'w'}$ as follows:

$$\begin{aligned} (\overline{u'w'})_z &= -\alpha \mathsf{p} = -\frac{\alpha}{w_g}\,\overline{u'w'} && (3.48)\\ \Rightarrow \quad \overline{u'w'}(z) &= \overline{u'w'}(0)\,\exp(-\alpha z/w_g) && (3.49)\\ \Rightarrow \quad \mathsf{p}(z) &= \mathsf{p}(0)\,\exp(-\alpha z/w_g), && (3.50) \end{aligned}$$

which together with (3.47) gives the mean-flow acceleration profile. Here, the ground-level $\mathsf{p}(0) = -h_0^2 N^2/(2U)$, as computed before.

3.8 Critical layers

We turn to the case of a slowly varying $U(z)$ and focus attention at the location of a critical line z_c such that $U(z_c) = 0$. We assume again that a ray-tracing, or JWKB approximation, is feasible, i.e. we look for a locally plane-wave solution with $k > 0$ and $\omega = 0$ constant but with variable $m(z)$ such that (3.30) is satisfied for all z. From (3.32) we anticipate that m will become very large near the critical line and we will for simplicity from now on assume that $m^2 \gg k^2$ everywhere. Gravity waves obeying this scaling are called *hydrostatic* gravity waves and for mountain waves this corresponds to $U \ll N/k$.

For hydrostatic mountain gravity waves we simply have

$$m(z) = \frac{N}{U(z)} \quad \text{and} \quad w_g(z) = \frac{kU^2}{N}. \tag{3.51}$$

This shows that $m \to \infty$ and $w_g \to 0$ as $z \to z_c$. The JWKB approximation will be valid if

$$m^2 \gg \left(\frac{U_z}{U}\right)^2 \quad \Rightarrow \quad \frac{N^2}{(U_z)^2} \gg 1. \tag{3.52}$$

The non-dimensional parameter $N^2/(U_z)^2$ is called the *Richardson* number and it is typically larger than unity in the atmosphere or oceans. Interestingly, we see that it is possible to have a gravity-wave critical layer at which JWKB theory remains valid, i.e. there are infinitely many wave oscillations below the critical line. This was not so in the Rossby-wave case. (A more detailed investigation in fact shows that the criterion is $N^2/(U_z)^2 > 1/4$, which is very mild and coincides with the linear stability criterion of the background shear flow.) It is easy to show that the travel time to the critical layer is again infinite.

The wave structure follows from (3.48) with the important difference that w_g is now a function of z. This yields

$$\overline{u'w'}(z) = \overline{u'w'}(0) \exp\left(-\alpha \int_0^z \frac{\mathrm{d}\bar{z}}{w_g(\bar{z})}\right) = \overline{u'w'}(0) \exp\left(-\frac{\alpha N}{k} \int_0^z \frac{\mathrm{d}\bar{z}}{U^2(\bar{z})}\right). \tag{3.53}$$

This shows that dissipation per unit vertical length is enhanced in regions where U^2 is small. If U is smooth at z_c then the integral diverges as $z \to z_c$ and we obtain that $\overline{u'w'}(z_c) = 0$, i.e. the wave has been completely absorbed in a critical layer just below the critical line. The mean-flow acceleration profile follows as

$$\begin{aligned} \overline{u}_t = \alpha \mathsf{p} = \frac{\alpha}{w_g} \overline{u'w'}(z) &= \frac{\alpha N}{kU^2} \overline{u'w'}(0) \exp\left(-\frac{\alpha N}{k} \int_0^z \frac{\mathrm{d}\bar{z}}{U^2(\bar{z})}\right) & (3.54) \\ \Rightarrow \quad \overline{u}_t &= -\frac{\alpha h_0^2 N^2}{2U} \exp\left(-\frac{\alpha N}{k} \int_0^z \frac{\mathrm{d}\bar{z}}{U^2(\bar{z})}\right), & (3.55) \end{aligned}$$

where the second form uses (3.39).

Whilst the JWKB approximation is valid in the critical layer we can not be assured that linear theory itself will be valid there. In fact, it is often not. To test for validity of linear theory we once again consider an overturning amplitude $a(z)$ such that

$$a(z) = \max_x |\zeta'_z| = \max_x |b'_z/N^2|. \tag{3.56}$$

If $a = 1$ then the stratification surfaces $\theta = N^2 z + b'$ overturn, i.e. $\theta_z = 0$ at some location, and linear theory must break down. Indeed, linear theory is based on $a \ll 1$. For sinusoidal waves it follows that

$$a^2 = 2m^2 \overline{\zeta'^2} = 2\frac{m^2}{\hat{\omega}^2} \overline{w'^2} = -2\frac{km}{\hat{\omega}^2} \overline{u'w'} = -2\frac{N}{kU^3} \overline{u'w'}. \tag{3.57}$$

Combined with (3.53) this indicates that a^2 will eventually go to zero at the critical line, but before that it will make an excursion to large values due to the occurrence of U^3 in the denominator. This shows that a^2 is less well behaved than $\overline{u'w'}$. Whether this excursion of a^2 to larger values will lead to $a \approx 1$ and hence to the breakdown of linear theory depends on the detailed circumstances. However, it is believed that in practice most gravity waves will break nonlinearly before they reach the critical line.

We have focused on mountain waves, which have zero absolute frequency ω. However, the above computations are easily adapted for waves with $\omega \neq 0$. Specifically, waves with phase speed $c = \omega/k$ will encounter critical layers where $U = c$, i.e. where the intrinsic phase speed $\hat{c} = c - U$ is zero. The above formulas remain valid after substituting for U by $U - c$ everywhere. As the sign of p is the sign of $\hat{c}$, it is easy to see from (3.46) that dissipating gravity waves will always accelerate the mean flow towards c. In other words, the mean-flow acceleration is always such that $\hat{c}$ is diminished. In the mountain wave case this means that the mean flow is driven towards $U = 0$, regardless of what the sign of $U(0)$ is. So wave dissipation drives the mean flow towards decreasing $\hat{c}$, which ultimately may lead to the occurrence of critical layers where $\hat{c} = 0$, and hence to increased wave dissipation, as we have seen. This is an important positive feedback cycle.

Finally, we note that wave reflection in the vertical is possible at altitudes where U reaches the upper speed limit N/k. As in the Rossby-wave case, there is no breakdown of linear theory there, although JWKB becomes invalid because $m \to 0$ at the reflection altitude.

3.9 Mean-flow feedback

Let us consider the mean-flow acceleration equation (3.54), but now written down for a wave with phase speed c:

$$\overline{u}_t(z,t) = \frac{\alpha N}{k(U(z)-c)^2}\,\overline{u'w'}(0)\,\exp\left(-\frac{\alpha N}{k}\int_0^z \frac{\mathrm{d}\bar{z}}{(U(\bar{z})-c)^2}\right). \tag{3.58}$$

From this we see that $\overline{u} - U = O(a^2 t)$, i.e. the mean-flow change grows linearly in time. For times $t = O(1)$ the mean-flow change is $O(a^2)$ as assumed in our small-amplitude scaling. However, for longer times $t = O(a^{-2})$ we can expect that the mean-flow change becomes $O(1)$, which is comparable to U. Of course, this defeats our scaling assumptions and our regular perturbation theory is not valid over such long, amplitude-dependent time scales. In principle, this requires the use of a singular perturbation theory capable of dealing with multiple time scales.

It is plausible (though difficult to prove rigorously) that the correct generalization of (3.58) valid for such long times is obtained by replacing U by $\overline{u}$ everywhere on the right-hand side. In essence, this allows the $O(1)$ background flow to evolve slowly in response to the $O(a^2)$ wave-induced forcing terms. This background evolution then feeds back into the structure of the waves themselves, and this produces an important nonlinear feedback cycle (e.g. Plumb (1977)).[4]

[4] The difficulty for rigorous theory is to ensure that the evolving background flow remains slowly varying relative to the waves.

Following this idea the resulting dynamical equation for $\overline{u}(z,t)$ is then

$$\boxed{\overline{u}_t(z,t) - \nu\overline{u}_{zz} = -(\overline{u'w'})_z = \frac{\alpha N}{k(\overline{u}-c)^2}\,\overline{u'w'}(0)\,\exp\left(-\frac{\alpha N}{k}\int_0^z \frac{\mathrm{d}\bar{z}}{(\overline{u}(\bar{z},t)-c)^2}\right)}, \tag{3.59}$$

which also includes vertical diffusion with constant diffusivity $\nu > 0$. This is a non-local equation for $\overline{u}$, meaning that $\overline{u}_t$ at some z depends on the values of $\overline{u}$ at all altitudes below z.

What do solutions of (3.59) look like, say for $c > 0$? It can be shown that if initially $\overline{u}(z,0) = 0$ and if the lower boundary condition $\overline{u}(0,t) = 0$, then the time evolution of (3.59) will lead towards a stable steady state such that

$$\nu\overline{u}_z = \overline{u'w'}(z) = \overline{u'w'}(0)\,\exp\left(-\frac{\alpha N}{k}\int_0^z \frac{\mathrm{d}\bar{z}}{(\overline{u}(\bar{z},t)-c)^2}\right). \tag{3.60}$$

This means that $\overline{u} = c$ everywhere apart from inside a thin boundary layer near the ground. The shear at the ground $\overline{u}_z(0) = \overline{u'w'}(0)/\nu$, and hence the depth of the boundary layer is approximately $c\nu/\overline{u'w'}(0)$. Thus a single wave with $c > 0$ leads to a stable mean-flow profile with $\overline{u} = c$ outside a viscous boundary layer. This was to expected, based on the understanding that dissipating waves always drag the mean flow towards the phase speed of the wave. When we consider the mean-flow response to several waves then interesting mean-flow oscillations can occur.

3.10 Several waves and the quasi-biennial oscillation

Linear waves can be freely superimposed at $O(a)$ but we must always be careful when we compute the $O(a^2)$ momentum flux $\overline{u'w'}$, which may be affected by correlations between wave modes. Specifically, consider two waves with parameters $\{k_1, c_1\}$ and $\{k_2, c_2\}$, respectively. If $|k_1| \neq |k_2|$ then the wave modes will be orthogonal with respect to x-averaging and hence

$$\overline{u'w'} = \overline{u_1'w_1'} + \overline{u_2'w_2'}. \tag{3.61}$$

On the other hand, if the wavenumber magnitudes are equal then $\overline{u'w'}$ will oscillate in time around a mean value given by (3.61) and the oscillations will have frequencies $|k(c_1 \pm c_2)|$. In practice one assumes that only the time-averaged value of $\overline{u'w'}$ matters for the long-time mean-flow forcing and hence (3.61) is assumed to hold.

Consider now the specific case where $k_1 = k_2 = k > 0$ and $c_1 = -c_2 = c > 0$ and the wave amplitudes are equal such that $\overline{u_1'w_1'}(0) = A > 0$ and $\overline{u_2'w_2'}(0) = -A$. This corresponds to a *standing wave* pattern $\psi' \propto \cos(kx)\cos(ckt)$ at fixed z. The mean-flow equation is given by (3.59) written down for forcing due to two waves, i.e.

$$\overline{u}_t - \nu\overline{u}_{zz} = \frac{\alpha N}{k}\left\{\frac{A}{(\overline{u}-c)^2}\exp\left(-\frac{\alpha N}{k}\int_0^z \frac{\mathrm{d}\bar{z}}{(\overline{u}-c)^2}\right) - \frac{A}{(\overline{u}+c)^2}\exp\left(-\frac{\alpha N}{k}\int_0^z \frac{\mathrm{d}\bar{z}}{(\overline{u}+c)^2}\right)\right\}. \tag{3.62}$$

This complicated-looking equation can be easily integrated numerically. Clearly, $\overline{u} = 0$ is a trivial equilibrium state as both waves then neutralize each other at all z, meaning that their respective dissipation exerts equal and opposite forces onto the mean flow.

However, it can be seen that this is not a stable equilibrium configuration. Indeed, if there is a small positive disturbance to $\overline{u}$ at some location then the first wave will be preferentially dissipated there, because $\overline{u} - c$ will have been reduced there. This means that $\overline{u}_t > 0$ there, amplifying the original disturbance and hence showing that the equilibrium is unstable. Furthermore, because the first wave dissipates stronger at this location, the second wave will dissipate stronger than the first wave in the region *above* the original positive disturbance. This means that the region above will experience an acceleration $\overline{u}_t < 0$. A modicum of further analysis reveals that the zero-wind line $\overline{u} = 0$ above the original disturbance travels downward in time as do the positive and negative mean-flow regions themselves.

In summary, the zero-flow equilibrium is unstable and spontaneously breaks down into an oscillatory flow pattern that travels downward. The frequency of these mean-flow oscillations depends on the details of the incident waves and is inversely proportional to $O(a^2)$, i.e. stronger waves lead to more rapid oscillations.

This spontaneous occurrence of a mean-flow oscillation is believed to be fundamental to the so-called "quasi-biennial oscillation" in the lower stratosphere (Baldwin et al. (2001)). This is a periodic reversal of the zonal mean winds between about 18-30 km in the equatorial stratosphere. The period of these carefully observed oscillations is about 26-27 months, which is *not* a sub-harmonic of the annual cycle. The zonal mean wind pattern is observed to travel downwards at a speed of about 1km per month. Extensions of the above theory for more than two waves and for more than one kind of wave (i.e. including Rossby waves) are believed to offer the best scientific explanation for these peculiar wind oscillations.

4 Circulation and pseudomomentum

Small-amplitude equations such as (3.28) can turn out to be leading-order versions of fully nonlinear conservation laws, i.e. laws that can in principle be formulated without restriction to small wave amplitude a. This points towards significant extensions of our small-amplitude theory. We will not pursue these extension here, but we will illustrate one way in which (3.28) can be shown to be a small-amplitude version of Kelvin's circulation theorem (e.g. Bretherton (1969)). This will provide a satisfying physical interpretation of the pseudomomentum p and opens a perspective towards extensions of wave–mean interaction theory beyond simple geometry. Such extensions are the subject of on-going research (e.g. Bühler and McIntyre (2004), Bühler and McIntyre (2003), Bühler (2000), McIntyre and Norton (1990)).

4.1 Kelvin's circulation theorem

Consider a closed oriented circuit C lying in the fluid. The circulation Γ around this circuit is defined by the closed line integral

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}. \tag{4.1}$$

If C is a material circuit (i.e. it moves with the flow) then the material time derivative of $\mathrm{d}\boldsymbol{x}$ is $\mathrm{d}\boldsymbol{u}$ and that of Γ can then be shown to be

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_C \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{x} + \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{u} = -\oint_C \frac{\boldsymbol{\nabla} p}{\rho} \cdot \mathrm{d}\boldsymbol{x} + 0, \tag{4.2}$$

where we neglect background rotation at the moment but allow for variable density ρ.

If the integrand is a perfect differential then the closed line integral vanishes and Γ does not change. This can happen in three physically interesting ways.

- First, ρ could be a global constant, as in homogeneous incompressible flow.
- Second, ρ could be a global function of pressure, as in so-called barotropic flow.
- Third, ρ could be general function of pressure and a second thermodynamic variable such as specific entropy. Then, *if C lies inside a surface of constant entropy* the integrand is a perfect differential on C and hence the integral vanishes. Furthermore, if the specific entropy is materially invariant then C will continue to lie within a surface of constant entropy, and hence $\Gamma = \text{const.}$ holds for circuits lying within entropy surfaces. This is the case of most interest in atmosphere and ocean fluid dynamics.

In all three cases we have the celebrated *Kelvin circulation theorem*

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0. \tag{4.3}$$

If there is background rotation with Coriolis vector $\boldsymbol{f}$ then the above considerations remain trivially true provided that $\boldsymbol{u}$ is replaced in (4.1) by the absolute velocity $\boldsymbol{u} + 0.5\boldsymbol{f} \times \boldsymbol{x}$. With this simple extension in mind we will continue to neglect background rotation.

In the Boussinesq system the stratification $\theta = b + N^2 z$ plays the same role as the entropy above. This can be shown directly from the Boussinesq equations, which yield

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\oint_C \frac{\boldsymbol{\nabla} p}{\rho_0} \cdot \mathrm{d}\boldsymbol{x} + \oint_C b\widehat{\boldsymbol{z}} \cdot \mathrm{d}\boldsymbol{x} = 0 + \oint_C b\mathrm{d}z \tag{4.4}$$

The remaining integral can be re-written as

$$\oint_C (b + N^2 z)\mathrm{d}z - \oint_C N^2 z \mathrm{d}z. \tag{4.5}$$

The second integral is clearly zero and if θ is constant on C then it can be pulled out of the first integral, which is then zero as well. Therefore circulation is indeed conserved on circuits lying within stratification surfaces. In a periodic domain a line traversing the the length of the domain can be viewed as a closed circuit and hence in the two-dimensional Boussinesq equations circulation is conserved on the kind of undulating material lines that were considered before.

Clearly, to make use of the circulation theorem we must have a way to follow material displacements induced by the waves. In linear wave theory this is achieved by defining the three-dimensional linear particle displacement vector

$$\boldsymbol{\xi}' = (\xi', \eta',' \zeta') \quad \text{via} \quad \mathrm{D}_t\boldsymbol{\xi}' = \boldsymbol{u}' \tag{4.6}$$

in terms of the linear wave velocities $\boldsymbol{u}'$. This defines $\boldsymbol{\xi}'$ up to an integration constant that has to be determined from other considerations. Often, the flow can be imagined to have started from rest and then the integration constant is zero. For a plane wave the simple relation $-i\hat{\omega}\boldsymbol{\xi}' = \boldsymbol{u}'$ holds. A particle that was initially at rest at location $\boldsymbol{x}$ will later be displaced by the waves to a position $\boldsymbol{x}+\boldsymbol{\xi}'$. Advected fields give access to some of the components of $\boldsymbol{\xi}'$. For instance, in the Boussinesq equations we have $\zeta' = -b'/N^2$.

Let us now denote by C^ξ a line of constant stratification θ, which is a material contour as well. In the undisturbed configuration (i.e. without waves) this contour is a flat line $z =$const., which we will denote by C. Kelvin's circulation theorem then tells us that the circulation along C^ξ is exactly constant. To $O(a)$ the displaced contour C^ξ is given by the "lifting" map

$$\begin{aligned} & \boldsymbol{x} \quad \rightarrow \quad \boldsymbol{x}+\boldsymbol{\xi}'(\boldsymbol{x},t) && (4.7)\\ \Leftrightarrow \quad & (x,z) \quad \rightarrow \quad (x+\xi'(x,z,t)\,,\; z+\zeta'(x,z,t)) && (4.8)\end{aligned}$$

based on the $O(a)$ linear displacements $\boldsymbol{\xi}'$. Here z is held constant, x varies from 0 to L, and the induced map "lifts" positions from the undisturbed contour C to the displaced contour C^ξ. The key step is now to express the conserved circulation

$$\Gamma = \oint_{C^\xi} u\mathrm{d}x + w\mathrm{d}z \tag{4.9}$$

in terms of an integral over the undisturbed contour C. The lifting map (4.7) reduces this to a simple problem of variable transformation in the line integral:

$$\Gamma = \oint_C u^\xi(\mathrm{d}x)^\xi + w^\xi(\mathrm{d}z)^\xi, \tag{4.10}$$

where we have used the nifty shorthand $\boldsymbol{u}^\xi(\boldsymbol{x},t) \equiv \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi}',t)$. Here the line element transforms as

$$\begin{aligned} (\mathrm{d}x)^\xi = \mathrm{d}x + \mathrm{d}\xi' \quad &= \quad \mathrm{d}x(1+\xi'_x) && (4.11)\\ (\mathrm{d}z)^\xi = \mathrm{d}z + \mathrm{d}\zeta' \quad &= \quad \mathrm{d}x\,\zeta'_x && (4.12)\end{aligned}$$

because $\mathrm{d}z = 0$ on the flat contour C. The integral over C is an integral over x at fixed z, i.e.

$$\begin{aligned} \Gamma = \int_0^L \left[u^\xi(1+\xi'_x) + w^\xi\,\zeta'_x\right]\mathrm{d}x \quad &= \quad L\,\overline{\left[u^\xi(1+\xi'_x) + w^\xi\,\zeta'_x\right]} && (4.13)\\ &= \quad L\overline{u}^L + L\,\overline{\left[u^\xi\xi'_x + w^\xi\,\zeta'_x\right]} && (4.14)\end{aligned}$$

where we have introduced the *Lagrangian-mean* velocity $\overline{u}^L \equiv \overline{u^\xi}$, which is a particle-following average velocity.[5] To compute Γ correct to $O(a^2)$ requires only $O(a)$ accuracy

[5] In general, $\overline{u}^L$ and the Eulerian-mean $\overline{u}$ differ by an $O(a^2)$ Stokes drift correction. However, the Stokes drift can be computed from the linear wave solution, so knowledge of $\overline{u}$ implies knowledge of $\overline{u}^L$ and vice versa. The decision of which mean velocity to use in a given problem can therefore be based on convenience.

in the remaining $\boldsymbol{u}^\xi$ terms, because $\boldsymbol{\xi}' = O(a)$. Also, $\boldsymbol{u}^\xi = \boldsymbol{u}' + O(a^2)$ if the background flow is constant and hence we obtain that

$$\Gamma = L\overline{u}^L + L\,\overline{[u'\xi'_x + w'\zeta'_x]} \equiv L\,(\overline{u}^L - \mathsf{p}) \tag{4.15}$$

provided we *define* the pseudomomentum to be

$$\boxed{\mathsf{p} \equiv -\overline{\xi'_x u'} - \overline{\zeta'_x w'}}\,. \tag{4.16}$$

This new Lagrangian definition is consistent with the earlier Eulerian pseudomomentum definition (3.25), i.e.

$$-\overline{\zeta'(u'_z - w'_x)} = -(\overline{\zeta' u'})_z + \overline{\zeta'_z u'} + \overline{\zeta' w'_x} = -(\overline{\zeta' u'})_z - \overline{\xi'_x u'} - \overline{\zeta'_x w'}. \tag{4.17}$$

The extra term $\overline{(\zeta' u')}_z$ vanishes for plane waves and due to its flux divergence form it does not upset the conservation law for p. Now let us consider what (4.15) implies. Because the circulation Γ is constant we must have

$$\overline{u}^L_t - \mathsf{p}_t = 0, \tag{4.18}$$

which is our usual mean-flow equation. This now arises naturally from the finite-amplitude conservation law for circulation. This is a different physical concept than momentum: pseudomomentum is closer related to circulation than to momentum.

4.2 Vectorial pseudomomentum

The pseudomomentum definition (4.16) suggests a natural extension to a vectorial pseudomomentum with components

$$\mathsf{p}_i \equiv -\overline{(\xi'_{j,i} u'_j)} \quad \text{(summation understood)} \tag{4.19}$$

such that

$$\Gamma = \oint_{C^\xi} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \oint_C (\overline{\boldsymbol{u}}^L - \boldsymbol{\mathsf{p}}) \cdot \mathrm{d}\boldsymbol{x} \tag{4.20}$$

holds by construction for circuits C that move with $\overline{\boldsymbol{u}}^L$. This leads to useful extensions of the classical wave–mean interaction theory to situations with non-simple geometry. One example of this is the generation of mean-flow vortices by breaking ocean waves on a beach (Bühler and Jacobson (2001)).

In general, what emerges from (4.19-4.20) is that pseudomomentum chose its name wisely: it is closely linked to fluid circulation and not to momentum. It measures the wave contribution to the circulation along stratification contours. The remaining contribution to the circulation is taken up by the Lagrangian-mean flow $\overline{\boldsymbol{u}}^L$.

5 Wave-driven global atmospheric circulations

Dissipating Rossby and gravity waves are essential contributors to the global zonal-mean circulation in the middle atmosphere, between 10-100 km or so (e.g. Andrews et al.

(1987), McIntyre (2000)). This region consists of the stratosphere between 10-60 km and the mesosphere above, between 60-100 km. Rossby waves dominate in the stratosphere and gravity waves in the mesosphere. The lower atmosphere below 10 km is called the troposphere, where our weather lives and where most of the atmospheric mass resides. However, the motion of the middle atmosphere is crucial for long-term trends in climate, such as the motion of long-lived chemicals responsible for the ozone hole. Now, a key to understanding the wave-induced circulation mechanisms is to note that on a rotating planet a mid-latitude zonal-mean westward (or retrograde) force reduces the absolute angular momentum of a ring of particles (i.e. a set of particles that initially shared the same altitude and latitude) and this drives that ring of particles poleward, i.e. towards the rotation axis. The mass flux of this poleward motion is compensated by sinking motion over the pole and rising motion over the equator. This works on both hemispheres and is sometimes called "gyroscopic pumping". Conversely, a zonal-mean eastward (or prograde) force increases the angular momentum and hence drives the ring of particles equatorward.

One can show that three-dimensional Rossby waves behave in fundamentally the same way as the two-dimensional waves we have investigated. In particular, dissipating Rossby waves always exert a retrograde force, and Rossby waves can be generated by flow over an undulating boundary, usually over mountains below. There is persistent Rossby-wave-induced retrograde forcing in the stratosphere in both hemispheres. This leads to poleward stratospheric motion at about 30 km, rising motion between 10-30 km over the equator, and sinking motion over the poles. This is called the stratospheric *Brewer–Dobson* circulation. Despite appearance, this circulation can not be explained by hot air rising over the equator and drifting polewards. This is because one has to explain the angular momentum change that allows particles to drift poleward on a rotating planet. This *requires* wave-induced forces to provide the necessary torque.

In the stratosphere Rossby waves are the most important waves, but higher up, in the mesosphere, the gravity waves dominate over Rossby waves. The density throughout the middle atmosphere decays roughly as

$$\rho(z) = \rho_0 \exp(-z/H_S) \tag{5.1}$$

where the density scale height $H_S \approx 7$km. It can be shown that this implies that Boussinesq gravity waves are modified such that a steady non-dissipating wave now obeys $\rho(z)\overline{u'w'} = \text{const}$. This implies that the particle velocities u' and w' *increase* with altitude. This leads to very large wave amplitudes at very high altitudes, especially in the so-called mesosphere, which begins around 60km or so. The large wave amplitudes lead to gravity-wave breaking in the mesosphere. Furthermore, the kinematic viscosity $\nu = \eta/\rho(z)$, where η is the dynamic viscosity, which does not vary much. This means that ν becomes very large and waves are subject to very strong diffusion. In practice, it is assumed that gravity waves must break down in the mesosphere due to a combination of nonlinear turbulence and viscous dissipation.

Mountain gravity waves that reach the mesosphere hence dissipate and exert a force on the mean flow that drives $\overline{u}$ to zero. This "gravity-wave drag" is known to be crucial for the observed structure of the zonal wind: without this drag there would be enormously large winds at these altitudes, which are not observed.

Now, there are increasing stratospheric temperatures from winter to summer hemisphere, meaning that there is a robust latitudinal gradient of zonal-mean temperature. On a rotating planet this is concomitant with a vertical shear of the zonal wind $\overline{u}$ (this link is described by the "thermal wind relation"). It turns out that this leads to strong prograde wind $\overline{u} > 0$ in the winter stratosphere and strong retrograde wind $\overline{u} < 0$ in the summer stratosphere. Gravity-wave drag that drives the wind to zero is hence retrograde in the winter hemisphere and prograde in the summer hemisphere! There is a second mechanism, which is not tied to zero-phase speed mountain waves but which leads to the same conclusion: critical-layer filtering in the stratosphere. The prograde winds in the winter stratosphere preferentially filter gravity waves with prograde phase velocities $c > 0$ and vice versa in the summer stratosphere. That results in the preferential transmission of retrograde gravity waves into the winter mesosphere and of prograde gravity waves into the summer mesosphere. The dissipation of these waves then gives the same result: retrograde drag in winter and prograde drag in summer.

The net result is a poleward flow in the winter mesosphere and an equatorward flow in the summer mesosphere. Together, this gives a net flow from summer to winter mesosphere, above about 60 km. There is no rising motion over the equator now, but there is sinking motion over the winter pole and rising motion over the summer pole to close the mass flux budget. This is called the mesospheric *Murgatroyd–Singleton* circulation. The rising motion over the summer pole is particularly important, because the adiabatic expansion of the rising air produces the *coldest* temperatures on Earth in the summer polar mesosphere even though this is the *sunniest* place on Earth as well. Temperatures as low -163^oC have been recorded and the extreme cold gives rise to "noctilucent clouds", which glow in electric blue at around 85 km and are made of ice crystals.

Finally, we note that gravity waves are far too small in scale (especially vertical scale) to be directly resolved in numerical models and hence gravity-wave drag must be put in by hand, based on the kind of theory we are studying here. This is an area of active research (Fritts and Alexander (2003), Kim et al. (2003)).

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