Non-Uniqueness of Weak Solutions to the Unforced Two-Dimensional Euler Equations with Vorticity in $L^1 \cap L^p$, I: A Compactness Criterion and the Linear Analysis

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Abstract

This paper and its companion Part II prove that the incompressible Euler equations in the plane, without external force, admit two distinct renormalized weak solutions with the same initial vorticity f, where f is radial and nonnegative and lies in $(L^1 \cap L^q)(\mathbb{R}^2)$ for every $q < \infty$ but is not bounded. One solution is radially symmetric and stationary. The other is 3-fold rotationally symmetric, is equimeasurable with f at every time, and departs from f in finite time; the two solutions are distinct in $L^\infty_t(L^1 \cap L^q)_x$ on $[0,T] \times \mathbb{R}^2$ for every $q \in (4/3, \infty)$ and every sufficiently large T. In this first part we prove two ingredients. One is a compactness criterion: for a radial datum, a sequence of 3-fold rotationally symmetric solutions with converging data, whose non-radial moment is bounded below at some bounded time, has a subsequence converging to a second distinct renormalized weak solution. The other is the linear analysis for the building blocks of our multiscale datum. We show that the linearization of the Euler equations around the basic building block plus a small point vortex has an unstable eigenvalue in a space of 3-fold rotationally symmetric perturbations. This latter argument depends on rigorous interval arithmetic that we discuss more fully in Part II.

1 Introduction

We consider the incompressible Euler equations in the plane, in vorticity form:

$$\partial_t \omega + v \cdot \nabla \omega = 0, \qquad v = K_{BS} * \omega, \qquad \omega(t=0) = \omega_0. \tag{1}$$

Here $K_{BS}(x) = x^\perp/(2\pi|x|^2)$, $x^\perp = (-x_2, x_1)$, denotes the Biot-Savart kernel, so that v is divergence free and $\partial_1 v_2 - \partial_2 v_1 = \omega$. There is no external force, and we call ω_0 the datum.

For a smooth decaying datum, Equation (1) has a unique global smooth solution (Wolibner [22], Hölder [13]; see Kato [14]; a self-contained proof for compactly supported data is in Appendix B of Part II [8]). Yudovich proved in [23], for bounded domains, that a weak solution with bounded vorticity is unique: its velocity is log-Lipschitz, which closes an energy estimate for the difference of two solutions. For the plane, with vorticity in $(L^1 \cap L^\infty)(\mathbb{R}^2)$ and square-integrable velocity, a complete proof is given in Theorem 0.3 and Appendix B.2 of [1]. For a datum in $(L^1 \cap L^p)(\Omega)$, $1 < p < \infty$, global weak solutions exist (Yudovich for a bounded domain Ω [23], see the introduction of [24]; DiPerna and Majda for $\Omega = \mathbb{R}^2$ [11]). When $p > 4/3$ the product $v\,\omega$ is locally integrable and

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Equation (1) has a weak formulation (Lemma 2.2, Definition 2.5), but the velocity is only Sobolev regular and Yudovich's uniqueness argument does not apply. Whether his theorem extends to vorticity in $L^\infty_t L^p_x$, $p < \infty$, is the question of Section 1.1.1 of [5]. This paper and Part II [8] prove that, without force in the plane, uniqueness fails among renormalized weak solutions with vorticity in $L^\infty_t(L^1 \cap L^q)_x$, for every $q \in (4/3, \infty)$ at once and for an explicit radial datum. The velocity of our datum is not square integrable (see below), whereas the planar theorem of [1] assumes it is, and the question of [5] is posed on $\mathbb{T}^2$.

Uniqueness is known in several classes of unbounded vorticity. Yudovich proved in Theorem 1 of [24], in a bounded domain D, uniqueness among solutions whose vorticity lies in every $L^p(D)$ with $p < \infty$, with $\|\omega(t)\|_{L^p(D)}$ bounded, for all large p, by a constant multiple of a growth function of p satisfying a divergence condition of Osgood type. Taniuchi introduced uniformly local versions of these spaces on $\mathbb{R}^2$ and constructed global solutions in them [18]; uniqueness there under the same kind of divergence condition is proved by Chen, Miao and Zheng [7] and stated as Theorem 1.4 of [9], both crediting Taniuchi, and Crippa and Stefani gave a Lagrangian proof of uniqueness in localized Yudovich spaces (Theorem 1.6 of [9]). Vishik proved uniqueness in Besov-type spaces containing the local space bmo, under an additional integrability condition [19]. Bernicot and Keraani [3], Bernicot and Hmidi [2], and Chen, Miao and Zheng [7] gave spaces of unbounded vorticity with global existence, uniqueness or propagation of the class; Elgindi, Murray and Said [12] gave a local-in-time existence and uniqueness class whose solutions can leave it in finite time. Our datum satisfies none of these uniqueness criteria: its $L^p(\mathbb{R}^2)$ norms grow exponentially in p, it is not in BMO, its dyadic blocks grow too fast for the spaces of [19], and it is not odd in the angle. We leave the verification of these facts to the interested reader. So Theorem 1.1 contradicts none of these results.

With a radial force, Vishik [20], [21] proved non-uniqueness with vorticity in an $L^p(\mathbb{R}^2)$ space, breaking the radial symmetry. In the account of Albritton, Bruè, Colombo, De Lellis, Giri, Janisch and Kwon [1], for every $p \in (2, \infty)$ there are a datum in $(L^1 \cap L^p)(\mathbb{R}^2)$ (possibly zero) with velocity in $L^2(\mathbb{R}^2)$ and a force in $L^1_t(L^1 \cap L^p)_x$ whose Biot-Savart velocity lies in $L^1_t L^2_x$, with uncountably many solutions with vorticity in $L^\infty_t(L^1 \cap L^p)_x$ and velocity in $L^\infty_t L^2_x$. The unstable manifold of a steady vortex gives non-uniqueness only at time $-\infty$; Vishik's self-similar background moves time $-\infty$ to time 0, but it is not an exact solution and requires the force (Section 0.1 of [1]). Without force, convex integration gives non-uniqueness for vorticity in $L^p(\mathbb{R}^2)$ and $L^p(\mathbb{T}^2)$: Mengual [15] showed, for $2 < p < \infty$, that a truncated power-law vortex, a steady solution with vorticity in $(L^1 \cap L^p)(\mathbb{R}^2)$, is the datum of infinitely many bounded admissible weak solutions, with velocity in $C_t L^2_x$. In that result only the initial vorticity lies in L^p: at positive times the vorticity of these solutions is known only as a distribution inside a growing disk, so they are not known to lie in the class $L^\infty_t L^p_x$ of the question above. Both solutions of Theorem 1.1, in contrast, have vorticity in $L^\infty_t(L^1 \cap L^q)_x$ for every $q \in (4/3, \infty)$, so Theorem 1.1 answers that question in the plane, in the class in which it is posed. Bruè, Colombo and Kumar (Theorem 1.2 of [5]) obtained, on $\mathbb{T}^2$, for some $p > 1$ and a dense set of initial velocities, infinitely many weak solutions with vorticity in $C([0,1]; L^p(\mathbb{T}^2))$, which do not conserve energy. Outside the L^p scale, convex integration also gives unforced non-uniqueness: [4] ($L^{1,\infty}$ vorticity, bounded energy) and [6] (Hardy spaces $H^p(\mathbb{R}^2)$, $2/3 < p < 1$).

In an instability construction without force, the background must be an exact solution. Ours is radial, hence stationary: infinitely many rescaled copies of one unstable vortex ring on disjoint annuli accumulating at the origin, with amplitudes increasing toward it. Inner rings rotate faster, and the task is to carry an instability from an inner ring outward through every ring in a time independent of the depth, uniformly in the number of interacting rings.

For $T > 0$ and $1 \le q \le \infty$ we write $X_q(T)$ for the space $L^\infty_t(L^1 \cap L^q)_x$ of functions on

$(0,T)\times\mathbb{R}^2$. Renormalized weak solutions in $X_q(T)$, $q>4/3$ (after DiPerna and Lions [10]), and equimeasurability are defined in Definitions 2.5 and 2.4. The ring Φ of Definition 4.1 is a closed formula in $\log|y|$ times a fixed smooth cutoff; it is radial, $\Phi\in C_c^\infty(\mathbb{R}^2)$, $0\le\Phi<1$, $\Phi\not\equiv 0$, and $\operatorname{supp}\Phi\subset\{a\le|y|\le 1\}$ for an explicit $a\in(0,1)$ (Lemma 4.2). For $A_1>0$, $\mu\in(1,2]$ and $\bar s\in(0,a/16]$ we put

$$\begin{gathered} f=\sum_{k\ge1}A_k\,\Phi(\cdot/r_k),\qquad A_k=A_1\mu^{k-1},\\ r_1=1,\qquad r_{k+1}=s_kr_k,\qquad s_k=\bar s(2\mu)^{-k}\qquad(k\ge1). \end{gathered}\tag{2}$$

The main theorem of the two papers is the following.

Theorem 1.1. *Let Φ and a be as in Definition 4.1. There are $\mu\in(1,\frac{17}{16}]$ and $\bar s\in(0,a/16]$ such that, for every $A_1>0$, the function f of Equation (2) has the following properties. It is radial and nonnegative, $f\in(L^1\cap L^q)(\mathbb{R}^2)$ for every $q\in[1,\infty)$, and $f\notin L^\infty(\mathbb{R}^2)$. There is $T_\star\in(0,\infty)$ such that, for every $T>T_\star$ and every $q\in(4/3,\infty)$, Equation (1) with $\omega_0=f$ has two distinct renormalized weak solutions in $X_q(T)$, in the sense of Definition 2.5:*

(i) *the stationary solution $\omega(t)=f$, $t\in[0,T]$;*

(ii) *a solution ω_∞ such that $\omega_\infty\in C([0,\infty);L^p(\mathbb{R}^2))$ for every $p\in[1,\infty)$ and, for every $t\ge0$, $\omega_\infty(t)$ is equimeasurable with f and 3-fold rotationally symmetric.*

More precisely, there is $t_0\in[0,T_\star)$ such that ω_∞ and f are distinct elements of $X_q(T')$ for every $T'>t_0$ and every $q\in(4/3,\infty)$.

We do not treat $q\le4/3$. No symmetry is imposed on the class; the 3-fold rotational symmetry of ω_∞ comes from the construction, where it removes the angular modes $\ell\notin3\mathbb{Z}$, among them the modes $\ell=\pm1$ that contain the translations of a ring, and leaves the modes $\ell\in3\mathbb{Z}$, whose spectra we control under Hypothesis 4.17.

Theorem 1.1 concerns one explicit family of data. It says nothing about the kinetic energy or the vanishing viscosity limit. The velocity of f is not square integrable: by Lemma 3.6, since $\operatorname{supp}f$ lies in the closed unit disk, $|K_{BS}*f(x)|=(2\pi|x|)^{-1}\int f$ for almost every x with $|x|\ge1$, and $\int f>0$.

The proof of Theorem 1.1 is divided between this paper and Part II [8]. This paper proves a non-uniqueness criterion (Section 3) and the linear analysis of the ring (Section 4), and it uses no machine computation. Part II proves that (N1)–(N3) of Hypothesis 4.17 imply Theorem 1.1 (Proposition 1.2 there), by a nonlinear construction followed by the criterion of Section 3, and it proves (N1)–(N3) (Theorem 1.3 there): (N1) and (N2) with rigorous interval arithmetic and (N3) by hand, in its Appendix A. Its Appendix B gives the existence theory for the global smooth solutions that the proof uses.

Section 3 follows the compactness theory of DiPerna and Lions for sequences of renormalized solutions, in a self-contained form for vorticity in $L^\infty_t(L^1\cap L^2)_x$, and proves a compactness criterion for non-uniqueness (Theorem 3.8). The form of it that Part II uses is the following conditional non-uniqueness theorem (Corollary 3.9 and Remark 3.10).

Theorem 1.2 (Conditional non-uniqueness)**.** *Let f be radial with $f\in(L^1\cap L^q)(\mathbb{R}^2)$ for every $q\in[1,\infty)$. Suppose that there are data $f^{(n)}\to f$ in $(L^1\cap L^2)(\mathbb{R}^2)$ and, for each n, a 3-fold rotationally symmetric renormalized weak solution $\omega^{(n)}$ of Equation (1) on $[0,\infty)$ with datum $f^{(n)}$*

and vorticity in $L^\infty_t(L^1 \cap L^2)_x$, with the following property: for some integer $m \geq 1$, some real $\chi \in C^\infty_c((0,\infty))$, some $c_0 > 0$ and $\bar{T} \in (0,\infty)$, and some times $T^{(n)} \in [0,\bar{T}]$,

$$\Big| \int_{\mathbb{R}^2} \omega^{(n)}(T^{(n)},x)\,\chi(|x|)\,(\bar{x})^m \mathrm{d}x \Big| \geq c_0 \qquad \textit{for all large } n,$$

with x written as a complex number. Then Equation (1) with $\omega_0 = f$ has, besides the stationary solution $\omega(t) = f$, a renormalized weak solution ω_∞ with vorticity in $L^\infty_t(L^1 \cap L^2)_x$ that is 3-fold rotationally symmetric and equimeasurable with f at every time and lies in $C([0,\infty); L^q(\mathbb{R}^2))$ for every $q \in [1,\infty)$. For some $t_0 \in [0,\bar{T})$, ω_∞ and f are distinct elements of $X_q(T')$ for every $T' > t_0$ and every $q \in (4/3,\infty)$.

Since the moment in Theorem 1.2 vanishes for the radial function f, its hypothesis says that the approximate solutions leave f by a fixed amount, in one non-radial moment, within a time bounded independently of n. The second solution is a limit of a subsequence of the $\omega^{(n)}$.

For the datum f of Theorem 1.1, Part II takes for $f^{(n)}$ the partial sum of the first n rings of f, with the innermost ring replaced by a seeded ring: a 3-fold rotationally symmetric copy of Φ, equimeasurable with it and close to it, that carries a small component along the unstable eigenfunction of Theorem 1.3 below. Figure 1 shows, for $n = 10$, the partial sum $f_{10} = \sum_{k=1}^{10} A_k \Phi(\cdot/r_k)$ of the actual datum f from which non-uniqueness occurs; $f^{(10)}$ differs from it only by the seed, which is invisible at this scale.

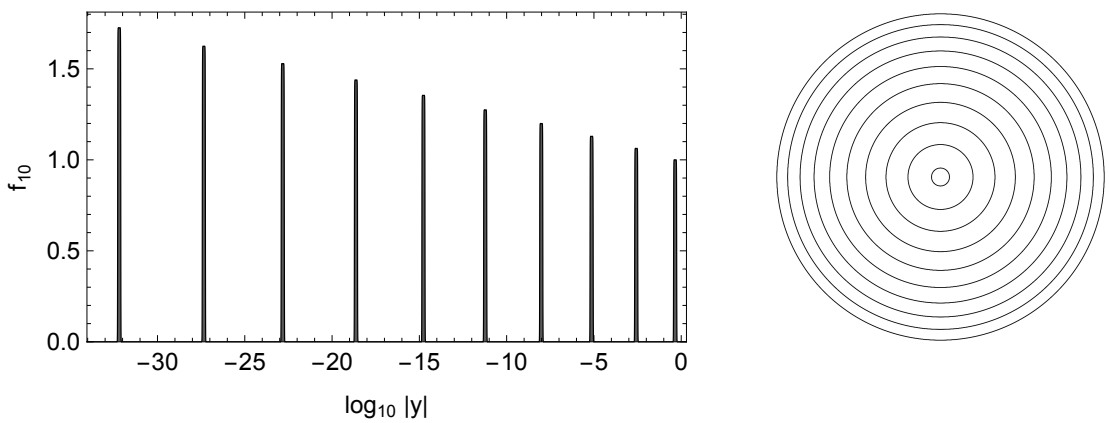

Figure 1: The partial sum $f_{10} = \sum_{k=1}^{10} A_k \Phi(\cdot/r_k)$ of the datum f of Equation (2), for the illustrative values $A_1 = 1$, $\mu = \frac{17}{16}$ and $\bar{s} = a/16$ (Theorem 1.1 uses some $\mu \in (1, \frac{17}{16}]$ and $\bar{s} \in (0, a/16]$ that are not computed; a smaller $\bar{s}$ separates the scales further), and for the function χ constructed in the proof of Lemma 4.2. Left: f_{10} against $\log_{10}|y|$; the rings sit at the scales $r_1 = 1 > r_2 > \cdots > r_{10} \approx 10^{-32}$, and their amplitudes $A_k = \mu^{k-1}$ grow toward the origin. Right: the same ten rings in a logarithmic polar picture, in which the distance from the center is an affine function of $\log|y|$, so that all of them are visible; each ring is drawn without its amplitude, darker where $\Phi(\cdot/r_k)$ is larger. Drawn with Mathematica from the exact formulas; the figure is an illustration and is not used in any proof.

The proof of Theorem 1.1 uses three explicit statements, collected in Section 4 as Hypothesis 4.17: (N1) and (N2) locate the zeros in the mode $\ell = 3$ of the Evans function of the ring, an explicit holomorphic function, for every $G \in [0, \frac{1}{50}]$, where $2\pi G$ is the circulation of a point vortex at the origin in the reference units of Definition 4.1; (N3) is an inequality between explicit numbers controlling the modes $\ell = \pm 6$ and $|\ell| \geq 9$. Apart from the thresholds stated in (N1)–(N3), the only

explicit numbers that the computer-assisted proof of these statements uses are the constants of the ring in Definition 4.1, which fix the building block of the multiscale datum; all other constants of this paper are used only through their existence. Without this hypothesis, Section 4 decomposes the linearization into angular modes (Lemma 4.5), reduces each mode to a linear second-order equation in $\xi = \log r$, and defines the Evans function of each mode $\ell \geq 1$ (Definition 4.4). The Evans function is, up to sign, the Wronskian of the solutions of this equation that decay at the two ends (Lemma 4.6), and its zeros in $\{\operatorname{Re}\lambda > 0\}$ are exactly the eigenvalues there (Lemma 4.7). A simple zero of the Evans function gives an algebraically simple eigenvalue with an explicit Riesz projection of rank one (Theorem 4.8). Section 4 also bounds the real part of every unstable eigenvalue of the mode ℓ, for a radial vorticity that increases and then decreases in r, by half the square root of an explicit double integral of the radial derivative of the vorticity over the region where it increases and the region where it decreases, which is nonincreasing in ℓ (Lemma 4.10). For the ring this integral is computed in closed form (Lemma 4.11), and (N3) makes the bound smaller than $\frac{9}{200}$ for $\ell = \pm 6$ and $|\ell| \geq 9$. The section also proves a semigroup bound, uniform in G, for all sufficiently large $|\ell|$ (Proposition 4.15), and constructs the semigroup of the linearization (Lemma 4.14).

The main result of the linear analysis is the linear instability of the ring, in the ring units of Definition 4.1, in the presence of a point vortex of small circulation $2\pi G$ at the origin, which stands for everything inside the ring. For an integer $s \geq 0$ let X^s be the space of 3-fold rotationally symmetric $w \in H^s(\mathbb{R}^2;\mathbb{C})$ supported in the annulus $\{\frac{a}{2} \leq |y| \leq 2\}$.

Theorem 1.3 (Linear instability of the ring)**.** *Assume* (N1)–(N3) *of Hypothesis 4.17, which Part II proves. For $G \geq 0$ let*

$$L(G)w = -\Big(K_{BS} * \Phi + \frac{G\, y^{\perp}}{|y|^2}\Big) \cdot \nabla w - (K_{BS} * w) \cdot \nabla \Phi$$

be the linearization of Equation (1) *at the steady state Φ plus a point vortex of circulation $2\pi G$ at the origin. Let $\overline{G} \approx 0.006024$ be the constant of Definition 4.1. Then, for every $G \in [0, \overline{G}]$ and every integer $s \geq 0$:*

(i) *$L(G)$, defined on X^{s+1}, extends to the generator of a C_0-semigroup $e^{tL(G)}$ on X^s. Its spectrum in $\{\operatorname{Re}\lambda > \frac{9}{200}\}$ is $\{\lambda_*(G), \overline{\lambda_*(G)}\}$, with $\operatorname{Re}\lambda_*(G) > \frac{211}{2500}$. The eigenvalue $\lambda_*(G)$ has the eigenfunction $\phi_*(G) = g(|y|)e^{3i\theta}$, with g smooth and supported in $[a, 1]$ and $\|\phi_*(G)\|_{L^2(\mathbb{R}^2)} = 1$, and it is algebraically simple on the functions in X^s of angular mode 3; $\overline{\lambda_*(G)}$ has the eigenfunction $\overline{\phi_*(G)}$, of angular mode -3.*

(ii) *$\lambda_*(G)$ depends continuously on G.*

(iii) *There is C, depending only on s, such that $\|e^{tL(G)}\|_{X^s \to X^s} \leq Ce^{\operatorname{Re}\lambda_*(G)\, t}$ for $t \geq 0$. For every $\sigma' \in (\frac{9}{200}, \min_{G'} \operatorname{Re}\lambda_*(G'))$ there is C', depending only on s and σ', such that $\|e^{tL(G)}(1 - \Pi(G))\|_{X^s \to X^s} \leq C' e^{\sigma' t}$ for $t \geq 0$, where $\Pi(G)$ is the spectral projection of $L(G)$ for $\{\lambda_*(G), \overline{\lambda_*(G)}\}$, whose range is spanned by $\phi_*(G)$ and $\overline{\phi_*(G)}$.*

(iv) *The moment $\int_{\mathbb{R}^2} \phi_*(G)(y)\, (\bar{y})^3 \mathrm{d}y$, with y written as a complex number, has modulus at least a constant $c > 0$ independent of G.*

Theorem 1.3 collects Proposition 4.19, Lemma 4.7(2), Propositions 4.21 and 4.22, and Lemma 4.23. In Section 4 these are stated with the precise eigendata that Part II uses (Definition 4.20).

Part II uses these conclusions in a nonlinear construction, which we describe heuristically. Each ring of f sees the rings inside it essentially as a point vortex of small circulation, so its linearization is

the operator $L(G)$ of Theorem 1.3 for a small G, and a perturbation along its unstable eigenfunction grows at a rate proportional to the amplitude of the ring: inner rings are faster. An approximation of f that keeps finitely many rings carries a small multiple of the unstable eigenfunction on its innermost ring. This perturbation grows until the ring saturates; the non-radial moment of the saturated ring then kicks the next ring outward along its unstable eigenfunction, and that ring grows and saturates in turn. Ring k needs a time of order k/A_k to saturate, and A_k grows geometrically toward the origin, so this relay reaches the outermost ring in a time bounded independently of the number of rings, and one bootstrap keeps every ring close to its expected evolution. This is why the datum is multiscale. A single unstable ring would not suffice: a perturbation of size ε of a steady state needs a time of order $\log(1/\varepsilon)$ to reach a fixed size, which tends to infinity as $\varepsilon \to 0$, so approximations that converge to the steady datum would not leave it in a bounded time; this is the reason why the unstable manifold of a steady vortex gives non-uniqueness only at time $-\infty$. In our datum the approximation differs from f only at the scale of its innermost ring, where the seed has a fixed size relative to the ring; it is small in $(L^1 \cap L^q)(\mathbb{R}^2)$ only because that ring is small. So the instability starts at order one on the time scale of that ring, which tends to 0 with the depth, and the relay carries it to the outermost ring in a bounded time. The instability is what produces a non-radial moment bounded away from zero. At saturation the perturbation of the outermost ring is, to leading order, a multiple of fixed size of its unstable eigenfunction and its conjugate, which lie in the angular modes 3 and -3. The moment $\int \omega\Psi$ with $\Psi(x) = \chi(|x|)(\bar{x})^3$, where $\chi = 1$ on the annulus of the ring, sees only the angular mode 3, like p_3^{out} (Definition 2.3), and on the unstable eigenfunction it is the moment of Theorem 1.3(iv), which is bounded away from zero uniformly in G. So this moment of the outermost ring has modulus at least a constant $c_0 > 0$ independent of the number of rings, while it vanishes for the radial datum f. This is the hypothesis of Theorem 1.2, which gives the second solution as a limit of the approximate solutions. Section 1.2 of Part II lists the definitions and results of this paper that it uses.

The main idea of a linear instability in physical coordinates driving the non-uniqueness mechanism, the use of non-monotone unstable radial vortices and of a multiscale datum, and the overall strategy of the proof are due to the author. Within this strategy, the detailed proofs and estimates, the code for the computer-assisted verification in Part II and the text were largely produced with Claude Opus 5.5 (Anthropic), under the direction of the author. The author has checked all proofs by hand and takes full responsibility for the paper.

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2 Preliminaries

2.1 Conventions

We identify $\mathbb{R}^2$ with $\mathbb{C}$, writing $x = (x_1, x_2) = x_1 + ix_2$ when convenient, and $\bar{x}$ for the complex conjugate; $|x|$ is the Euclidean norm; for a matrix or an array of numbers, $|\cdot|$ is the Euclidean norm of the array of its entries, and for a matrix M we write $\|M\|_{\mathrm{op}}$ for its operator norm. For $r > 0$, $B_r = \{|x| < r\}$ and $\overline{B}_r$ is its closure; $|S|$ is the Lebesgue measure of a measurable set $S \subset \mathbb{R}^2$. The overline thus denotes complex conjugation, of numbers and of functions, and the closure of a set; it is also part of the names of some fixed objects, such as the radial background $\overline{\omega}$ of Section 4 and the constants $\overline{G}$, $\bar{s}$ and $\bar{T}$. Which meaning is intended will always be clear from the context. We use

polar coordinates $x = r(\cos\theta, \sin\theta)$, with $\mathbf{e}_r = (\cos\theta, \sin\theta)$ and $\mathbf{e}_\theta = (-\sin\theta, \cos\theta)$. For $\theta \in \mathbb{R}$, O_θ is the rotation of $\mathbb{R}^2$ by the angle θ (multiplication by $e^{i\theta}$), and $\mathsf{O} = \mathsf{O}_{2\pi/3}$. For an integer $m \geq 1$, a function g on $\mathbb{R}^2$ is *m-fold rotationally symmetric* if and only if $g \circ \mathsf{O}_{2\pi/m} = g$ almost everywhere, and *radial* if and only if $g \circ \mathsf{O}_\theta = g$ almost everywhere for all θ. If $\theta \mapsto g(re^{i\theta})$ is integrable on $[0, 2\pi]$ for almost every $r > 0$ (by Fubini's theorem in polar coordinates, this holds for every $g \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$) and $\ell \in \mathbb{Z}$, the **angular mode** ℓ of g is $g_\ell(r) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})e^{-i\ell\theta}\mathrm{d}\theta$, defined for almost every $r > 0$. If $g \in L^2(\mathbb{R}^2)$, then $g = \sum_{\ell\in\mathbb{Z}} g_\ell(r)e^{i\ell\theta}$ in $L^2(\mathbb{R}^2)$, and g is m-fold rotationally symmetric if and only if $g_\ell = 0$ for all $\ell \notin m\mathbb{Z}$.

The Biot-Savart kernel is $K_{BS}(x) = x^\perp/(2\pi|x|^2)$ with $x^\perp = (-x_2, x_1)$, and $\nabla^\perp = (-\partial_2, \partial_1)$. For a vorticity ω we write $v = K_{BS} * \omega$ whenever the integral converges. If $\omega \in C_c^\infty(\mathbb{R}^2)$, then the **stream function** $\psi(x) = \frac{1}{2\pi}\int \log|x-y|\,\omega(y)\mathrm{d}y$ satisfies $v = \nabla^\perp\psi$, $\Delta\psi = \omega$ and $\partial_1 v_2 - \partial_2 v_1 = \omega$ (Lemma 2.1(d) below). In polar coordinates these sign conventions read $v \cdot \mathbf{e}_r = -r^{-1}\partial_\theta\psi$ and $v \cdot \mathbf{e}_\theta = \partial_r\psi$. For a vector field u we write ∇u for the matrix $(\partial_l u_i)_{i,l=1,2}$, so that $\operatorname{tr}\nabla u = \operatorname{div} u$.

Functions are complex valued unless we say otherwise; vorticities and velocities are real. For a multi-index $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}_0^2$ we write $\partial^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}$, $|\gamma| = \gamma_1 + \gamma_2$, $\zeta^{2\gamma} = \zeta_1^{2\gamma_1}\zeta_2^{2\gamma_2}$ for $\zeta \in \mathbb{R}^2$, $\beta \leq \gamma$ if $\beta_1 \leq \gamma_1$ and $\beta_2 \leq \gamma_2$, and $\binom{\gamma}{\beta} = \binom{\gamma_1}{\beta_1}\binom{\gamma_2}{\beta_2}$; here $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $m\mathbb{N} = \{m, 2m, \dots\}$. For an integer $k \geq 0$ and an open set $V \subset \mathbb{R}^2$, $H^k(V)$ is the Sobolev space with norm $\|g\|^2_{H^k(V)} = \sum_{|\gamma|\leq k}\|\partial^\gamma g\|^2_{L^2(V)}$, and $H^k = H^k(\mathbb{R}^2)$. Sobolev norms of vector and matrix fields are summed over the entries; for instance $\|\nabla u\|^2_{H^k} = \sum_{i,l}\|\partial_l u_i\|^2_{H^k}$. For a vector or matrix field F on V and $1 \leq q \leq \infty$ we put $\|F\|_{L^q(V)} = \|\,|F|\,\|_{L^q(V)}$, where $|F|$ is the Euclidean norm of the array of entries; for $q = 2$ this agrees with the sum over the entries. For a vector field F on V we put $\|F\|_{W^{1,\infty}(V)} = \operatorname{ess\,sup}_V |F| + \operatorname{ess\,sup}_V |\nabla F|$.

For a function or vector field h of class C^q near a point x, we write $\nabla^q h(x)$ for the array of all partial derivatives of order q of the components of h at x, one entry for each ordered q-tuple of directions, so that $|\nabla^q h(x)|$ is its Euclidean norm. For an integer $k \geq 0$, a closed set $U \subset \mathbb{R}^2$ and h of class C^k on a neighborhood of U, we put $\|h\|_{C^k(U)} = \sum_{q=0}^k \sup_{x\in U}|\nabla^q h(x)|$; in particular $|\partial^\gamma h(x)| \leq \|h\|_{C^k(U)}$ for $x \in U$ and $|\gamma| \leq k$, and $C^k(U)$ denotes the space of such h with finite norm. The Fourier transform is $\hat{g}(\zeta) = \int g(x)e^{-ix\cdot\zeta}\mathrm{d}x$, so that $\|\partial^\gamma g\|^2_{L^2(\mathbb{R}^2)} = (2\pi)^{-2}\int \zeta^{2\gamma}|\hat{g}(\zeta)|^2\mathrm{d}\zeta$ by Plancherel's theorem.

We put $\langle t\rangle = (1 + t^2)^{1/2}$ for $t \in \mathbb{R}$ and $\log_+ x = \max\{\log x, 0\}$ for $x > 0$. For $x > 0$ and an integer $p \geq 0$ we put

$$I_p(x) = \int_0^\infty e^{-xu}(1+u)^p\mathrm{d}u, \tag{3}$$

a finite (as $x > 0$), decreasing function of x.

Constants that enter a later choice of parameters carry their own names, and their dependence is stated where they are defined. The letter C denotes a constant that enters no later choice; its dependence is stated each time, and within one proof its value may change from line to line. For a function u and a nonnegative function g, $u = O(g)$ as $r \to r_0$ means that $|u| \leq Cg$ in a neighborhood of r_0 for some constant C.

Let I be an interval and X a Banach space. A function $F\colon I \to X$ is *strongly measurable* if it is an almost everywhere limit of simple functions $\sum_{i=1}^k \mathbf{1}_{E_i}(t)x_i$, with $E_i \subset I$ measurable and $x_i \in X$, and *weakly measurable* if $t \mapsto \Lambda(F(t))$ is measurable for every $\Lambda \in X^*$. We use Pettis's theorem [17]: if X is separable, then F is strongly measurable if and only if it is weakly measurable. The Bochner spaces $L^\infty(I; X)$ and $L^1(I; X)$ consist of the strongly measurable F for which $t \mapsto \|F(t)\|_X$ is essentially bounded, respectively integrable, modulo equality almost everywhere.

Let $\Omega \subset \mathbb{R}^2$ be a measurable set. For $b \in [1, \infty)$, $L^\infty(I; L^b(\Omega))$ and $L^1(I; L^b(\Omega))$ are the Bochner spaces, whose elements we identify with measurable functions on $I \times \Omega$; a measurable u on $I \times \Omega$

lies in them if and only if $t \mapsto \|u(t)\|_{L^b(\Omega)}$ is essentially bounded, respectively integrable, by Pettis's theorem: $L^b(\Omega)$ is separable, and for $\phi \in L^{b'}(\Omega)$, $1/b+1/b' = 1$, the function $t \mapsto \int u(t)\phi$, defined for almost every t, is measurable by Tonelli's theorem applied to the positive and negative parts of the real and imaginary parts of $u\phi$. We write $L^\infty(I; L^\infty(\Omega))$ for $L^\infty(I \times \Omega)$, and $L^1(I; L^\infty(\Omega))$ for the set of measurable u on $I \times \Omega$ with $\int_I \|u(t)\|_{L^\infty(\Omega)} \mathrm{d}t < \infty$. Intersections such as $L^\infty(I; (L^1 \cap L^\infty)(\Omega))$ are intersections of these spaces; a space with $W^{1,a}(\Omega)$ in place of $L^b(\Omega)$ consists of the w for which w and its distributional gradient in x lie in the corresponding space with $L^a(\Omega)$; the subscript loc on the time interval means that the condition holds on $J \times \Omega$ for every compact interval $J \subset I$, on $\mathbb{R}^2$ that it holds on $I \times B_R$ for every $R > 0$, and on both that it holds on $J \times B_R$ for all such J and R.

For $T > 0$ and $1 \le q \le \infty$, the space $X_q(T)$ of Section 1, written $L^\infty_t(L^1 \cap L^q)_x$ there, is the Bochner space $L^\infty\big(0, T; (L^1 \cap L^q)(\mathbb{R}^2)\big)$ just described, with $\|\omega\|_{(L^1\cap L^q)(\mathbb{R}^2)} = \|\omega\|_{L^1(\mathbb{R}^2)} + \|\omega\|_{L^q(\mathbb{R}^2)}$; and $\mathcal{D}_T = \{\varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^2) : \operatorname{supp}\varphi \subset (-\infty, T) \times \mathbb{R}^2\}$. Lip_0 is the set of Lipschitz functions $\Upsilon\colon \mathbb{R} \to \mathbb{R}$ with $\Upsilon(0) = 0$. For $g \in L^1(\mathbb{R}^2)$ we put $\Gamma_g(r) = \int_{B_r} g$.

A C_0-semigroup $(S(t))_{t\ge 0}$ on a Banach space X is a family of bounded operators with $S(0)$ the identity, $S(t+s) = S(t)S(s)$, and $t \mapsto S(t)x$ continuous for all $x \in X$; its generator A is $Ax = \lim_{h\downarrow 0} h^{-1}(S(h)x - x)$ on the set $D(A)$ of x for which the limit exists.

2.2 The Biot-Savart Law

We split $K_{BS} = K^{\mathrm{nr}}_{BS} + K^{\mathrm{fr}}_{BS}$ with $K^{\mathrm{nr}}_{BS} = \mathbf{1}_{B_1} K_{BS}$. By Lemma 2.1(a) below, $\|K^{\mathrm{nr}}_{BS}\|^s_{L^s(\mathbb{R}^2)} = (2\pi)^{1-s}/(2-s)$ for $s \in [1, 2)$, and $|K^{\mathrm{fr}}_{BS}(z)| \le \min\{1, |z|^{-1}\}/(2\pi)$ for $z \in \mathbb{R}^2$.

Lemma 2.1. *The kernel K_{BS} has the following properties.*

(a) *For $z \ne 0$, $i, l \in \{1, 2\}$ and $\rho > 0$, $|K_{BS}(z)| = (2\pi|z|)^{-1}$, $|\partial_l (K_{BS})_i(z)| \le (2\pi|z|^2)^{-1}$ and $\int_{B_\rho} |K_{BS}| \mathrm{d}z = \rho$.*

(b) *Let $g \in (L^1 \cap L^\infty)(\mathbb{R}^2)$. Then for every $x \in \mathbb{R}^2$,*

$$\big(|K_{BS}| * |g|\big)(x) \le \Big(\frac{2\|g\|_{L^1(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R}^2)}}{\pi}\Big)^{1/2}.$$

In particular the integral defining $K_{BS} * g(x)$ converges absolutely for every x, and $\|K_{BS} * g\|_{L^\infty(\mathbb{R}^2)} \le \big(2\|g\|_{L^1(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R}^2)}/\pi\big)^{1/2}$.

(c) *Let $\rho, R > 0$ and let $g \in L^2(\mathbb{R}^2)$ vanish outside B_ρ. Then $K_{BS} * g$ is defined almost everywhere, and $\|K_{BS} * g\|_{L^2(B_R)} \le (R + \rho)\|g\|_{L^2(\mathbb{R}^2)}$.*

(d) *Let $g \in C^\infty_c(\mathbb{R}^2)$ and $u = K_{BS} * g$. Then $u \in C^\infty(\mathbb{R}^2)$, $\partial^\gamma u = K_{BS} * \partial^\gamma g$ for every γ, $u = \nabla^\perp \psi$ with $\psi(x) = \frac{1}{2\pi}\int \log|x - y|\, g(y)\mathrm{d}y$, $\Delta\psi = g$, $\operatorname{div} u = 0$ and $\partial_1 u_2 - \partial_2 u_1 = g$. Moreover $\sum_{i,l=1}^2 \|\partial_l \partial^\gamma u_i\|^2_{L^2(\mathbb{R}^2)} = \|\partial^\gamma g\|^2_{L^2(\mathbb{R}^2)}$ for every γ, and hence $\|\nabla u\|_{H^k} = \|g\|_{H^k}$ for every integer $k \ge 0$.*

(e) *Let $g \in L^1(\mathbb{R}^2)$ and $R > 0$. Then $\int_{B_R} \big(|K_{BS}| * |g|\big)\mathrm{d}x \le R\|g\|_{L^1(\mathbb{R}^2)}$. In particular the integral defining $K_{BS} * g$ converges absolutely almost everywhere, $K_{BS} * g \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, and $\operatorname{div}(K_{BS} * g) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.*

Proof. (a) The first identity holds because $|x^\perp| = |x|$, and integration in polar coordinates gives $\int_{B_\rho} (2\pi|z|)^{-1}\mathrm{d}z = \rho$. Since $2\pi K_{BS}(z) = (-z_2, z_1)/|z|^2$, the derivatives $2\pi\partial_l(K_{BS})_i$ are, up to sign,

the numbers $\partial_l(z_k/|z|^2) = \delta_{lk}|z|^{-2} - 2z_l z_k|z|^{-4}$, whose modulus is $|z_{k'}^2 - z_k^2|/|z|^4 \le |z|^{-2}$ if $l = k$ (where $k' \neq k$) and $2|z_1 z_2|/|z|^4 \le |z|^{-2}$ if $l \neq k$.

(b) We may assume $g \neq 0$. For arbitrary $r > 0$, (a) gives, after splitting the integral into the near field $\{|z| < r\}$ and the far field $\{|z| \ge r\}$,

$$\begin{aligned} 2\pi\big(|K_{BS}| * |g|\big)(x) &\le \|g\|_{L^\infty(\mathbb{R}^2)} \int_{B_r} \frac{\mathrm{d}z}{|z|} + \frac{1}{r}\int_{|z|\ge r} |g(x-z)|\mathrm{d}z \\ &\le 2\pi r\|g\|_{L^\infty(\mathbb{R}^2)} + \frac{\|g\|_{L^1(\mathbb{R}^2)}}{r}, \end{aligned}$$

and choosing $r = (\|g\|_{L^1(\mathbb{R}^2)}/(2\pi\|g\|_{L^\infty(\mathbb{R}^2)}))^{1/2}$ gives

$$2\pi(|K_{BS}| * |g|)(x) \le 2(2\pi\|g\|_{L^1(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R}^2)})^{1/2},$$

which is the claim. The rest follows from $|K_{BS} * g| \le |K_{BS}| * |g|$.

(c) For $|x| < R$ and $|x - z| < \rho$, we have $|z| < \rho + R$, so, as g vanishes outside B_ρ, $K_{BS} * g(x) = \int_{B_{R+\rho}} K_{BS}(z)g(x-z)\mathrm{d}z$ for $x \in B_R$, and Minkowski's integral inequality and (a) give $\big\| \int_{B_{R+\rho}} |K_{BS}(z)||g(\cdot - z)|\mathrm{d}z\big\|_{L^2(B_R)} \le (R+\rho)\|g\|_{L^2(\mathbb{R}^2)}$. In particular the integral converges absolutely almost everywhere on B_R, for every $R > 0$.

(d) Let $\operatorname{supp} g \subset B_\rho$. Since K_{BS} and $\log|\cdot|$ are locally integrable, differentiation under the integral sign shows that $u, \psi \in C^\infty$, $\partial^\gamma u = K_{BS} * \partial^\gamma g$ and $\partial^\gamma \psi = \frac{1}{2\pi}\log|\cdot| * \partial^\gamma g$. Writing $\partial_j g(x-z) = -\partial_{z_j}[g(x-z)]$ and integrating by parts on $\{|z| > \varepsilon\}$, with a boundary term at most $\varepsilon|\log\varepsilon|\sup|g| \to 0$, we get $\partial_j\psi(x) = \int \frac{z_j}{2\pi|z|^2} g(x-z)\mathrm{d}z$. Hence $u = \nabla^\perp\psi$, so $\operatorname{div} u = -\partial_1\partial_2\psi + \partial_2\partial_1\psi = 0$ and $\Delta\psi = \partial_1 u_2 - \partial_2 u_1$. Fix x, put $h(z) = g(x-z)$, and use polar coordinates $z = re^{i\theta}$, in which $z \cdot \nabla_z = r\partial_r$. Since $\partial_l u = K_{BS} * \partial_l g$ and $(\partial_l g)(x-z) = -\partial_{z_l} h(z)$,

$$(\partial_1 u_2 - \partial_2 u_1)(x) = -\int \frac{z \cdot \nabla_z h}{2\pi|z|^2}\mathrm{d}z = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \partial_r h \mathrm{d}r\mathrm{d}\theta = g(x),$$

because $(K_{BS})_2\partial_1 h - (K_{BS})_1\partial_2 h = z \cdot \nabla_z h/(2\pi|z|^2)$ and $\int_0^\infty \partial_r h\mathrm{d}r = -h(0) = -g(x)$; the integrand is integrable, as $|\partial_r h| \le |\nabla h|$ and h has compact support. Hence $\Delta\psi = g$.

For $|x| \ge 2\rho$ and $|y| < \rho$ we have $|x - y| \ge |x|/2$; differentiating under the integral sign, with $|\nabla \log|z|| = |z|^{-1}$ and $|\partial_i\partial_j \log|z|| = |\partial_j(z_i/|z|^2)| \le |z|^{-2}$ (proof of (a)), gives $|\nabla\psi(x)| \le \|g\|_{L^1(\mathbb{R}^2)}/(\pi|x|)$ and $|\partial_i\partial_j\psi(x)| \le 2\|g\|_{L^1(\mathbb{R}^2)}/(\pi|x|^2)$ for $|x| \ge 2\rho$. Two integrations by parts on B_r give $\int_{B_r}\sum_{i,j}(\partial_i\partial_j\psi)^2\mathrm{d}x = \int_{B_r}(\Delta\psi)^2\mathrm{d}x + \mathcal{B}_r$, where $\mathcal{B}_r$ is a sum of integrals over ∂B_r of products of a first and a second derivative of ψ; for $r \ge 2\rho$ the bounds above give $|\mathcal{B}_r| \le Cr \cdot r^{-1}r^{-2}\|g\|_{L^1(\mathbb{R}^2)}^2$ with C absolute. Letting $r \to \infty$ gives $\sum_{i,j}\|\partial_i\partial_j\psi\|_{L^2(\mathbb{R}^2)}^2 = \|g\|_{L^2(\mathbb{R}^2)}^2$. The entries of ∇u are, up to sign, the second derivatives of ψ, so $\sum_{i,l}\|\partial_l u_i\|_{L^2(\mathbb{R}^2)}^2 = \|g\|_{L^2(\mathbb{R}^2)}^2$. Replacing g by $\partial^\gamma g$, whose velocity is $\partial^\gamma u$, and summing over $|\gamma| \le k$ gives the rest. For complex g we apply all of this to the real and imaginary parts.

(e) Let D be a disk of radius R. Then $|D \setminus B_R| = |B_R \setminus D|$. Since $|K_{BS}(z)| = (2\pi|z|)^{-1}$ by (a), $|K_{BS}| \le (2\pi R)^{-1}$ on $D \setminus B_R$ and $|K_{BS}| > (2\pi R)^{-1}$ on $B_R \setminus D$; so $\int_D |K_{BS}| \le \int_{B_R}|K_{BS}| = R$, by (a). For fixed y, the substitution $z = x - y$ maps B_R onto the disk $D_y = \{z : |z + y| < R\}$ of radius R, so Tonelli's theorem gives $\int_{B_R}(|K_{BS}| * |g|)\mathrm{d}x = \int |g(y)|\int_{D_y}|K_{BS}(z)|\mathrm{d}z\mathrm{d}y \le R\|g\|_{L^1(\mathbb{R}^2)}$. Hence $|K_{BS}| * |g|$ is finite almost everywhere, and $K_{BS} * g \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ because $|K_{BS} * g| \le |K_{BS}| * |g|$. Take $g_n \in C_c^\infty(\mathbb{R}^2)$ with $g_n \to g$ in $L^1(\mathbb{R}^2)$. This bound, applied to $g_n - g$, gives $K_{BS} * g_n \to K_{BS} * g$ in $L^1(B_R)$ for every $R > 0$, hence in $\mathcal{D}'(\mathbb{R}^2)$; since $\operatorname{div}(K_{BS} * g_n) = 0$ by (d), $\operatorname{div}(K_{BS} * g) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. $\square$

Lemma 2.2. *Let $4/3 < p < \infty$, $p' = p/(p-1)$, and $R > 0$. There is $C = C(p, R)$ such that the following hold.*

(a) *For $\omega \in (L^1 \cap L^p)(\mathbb{R}^2)$ the integral defining $K_{BS} * \omega$ converges absolutely almost everywhere and, with $v = K_{BS} * \omega$,*

$$\|v\|_{L^{p'}(B_R)} + \|v\|_{L^2(B_R)} \le C\|\omega\|_{(L^1\cap L^p)(\mathbb{R}^2)}, \qquad \int_{\mathbb{R}^2} |\omega| \left(|K_{BS}| * |\omega|\right) \mathrm{d}x \le C\|\omega\|^2_{(L^1\cap L^p)(\mathbb{R}^2)};$$

the constant in the second inequality can be chosen depending only on p, independently of R.

(b) *Let ω_n be real and $\omega_n \rightharpoonup \omega$ weakly in $L^p(\mathbb{R}^2)$, with $\sup_n \|\omega_n\|_{L^1(\mathbb{R}^2)} \le M$. Then $\|\omega\|_{L^1(\mathbb{R}^2)} \le M$ and $K_{BS} * \omega_n \to K_{BS} * \omega$ strongly in $L^{p'}(B_R)$.*

(c) *If $\omega \in (L^1 \cap L^2)(\mathbb{R}^2)$, then $v = K_{BS} * \omega$ satisfies $\nabla v \in L^2(\mathbb{R}^2)$ and $\|\nabla v\|_{L^2(\mathbb{R}^2)} = \|\omega\|_{L^2(\mathbb{R}^2)}$.*

Proof. (a) Put $r(p) = p'$ and $s(p) = p/(2p-2)$ if $p \le 2$, and $r(p) = p$ and $s(p) = 1$ if $p > 2$. Since $p > 4/3$, we have $s(p) \in [1, 2)$, so $K^{\mathrm{nr}}_{BS} \in L^{s(p)}(\mathbb{R}^2)$. Moreover $1 + 1/r(p) = 1/s(p) + 1/p$, so Young's inequality gives $\||K^{\mathrm{nr}}_{BS}| * |\omega|\|_{L^{r(p)}(\mathbb{R}^2)} \le \|K^{\mathrm{nr}}_{BS}\|_{L^{s(p)}(\mathbb{R}^2)}\|\omega\|_{L^p(\mathbb{R}^2)}$. Also $\||K^{\mathrm{fr}}_{BS}| * |\omega|\|_{L^\infty(\mathbb{R}^2)} \le \|\omega\|_{L^1(\mathbb{R}^2)}/(2\pi)$. Hence $|K_{BS}| * |\omega|$ is finite almost everywhere. Since $r(p) \ge \max\{2, p'\}$, Hölder's inequality on B_R gives the velocity bound. For the last bound, $r(p)' \in [1, p]$, so $\|\omega\|_{L^{r(p)'}(\mathbb{R}^2)} \le \|\omega\|_{(L^1\cap L^p)(\mathbb{R}^2)}$ by interpolation, and $\int |\omega|(|K_{BS}| * |\omega|) \le \|\omega\|_{L^{r(p)'}(\mathbb{R}^2)}\||K^{\mathrm{nr}}_{BS}| * |\omega|\|_{L^{r(p)}(\mathbb{R}^2)} + \|\omega\|^2_{L^1(\mathbb{R}^2)}/(2\pi)$.

(b) For $L > 0$ and $g \in L^\infty(B_L)$, the function $g\mathbf{1}_{B_L}$ lies in $L^{p'}(\mathbb{R}^2)$, so $|\int_{B_L} \omega g| = \lim_n |\int_{B_L} \omega_n g| \le M\|g\|_{L^\infty(\mathbb{R}^2)}$. The limit ω is real, as a weak limit of real functions. Taking $g = \operatorname{sgn}\omega$ (with $\operatorname{sgn} s = s/|s|$ for $s \ne 0$ and $\operatorname{sgn} 0 = 0$) and letting $L \to \infty$ gives $\|\omega\|_{L^1(\mathbb{R}^2)} \le M$.

By hypothesis and the bound just proved, ω_n and ω lie in $(L^1 \cap L^p)(\mathbb{R}^2)$, with $L^1(\mathbb{R}^2)$ norms at most M. Put $N = \sup_n \|\omega_n\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^p(\mathbb{R}^2)} < \infty$ (weakly convergent sequences are bounded), let $0 < \varepsilon < 1 < L$, and split $K_{BS} = K^{(1)} + K^{(2)} + K^{(3)}$ with $K^{(1)} = K_{BS}\mathbf{1}_{B_\varepsilon}$, $K^{(2)} = K_{BS}\mathbf{1}_{B_L\setminus B_\varepsilon}$ and $K^{(3)} = K_{BS}\mathbf{1}_{\mathbb{R}^2\setminus B_L}$. By Young's inequality, as in (a), and Hölder's inequality on B_R, $\|K^{(1)} * (\omega_n - \omega)\|_{L^{p'}(B_R)} \le |B_R|^{1/p'-1/r(p)}\|K^{(1)}\|_{L^{s(p)}(\mathbb{R}^2)}N$, and $\|K^{(1)}\|_{L^{s(p)}(\mathbb{R}^2)} \to 0$ as $\varepsilon \to 0$ because $s(p) < 2$. Next, $|K^{(3)} * (\omega_n - \omega)| \le 2M/(2\pi L)$ everywhere, since $\|\omega_n\|_{L^1(\mathbb{R}^2)}, \|\omega\|_{L^1(\mathbb{R}^2)} \le M$; so its $L^{p'}(B_R)$ norm is at most $|B_R|^{1/p'}M/(\pi L)$, uniformly in n. Finally, for $x \in B_R$ the function $K^{(2)}(x - \cdot)$ is bounded by $(2\pi\varepsilon)^{-1}$, since $|K_{BS}(z)| = (2\pi|z|)^{-1}$ and $|z| \ge \varepsilon$ on the support of $K^{(2)}$, and vanishes outside $\{y : |x - y| < L\}$; hence it lies in $L^{p'}(\mathbb{R}^2)$, so $K^{(2)} * \omega_n(x) \to K^{(2)} * \omega(x)$, with $|K^{(2)} * \omega_n(x)| \le \|K^{(2)}\|_{L^{p'}(\mathbb{R}^2)}N$; dominated convergence gives $K^{(2)} * \omega_n \to K^{(2)} * \omega$ in $L^{p'}(B_R)$. Letting first $n \to \infty$ and then $\varepsilon \to 0$ and $L \to \infty$ gives (b).

(c) Choose $\omega_j \in C^\infty_c(\mathbb{R}^2)$ with $\omega_j \to \omega$ in $L^1(\mathbb{R}^2)$ and in $L^2(\mathbb{R}^2)$, and put $v_j = K_{BS} * \omega_j$. By Lemma 2.1(d), applied to $\omega_j - \omega_i$ and to ω_j, $\|\nabla v_j - \nabla v_i\|_{L^2(\mathbb{R}^2)} = \|\omega_j - \omega_i\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla v_j\|_{L^2(\mathbb{R}^2)} = \|\omega_j\|_{L^2(\mathbb{R}^2)}$, so $\nabla v_j \to F$ in $L^2(\mathbb{R}^2)$ with $\|F\|_{L^2(\mathbb{R}^2)} = \lim_j \|\omega_j\|_{L^2(\mathbb{R}^2)} = \|\omega\|_{L^2(\mathbb{R}^2)}$. By Lemma 2.1(e), $v_j \to v$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$, so $\nabla v_j \to \nabla v$ in $\mathcal{D}'$, and $\nabla v = F$. □

Definition 2.3. *Let $m \ge 1$ be an integer and let $g \in L^1(\mathbb{R}^2; \mathbb{C})$ have compact support in $\mathbb{R}^2 \setminus \{0\}$. Writing $y \in \mathbb{R}^2$ as a complex number, we put*

$$p^{\mathrm{out}}_m[g] = \frac{1}{2\pi}\int g(y)\,(\bar{y})^m \mathrm{d}y, \quad p^{\mathrm{in}}_m[g] = \frac{1}{2\pi}\int g(y)\,y^{-m}\mathrm{d}y, \quad Q_m[g] = \int g(y)\,y^m \mathrm{d}y.$$

These are the multipole moments of g. In polar coordinates $y = se^{i\vartheta}$, since $(\bar{y})^m = s^m e^{-im\vartheta}$, $y^{-m} = s^{-m}e^{-im\vartheta}$ and $y^m = s^m e^{im\vartheta}$, they read, with the angular modes g_ℓ of Section 2,

$$p_m^{\text{out}}[g] = \int_0^\infty g_m(s)\, s^{m+1}\mathrm{d}s, \qquad p_m^{\text{in}}[g] = \int_0^\infty g_m(s)\, s^{1-m}\mathrm{d}s, \qquad Q_m[g] = 2\pi \int_0^\infty g_{-m}(s)\, s^{m+1}\mathrm{d}s;$$

so they see only the angular modes $\pm m$ of g. Let g be real with $\operatorname{supp} g \subset \{\rho_1 \le |y| \le \rho_2\}$, $0 < \rho_1 \le \rho_2$, and let $\psi(x) = \frac{1}{2\pi}\int \log|x-y|\, g(y)\mathrm{d}y$ be its stream function. Expanding $\log|x-y| = \log|x| + \operatorname{Re}\log(1-y/x)$ for $|x| > \rho_2$, and $\log|x-y| = \log|y| + \operatorname{Re}\log(1-x/y)$ for $|x| < \rho_1$, in powers of y/x and x/y gives

$$\psi(x) = \frac{\int g}{2\pi}\log|x| - \operatorname{Re}\sum_{m\ge 1}\frac{p_m^{\text{out}}[g]}{m}(\bar{x})^{-m} \qquad (|x| > \rho_2),$$

$$\psi(x) = \frac{1}{2\pi}\int g(y)\log|y|\mathrm{d}y - \operatorname{Re}\sum_{m\ge 1}\frac{p_m^{\text{in}}[g]}{m}\, x^m \quad (|x| < \rho_1),$$

since $\overline{Q_m[g]} = 2\pi p_m^{\text{out}}[g]$ for real g. Thus $p_m^{\text{out}}[g]$ and $p_m^{\text{in}}[g]$ give the field of g in the angular modes $\pm m$ outside and inside its support.

2.3 Weak Solutions

Definition 2.4. *Two real functions $g, h \in L^1(\mathbb{R}^2)$ are equimeasurable if and only if $|\{g > s\}| = |\{h > s\}|$ and $|\{g < -s\}| = |\{h < -s\}|$ for every $s > 0$.*

Definition 2.5. *Let $4/3 < q < \infty$, $T > 0$ and $\omega^{\text{in}} \in (L^1 \cap L^q)(\mathbb{R}^2)$. A weak solution with datum ω^{in} in $X_q(T)$ is a function $\omega \in X_q(T)$ such that, with $v(t) = K_{BS} * \omega(t)$,*

$$\int_0^T\!\!\int \omega\,(\partial_t\varphi + v\cdot\nabla\varphi)\mathrm{d}x\mathrm{d}t + \int \omega^{\text{in}}\varphi(0,\cdot)\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathcal{D}_T. \tag{4}$$

It is renormalized if and only if, in addition, for every $\Upsilon \in C^1(\mathbb{R})$ with Υ' bounded and $\Upsilon(0) = 0$,

$$\int_0^T\!\!\int \Upsilon(\omega)\,(\partial_t\varphi + v\cdot\nabla\varphi)\mathrm{d}x\mathrm{d}t + \int \Upsilon(\omega^{\text{in}})\varphi(0,\cdot)\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathcal{D}_T. \tag{5}$$

For $T_0 \in (0,\infty]$, a (renormalized) weak solution on $[0,T_0)$ is a function on $[0,T_0)$ whose restriction to $[0,T)$ is a (renormalized) weak solution in $X_q(T)$ for every $T < T_0$.

The integrals in Equations (4) and (5) converge absolutely, because $|\Upsilon(\omega)| \le \|\Upsilon'\|_{L^\infty(\mathbb{R})}|\omega|$ and $|\omega v| \le |\omega|(|K_{BS}| * |\omega|)$, and Lemma 2.2(a), with $p = q$ and $R = 1$, gives $\|\omega(t)v(t)\|_{L^1(\mathbb{R}^2)} \le C(q,1)\operatorname{ess\,sup}_{0<s<T}\|\omega(s)\|^2_{(L^1\cap L^q)(\mathbb{R}^2)}$ for almost every $t \in (0,T)$.

3 Compactness and a Non-Uniqueness Criterion

Compactness results for sequences of renormalized solutions go back to DiPerna and Lions [10]; we give a self-contained proof of the one we need. We prove it only for vorticity in $L^\infty_t(L^1\cap L^2)_x$, which is all that the proof of Theorem 1.1 needs. The arguments control ∇v by ω: for vorticity in $L^2(\mathbb{R}^2)$ this is the identity $\|\nabla v\|_{L^2(\mathbb{R}^2)} = \|\omega\|_{L^2(\mathbb{R}^2)}$ of Lemma 2.2(c), while for vorticity in $L^q(\mathbb{R}^2)$, $q \ne 2$, it would require the $L^q(\mathbb{R}^2)$ bounds of the Calderón-Zygmund (Riesz transform) estimate. Equations (4) and (5) do not involve q, so, for $q \in (4/3,\infty)$, a renormalized weak solution in

$X_2(T)$ that lies in $X_q(T)$, with datum in $(L^1 \cap L^q)(\mathbb{R}^2)$, is a renormalized weak solution in $X_q(T)$; Theorem 3.8(ii) gives this membership.

All functions in this section are real valued, except Ψ in Corollary 3.9. Let $\varsigma \in C_c^\infty(B_2)$ with $0 \le \varsigma \le 1$ and $\varsigma = 1$ on B_1, and put $\varsigma_R = \varsigma(\cdot/R)$, so that $|\nabla \varsigma_R| \le \|\nabla \varsigma\|_{L^\infty(\mathbb{R}^2)}/R$. Let $\rho \in C_c^\infty(B_1)$ and $\rho^{\mathrm{t}} \in C_c^\infty((-1,1))$ be nonnegative with integral 1, and put $\rho_\varepsilon = \varepsilon^{-2}\rho(\cdot/\varepsilon)$ and $\rho_\delta^{\mathrm{t}} = \delta^{-1}\rho^{\mathrm{t}}(\cdot/\delta)$; convolution with ρ_ε acts in the space variable only, and convolution with ρ_δ^{t} in the time variable only. Let $\varkappa \in C^\infty(\mathbb{R})$ be nondecreasing with $\varkappa = 0$ on $(-\infty, 0]$ and $\varkappa = 1$ on $[1, \infty)$. We write C_{v} for the constant $C(2,1)$ of Lemma 2.2(a); thus $\int |g|(|K_{BS}| * |g|)\mathrm{d}x \le C_{\mathrm{v}} \|g\|^2_{(L^1 \cap L^2)(\mathbb{R}^2)}$ for $g \in (L^1 \cap L^2)(\mathbb{R}^2)$.

3.1 The Linear Transport Equation

Let $T > 0$ and $t_0 \in [0, T)$, let w be a real measurable vector field and u a real measurable function on $(t_0, T) \times \mathbb{R}^2$, and let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ be real, such that u, w and uw are integrable on $(t_0, T') \times B_R$ for all $T' < T$ and $R > 0$. We say that u satisfies Equation (6) on $[t_0, T)$ with field w and datum u_0 if

$$\int_{t_0}^T \int u\,(\partial_t \varphi + w \cdot \nabla \varphi)\mathrm{d}x\mathrm{d}t + \int u_0\, \varphi(t_0, \cdot)\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathcal{D}_T; \tag{6}$$

every integral here converges. Thus $\omega \in X_2(T)$ is a weak solution of the Euler equations with datum $\omega^{\mathrm{in}} \in (L^1 \cap L^2)(\mathbb{R}^2)$ if and only if it satisfies Equation (6) on $[0, T)$ with field $v = K_{BS} * \omega$ and datum ω^{in}, and it is renormalized if and only if, in addition, $\Upsilon(\omega)$ satisfies Equation (6) on $[0, T)$ with field v and datum $\Upsilon(\omega^{\mathrm{in}})$ for every $\Upsilon \in C^1(\mathbb{R})$ with Υ' bounded and $\Upsilon(0) = 0$. The integrability conditions hold by Lemma 2.2(a).

Lemma 3.1. *Let $t_0 < T$, $B \in L^\infty(t_0, T)$ and $b_0 \in \mathbb{R}$, and suppose that, for every $\alpha \in C_c^\infty((-\infty, T))$, $\int_{t_0}^T \alpha'(t)B(t)\mathrm{d}t + \alpha(t_0)b_0 = 0$. Then $B = b_0$ almost everywhere on (t_0, T).*

Proof. For $0 < \delta < (T - t_0)/2$ and $t \in (t_0 + \delta, T - \delta)$ put $B_\delta(t) = \int B(s)\rho_\delta^{\mathrm{t}}(t - s)\mathrm{d}s$. The function $\alpha(s) = \rho_\delta^{\mathrm{t}}(t - s)$ vanishes outside $(t - \delta, t + \delta)$, which lies in (t_0, T) because $t - t_0 > \delta$ and $T - t > \delta$; so $\alpha \in C_c^\infty((t_0, T))$ and $\alpha(t_0) = 0$, and the hypothesis gives $B_\delta'(t) = -\int B(s)\,\partial_s\big(\rho_\delta^{\mathrm{t}}(t - s)\big)\mathrm{d}s = 0$, so B_δ is constant on $(t_0 + \delta, T - \delta)$. Since $B_\delta \to B$ in $L^1_{\mathrm{loc}}((t_0, T))$ as $\delta \to 0$, the constants converge to some $c \in \mathbb{R}$ and $B = c$ almost everywhere. For α with $\alpha(t_0) = 1$ we have $\int_{t_0}^T \alpha' c\mathrm{d}t = -c$, because α vanishes near T; so the hypothesis gives $b_0 = c$. □

Lemma 3.2. *Let $T > 0$. Let w be a vector field with $w \in L^\infty(0, T; L^2(B_R))$ for every $R > 0$, let $u \in L^\infty(0, T; L^2(\mathbb{R}^2))$ and $u_0 \in L^2(\mathbb{R}^2)$, and suppose that u satisfies Equation (6) on $[0, T)$ with field w and datum u_0. Put $N = \operatorname{ess\,sup}_t \|u(t)\|_{L^2(\mathbb{R}^2)}$. Then u has a representative $\tilde{u}\colon [0, T] \to L^2(\mathbb{R}^2)$ with the following properties.*

(a) *$\tilde{u}$ is weakly continuous, $\tilde{u}(0) = u_0$, and $\|\tilde{u}(t)\|_{L^2(\mathbb{R}^2)} \le N$ for every t. If also $u \in L^\infty(0, T; L^1(\mathbb{R}^2))$, then $\|\tilde{u}(t)\|_{L^1(\mathbb{R}^2)} \le \operatorname{ess\,sup}_s \|u(s)\|_{L^1(\mathbb{R}^2)}$ for every t.*

(b) *For $\psi \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp}\psi \subset B_R$ and every $t \in [0, T]$, $\int \tilde{u}(t)\psi = \int u_0\psi + \int_0^t \int u\, w \cdot \nabla\psi$. In particular, for $s, t \in [0, T]$,*

$$\Big|\int (\tilde{u}(t) - \tilde{u}(s))\psi\Big| \le N\, \|w\|_{L^\infty(0,T;L^2(B_R))} \|\nabla\psi\|_{L^\infty(\mathbb{R}^2)} |t - s|.$$

(c) *For every $t_0 \in [0, T)$, u satisfies Equation (6) on $[t_0, T)$ with field w and datum $\tilde{u}(t_0)$.*

Proof. For $j \in \mathbb{N}$, $C_c^\infty(B_j)$ is a subset of the separable Banach space $C^1(\overline{B_j})$, hence separable: there is a countable set $\mathcal{F}_j \subset C_c^\infty(B_j)$ such that every $\psi \in C_c^\infty(B_j)$ is the limit in C^1 of elements of $\mathcal{F}_j$. The countable union $\mathcal{F} = \bigcup_{j\in\mathbb{N}} \mathcal{F}_j \subset C_c^\infty(\mathbb{R}^2)$ does not depend on j, and every $\psi \in C_c^\infty(B_j)$, $j \in \mathbb{N}$, is the limit in C^1 of elements of $\mathcal{F} \cap C_c^\infty(B_j)$. C_c^∞ is dense in $L^2(\mathbb{R}^2)$, and convergence in C^1 with supports in a fixed B_j implies convergence in $L^2(\mathbb{R}^2)$; so $\mathcal{F}$ is dense in $L^2(\mathbb{R}^2)$. Write $W_R = \|w\|_{L^\infty(0,T;L^2(B_R))}$.

For $\psi \in \mathcal{F}$ with $\operatorname{supp}\psi \subset B_R$ put $a_\psi(t) = \int u(t)\psi$ and $c_\psi(t) = \int u(t)\, w(t) \cdot \nabla\psi$. These are measurable by Fubini's theorem, and $|c_\psi| \le NW_R\|\nabla\psi\|_{L^\infty(\mathbb{R}^2)}$ almost everywhere by Hölder's inequality, so $A_\psi(t) = \int u_0\psi + \int_0^t c_\psi$ is Lipschitz on $[0,T]$. For $\alpha \in C_c^\infty((-\infty,T))$, Equation (6) with the test function $\alpha(t)\psi(x)$ gives $\int_0^T \alpha' a_\psi + \int_0^T \alpha c_\psi + \alpha(0)\int u_0\psi = 0$. Since α vanishes near T and $A'_\psi = c_\psi$ almost everywhere, integration by parts gives $\int_0^T \alpha c_\psi = -\int_0^T \alpha' A_\psi - \alpha(0)\int u_0\psi$. Hence $\int_0^T \alpha'(a_\psi - A_\psi) = 0$, and Lemma 3.1, with $t_0 = 0$ and $b_0 = 0$, gives $a_\psi = A_\psi$ almost everywhere. Let $\mathcal{N}$ be the union of $\{0,T\}$, of the exceptional null sets of these identities for $\psi \in \mathcal{F}$, of the set of t with $u(t) \notin L^2(\mathbb{R}^2)$ or $\|u(t)\|_{L^2(\mathbb{R}^2)} > N$, and, if $u \in L^\infty(0,T;L^1(\mathbb{R}^2))$, of the set of t with $\|u(t)\|_{L^1(\mathbb{R}^2)} > N_1 := \operatorname{ess\,sup}_s \|u(s)\|_{L^1(\mathbb{R}^2)}$. As $\mathcal{F}$ is countable, $\mathcal{N}$ is a null set.

For $t \notin \mathcal{N}$ put $\tilde{u}(t) = u(t)$. For $t \in \mathcal{N}$ take $t_j \notin \mathcal{N}$ with $t_j \to t$. Then $\|u(t_j)\|_{L^2(\mathbb{R}^2)} \le N$ and $\int u(t_j)\psi = A_\psi(t_j) \to A_\psi(t)$ for $\psi \in \mathcal{F}$. Since $\mathcal{F}$ is dense in $L^2(\mathbb{R}^2)$, $\int u(t_j)\phi$ converges for every $\phi \in L^2(\mathbb{R}^2)$, to a linear functional of norm at most N; by the Riesz representation theorem in $L^2(\mathbb{R}^2)$, it is $\phi \mapsto \int \tilde{u}(t)\phi$ for a unique $\tilde{u}(t) \in L^2(\mathbb{R}^2)$, which does not depend on the sequence, because $\int \tilde{u}(t)\psi = A_\psi(t)$ for $\psi \in \mathcal{F}$. Thus $\tilde{u} = u$ almost everywhere, $\|\tilde{u}(t)\|_{L^2(\mathbb{R}^2)} \le N$ for every t, and $\int \tilde{u}(t)\psi = A_\psi(t)$ for all $t \in [0,T]$ and $\psi \in \mathcal{F}$.

We prove (a) and (b). For $\phi \in L^2(\mathbb{R}^2)$, $\psi \in \mathcal{F}$ and $s,t \in [0,T]$, $|\int(\tilde{u}(t) - \tilde{u}(s))\phi| \le 2N\|\phi - \psi\|_{L^2(\mathbb{R}^2)} + |A_\psi(t) - A_\psi(s)|$. Given ϕ and $\eta > 0$, first choose $\psi \in \mathcal{F}$ with $2N\|\phi - \psi\|_{L^2(\mathbb{R}^2)} < \eta/2$, which is possible as $\mathcal{F}$ is dense in $L^2(\mathbb{R}^2)$; then, since A_ψ is Lipschitz, $|A_\psi(t) - A_\psi(s)| < \eta/2$ whenever $|t-s|$ is small enough. So $t \mapsto \int \tilde{u}(t)\phi$ is continuous for every $\phi \in L^2(\mathbb{R}^2)$, that is, $\tilde{u}$ is weakly continuous; and $\tilde{u}(0) = u_0$, because $A_\psi(0) = \int u_0\psi$. If $u \in L^\infty(0,T;L^1(\mathbb{R}^2))$ and $t \in \mathcal{N}$, then $u(t_j) \rightharpoonup \tilde{u}(t)$ weakly in $L^2(\mathbb{R}^2)$ and $\|u(t_j)\|_{L^1(\mathbb{R}^2)} \le N_1$, so $\|\tilde{u}(t)\|_{L^1(\mathbb{R}^2)} \le N_1$ by Lemma 2.2(b) with $p = 2$. For (b), let $\psi \in C_c^\infty(B_j)$ and $\psi_i \in \mathcal{F} \cap C_c^\infty(B_j)$ with $\psi_i \to \psi$ in C^1. Then $\int \tilde{u}(t)\psi_i \to \int \tilde{u}(t)\psi$, since $\|\psi_i - \psi\|_{L^2(\mathbb{R}^2)} \le |B_j|^{1/2}\|\psi_i - \psi\|_{L^\infty(\mathbb{R}^2)}$, and $A_{\psi_i}(t) \to \int u_0\psi + \int_0^t \int u\, w \cdot \nabla\psi$, since $\int u_0\psi_i \to \int u_0\psi$ and $|\int_0^t \int u\, w \cdot \nabla(\psi_i - \psi)| \le TNW_j\|\nabla(\psi_i - \psi)\|_{L^\infty(\mathbb{R}^2)}$. This is the identity in (b), and the bound on c_ψ gives the Lipschitz bound.

For (c), the case $t_0 = 0$ is the hypothesis, since $\tilde{u}(0) = u_0$. Let $t_0 \in (0,T)$ and $\varphi \in \mathcal{D}_T$, and for $0 < h < T - t_0$ put $\varkappa_h(t) = \varkappa((t - t_0)/h)$. Equation (6) with the test function $\varkappa_h\varphi \in \mathcal{D}_T$, which vanishes for $t \le t_0$, gives

$$\int_0^T \int \varkappa_h\, u\, (\partial_t\varphi + w \cdot \nabla\varphi)\mathrm{d}x\mathrm{d}t + \int_0^T \varkappa'_h(t)\, g(t)\mathrm{d}t = 0, \qquad g(t) = \int \tilde{u}(t)\varphi(t)\mathrm{d}x.$$

The function g is continuous, since $|g(t) - g(s)| \le N\|\varphi(t) - \varphi(s)\|_{L^2(\mathbb{R}^2)} + |\int(\tilde{u}(t) - \tilde{u}(s))\varphi(s)|$ and $\tilde{u}$ is weakly continuous. As $h \to 0$, the second term tends to $g(t_0)$: since $\varkappa'_h \ge 0$, $\int \varkappa'_h = 1$ and $\operatorname{supp}\varkappa'_h \subset [t_0, t_0 + h]$, the functions $\varkappa'_h$ converge to the Dirac mass at t_0 in the sense of distributions, indeed against every continuous function, and g is continuous; and, since $\varkappa_h \to \mathbf{1}_{(t_0,\infty)}$ pointwise, the first term tends to $\int_{t_0}^T \int u\,(\partial_t\varphi + w \cdot \nabla\varphi)$ by dominated convergence, with the integrable dominating function $|u|(|\partial_t\varphi| + |w||\nabla\varphi|)$. This is (c). $\square$

3.2 Weakly Continuous Representatives

Let ω be a weak solution in $X_2(T)$ with datum ω^{in}, and put $v = K_{BS} * \omega$. By Lemma 2.2(a), $g \mapsto K_{BS} * g$ is linear and bounded from $(L^1 \cap L^2)(\mathbb{R}^2)$ to $L^2(B_R)$, and such a map preserves strong measurability; so $v \in L^\infty(0,T;L^2(B_R))$ for every $R > 0$, and $\|\omega(t)v(t)\|_{L^1(\mathbb{R}^2)} \le C_{\mathrm{v}} \|\omega(t)\|^2_{(L^1\cap L^2)(\mathbb{R}^2)}$ for almost every t, so $\omega v \in L^1((0,T)\times\mathbb{R}^2)$. Hence Lemma 3.2 applies with $u = \omega$, $w = v$ and $u_0 = \omega^{\mathrm{in}}$. From now on every such weak solution is identified with its weakly continuous representative, so that $\omega(t)$ is defined for every $t \in [0,T]$, $\omega(0) = \omega^{\mathrm{in}}$, and $\|\omega(t)\|_{L^1(\mathbb{R}^2)}$ and $\|\omega(t)\|_{L^2(\mathbb{R}^2)}$ are bounded by the corresponding essential suprema for every t. Two weakly continuous functions that agree for almost every t agree for every t; hence the representatives on the intervals $[0,T]$, $T < T_0$, of a weak solution on $[0,T_0)$ with $q = 2$ define a weakly continuous representative on $[0,T_0)$. Moreover $\|\nabla v(t)\|_{L^2(\mathbb{R}^2)} = \|\omega(t)\|_{L^2(\mathbb{R}^2)}$ by Lemma 2.2(c) and $\operatorname{div} v(t) = 0$ by Lemma 2.1(e); thus $v \in L^\infty(0,T;W^{1,2}(B_R))$ for every $R > 0$, and v is divergence-free.

3.3 Renormalization

For an open interval I, a vector field w and a function u on $I \times \mathbb{R}^2$ with $u, wu \in L^1_{\mathrm{loc}}(I\times\mathbb{R}^2)$, put

$$C_\varepsilon[u,w] = w \cdot \nabla(u * \rho_\varepsilon) - \big(\operatorname{div}(wu)\big) * \rho_\varepsilon,$$

where $\big(\operatorname{div}(wu)\big) * \rho_\varepsilon(t,x) = \int (wu)(t,y) \cdot \nabla\rho_\varepsilon(x-y)\mathrm{d}y$. Both terms of C_ε are defined by these formulas, not derived: the divergence $\operatorname{div}(wu)$ is a distribution, and its convolution in x with ρ_ε, which acts by pairing it with $\rho_\varepsilon(x-\cdot)$, is the function displayed above, since $wu \in L^1_{\mathrm{loc}}(I\times\mathbb{R}^2)$. The function $u * \rho_\varepsilon(t,\cdot)$ is smooth, and $w \cdot \nabla(u*\rho_\varepsilon)$ is its pointwise product with w. Writing $w(t,x)\cdot\nabla(u*\rho_\varepsilon)(t,x) = \int u(t,y)\, w(t,x)\cdot\nabla\rho_\varepsilon(x-y)\mathrm{d}y$ and substituting $y = x - \varepsilon z$, we get, for almost every (t,x),

$$\begin{aligned} C_\varepsilon[u,w](t,x) &= \int u(t,y)\big(w(t,x) - w(t,y)\big)\cdot\nabla\rho_\varepsilon(x-y)\mathrm{d}y \\ &= \int_{B_1} u(t,x-\varepsilon z)\,\frac{w(t,x)-w(t,x-\varepsilon z)}{\varepsilon}\cdot\nabla\rho(z)\mathrm{d}z, \end{aligned} \tag{7}$$

because $\nabla\rho_\varepsilon(\varepsilon z) = \varepsilon^{-3}\nabla\rho(z)$. The map $(u,w)\mapsto C_\varepsilon[u,w]$ is bilinear.

Lemma 3.3 (cf. Lemma II.1 in [10])**.** *Let $I \subset \mathbb{R}$ be an open interval, let $w \in L^1_{\mathrm{loc}}(I; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2))$ with $\operatorname{div} w = 0$, and let $u \in L^\infty_{\mathrm{loc}}(I; L^2_{\mathrm{loc}}(\mathbb{R}^2))$. Then $C_\varepsilon[u,w] \to 0$ in $L^1_{\mathrm{loc}}(I\times\mathbb{R}^2)$ as $\varepsilon \to 0$.*

Proof. Note that $wu \in L^1_{\mathrm{loc}}(I\times\mathbb{R}^2)$, by the Cauchy-Schwarz inequality in x. Fix a compact interval $J \subset I$ and $R > 0$, and write $\|\cdot\|_{q,r}$ for the norm of $L^q(B_r)$.

We first prove a bound that is uniform in $\varepsilon \in (0,1]$ and does not use $\operatorname{div} w = 0$. If $w(t,\cdot) \in C^1(\mathbb{R}^2;\mathbb{R}^2)$, then $w(x) - w(x-\varepsilon z) = \varepsilon\int_0^1 \nabla w(x - s\varepsilon z)z\mathrm{d}s$, and Equation (7) with $|z| \le 1$ and the Cauchy-Schwarz inequality in x give

$$\begin{aligned} \|C_\varepsilon[u,w](t)\|_{1,R} &\le \|\nabla\rho\|_{L^\infty(\mathbb{R}^2)} \int_{B_1}\int_0^1\int_{B_R} |u(t,x-\varepsilon z)|\,|\nabla w(t,x-s\varepsilon z)|\mathrm{d}x\mathrm{d}s\mathrm{d}z \\ &\le \pi\|\nabla\rho\|_{L^\infty(\mathbb{R}^2)}\|u(t)\|_{2,R+1}\|\nabla w(t)\|_{2,R+1}. \end{aligned}$$

For general w, apply this to $w_\delta = w * \rho_\delta$, $\delta \in (0,1]$, for which $\|\nabla w_\delta(t)\|_{2,R+1} \le \|\nabla w(t)\|_{2,R+2}$ by Young's inequality. For fixed ε, the first line of Equation (7) and the inequalities of Hölder and Young, with $\|\nabla\rho_\varepsilon\|_{L^1(\mathbb{R}^2)} = \varepsilon^{-1}\|\nabla\rho\|_{L^1(\mathbb{R}^2)}$, give

$$\|C_\varepsilon[u, w - w_\delta](t)\|_{1,R} \le 2\varepsilon^{-1}\|\nabla\rho\|_{L^1(\mathbb{R}^2)}\|u(t)\|_{2,R+1}\|(w-w_\delta)(t)\|_{2,R+1},$$

which tends to 0 as $\delta \to 0$ for almost every t, because mollification converges in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Hence, for almost every $t \in J$ and all $\varepsilon \in (0,1]$,

$$\|C_\varepsilon[u,w](t)\|_{1,R} \le C_\rho \, \|u(t)\|_{2,R+1} \, \|\nabla w(t)\|_{2,R+2}, \qquad C_\rho = \pi \|\nabla \rho\|_{L^\infty(\mathbb{R}^2)}. \tag{8}$$

We now use $\operatorname{div} w = 0$. For $\delta \in (0,1]$ put $u_\delta(t) = u(t) * \rho_\delta$. For almost every $t \in J$, $u_\delta(t)$ is smooth with $\|\nabla u_\delta(t)\|_{\infty,R+1} \le \|\nabla \rho_\delta\|_{L^\infty(\mathbb{R}^2)} \|u(t)\|_{1,R+2}$, and, for $\phi \in C_c^\infty(\mathbb{R}^2)$, $\int w u_\delta \cdot \nabla \phi = \int w \cdot \nabla(u_\delta \phi) - \int \phi \, w \cdot \nabla u_\delta = -\int \phi \, w \cdot \nabla u_\delta$, because $\operatorname{div} w(t) = 0$. Thus $\operatorname{div}(w u_\delta) = w \cdot \nabla u_\delta$, and

$$C_\varepsilon[u_\delta, w] = w \cdot \big((\nabla u_\delta) * \rho_\varepsilon - \nabla u_\delta\big) + \big(w \cdot \nabla u_\delta - (w \cdot \nabla u_\delta) * \rho_\varepsilon\big).$$

As $\varepsilon \to 0$, for almost every $t \in J$ the first term tends to 0 in $L^1(B_R)$, because $\nabla u_\delta(t)$ is uniformly continuous on $\overline{B_{R+1}}$, and so does the second, because $w \cdot \nabla u_\delta(t) \in L^1(B_{R+1})$. Since $\|C_\varepsilon[u_\delta, w](t)\|_{1,R} \le 4\|w(t)\|_{1,R+1} \|\nabla u_\delta(t)\|_{\infty,R+1}$, which is integrable on J because $w \in L^1(J; L^1(B_{R+1}))$ and $u \in L^\infty(J; L^1(B_{R+2}))$, $C_\varepsilon[u_\delta, w] \to 0$ in $L^1(J \times B_R)$ for each δ. Finally, by bilinearity and Equation (8),

$$\sup_{\varepsilon \le 1} \|C_\varepsilon[u,w] - C_\varepsilon[u_\delta, w]\|_{L^1(J \times B_R)} \le C_\rho \int_J \|(u - u_\delta)(t)\|_{2,R+1} \|\nabla w(t)\|_{2,R+2} \mathrm{d}t,$$

Call the right side $\eta(\delta)$; it does not depend on ε, and $\eta(\delta) \to 0$ as $\delta \to 0$ by dominated convergence, since $\|(u - u_\delta)(t)\|_{2,R+1} \to 0$ for almost every t and is at most $2\|u\|_{L^\infty(J; L^2(B_{R+2}))}$. Thus $\|C_\varepsilon[u,w]\|_{L^1(J \times B_R)} \le \|C_\varepsilon[u_\delta, w]\|_{L^1(J \times B_R)} + \eta(\delta)$ for all $\varepsilon \in (0,1]$ and $\delta \in (0,1]$. Letting first $\varepsilon \to 0$ with δ fixed gives $\limsup_{\varepsilon \to 0} \|C_\varepsilon[u,w]\|_{L^1(J \times B_R)} \le \eta(\delta)$, and letting then $\delta \to 0$ gives $C_\varepsilon[u,w] \to 0$ in $L^1(J \times B_R)$. □

Lemma 3.4. *Let $T > 0$ and $t_0 \in [0,T)$. Let $w \in L^\infty(0,T; W^{1,2}_{\mathrm{loc}})$ with $\operatorname{div} w = 0$, let $u \in X_2(T)$ with $uw \in L^1((0,T) \times \mathbb{R}^2)$, and let $u_0 \in (L^1 \cap L^2)(\mathbb{R}^2)$. Assume that u satisfies Equation (6) on $[t_0, T)$ with field w and datum u_0. Then, for every $\Upsilon \in \mathrm{Lip}_0$:*

(i) *$\Upsilon(u)$ satisfies Equation (6) on $[t_0, T)$ with field w and datum $\Upsilon(u_0)$;*

(ii) *$\int \Upsilon(u(t)) \mathrm{d}x = \int \Upsilon(u_0) \mathrm{d}x$ for almost every $t \in (t_0, T)$.*

Proof. We first prove (i) for $\Upsilon \in C^1(\mathbb{R})$ with $|\Upsilon'| \le L$ and $\Upsilon(0) = 0$. Put $I = (t_0 - 1, T)$, and redefine w and u on $(t_0 - 1, t_0]$ by $w = 0$ and $u = u_0$; on (t_0, T) they are unchanged. For $\varphi \in C_c^\infty(I \times \mathbb{R}^2)$, extended by zero to an element of $\mathcal{D}_T$, the interval $(t_0 - 1, t_0)$ contributes $\int_{t_0-1}^{t_0} \int u_0 \partial_t \varphi = \int u_0 \varphi(t_0)$ to $\int_I \int u(\partial_t \varphi + w \cdot \nabla \varphi)$, and by Equation (6) the interval (t_0, T) contributes $-\int u_0 \varphi(t_0)$. Thus $\partial_t u + \operatorname{div}(wu) = 0$ in $\mathcal{D}'(I \times \mathbb{R}^2)$, and the extended pair satisfies the hypotheses of Lemma 3.3 on I.

Let $u_\varepsilon = u * \rho_\varepsilon$. Then $\nabla u_\varepsilon \in L^\infty_{\mathrm{loc}}(I \times \mathbb{R}^2)$, and convolving the equation $\partial_t u + \operatorname{div}(wu) = 0$ in x with ρ_ε gives $\partial_t u_\varepsilon = -\big(\operatorname{div}(wu)\big) * \rho_\varepsilon \in L^1_{\mathrm{loc}}(I \times \mathbb{R}^2)$, where, as $wu \in L^1_{\mathrm{loc}}(I \times \mathbb{R}^2)$, the right side is the function defined at the start of this subsection, so $u_\varepsilon \in W^{1,1}_{\mathrm{loc}}(I \times \mathbb{R}^2)$ and $\partial_t u_\varepsilon + w \cdot \nabla u_\varepsilon = C_\varepsilon[u,w]$. For almost every t, $\Upsilon(u_\varepsilon(t))$ is a C^1 function of x with locally bounded gradient, so $\operatorname{div}\big(w \Upsilon(u_\varepsilon)\big) = w \cdot \nabla \Upsilon(u_\varepsilon)$ because $\operatorname{div} w = 0$; with the chain rule in $W^{1,1}_{\mathrm{loc}}$ this gives

$$\partial_t \Upsilon(u_\varepsilon) + \operatorname{div}\big(w \Upsilon(u_\varepsilon)\big) = \Upsilon'(u_\varepsilon) C_\varepsilon[u,w] \qquad \text{in } \mathcal{D}'(I \times \mathbb{R}^2).$$

As $\varepsilon \to 0$, the right side tends to 0 in $L^1_{\mathrm{loc}}(I \times \mathbb{R}^2)$ by Lemma 3.3, since $|\Upsilon'| \le L$. For almost every t, $u_\varepsilon(t) \to u(t)$ in $L^2(B_R)$, with the bound $2\|u\|_{L^\infty(I; L^2(\mathbb{R}^2))}$; since $|\Upsilon(u_\varepsilon) - \Upsilon(u)| \le L|u_\varepsilon - u|$ and $w \in L^\infty(I; L^2(B_R))$, dominated convergence gives $\Upsilon(u_\varepsilon) \to \Upsilon(u)$ and $w\Upsilon(u_\varepsilon) \to w\Upsilon(u)$

in $L^1_{\mathrm{loc}}(I\times\mathbb{R}^2)$. Therefore $\partial_t\Upsilon(u)+\operatorname{div}\big(w\Upsilon(u)\big)=0$ in $\mathcal{D}'(I\times\mathbb{R}^2)$. Now let $\varphi\in\mathcal{D}_T$ and $\kappa_0(t)=\varkappa\big(3(t-t_0)+2\big)$, so that $\kappa_0=0$ on $(-\infty,t_0-2/3]$ and $\kappa_0=1$ on $[t_0-1/3,\infty)$. Test the last equation with $\kappa_0\varphi\in C_c^\infty(I\times\mathbb{R}^2)$. On (t_0-1,t_0), $w=0$ and $\Upsilon(u)=\Upsilon(u_0)$, so this interval contributes $\int_{t_0-1}^{t_0}\int\Upsilon(u_0)\partial_t(\kappa_0\varphi)=\int\Upsilon(u_0)\varphi(t_0)$, while $\kappa_0=1$ on (t_0,T). This is (i) for Υ.

Let now $\Upsilon\in\mathrm{Lip}_0$ have Lipschitz constant L, and put $\Upsilon_j=\Upsilon*\rho^{\mathrm{t}}_{1/j}-(\Upsilon*\rho^{\mathrm{t}}_{1/j})(0)$. Then $\Upsilon_j\in C^\infty(\mathbb{R})$, $|\Upsilon_j'|\le L$, $\Upsilon_j(0)=0$, and $|\Upsilon_j-\Upsilon|\le 2L/j$, because $|\Upsilon*\rho^{\mathrm{t}}_{1/j}-\Upsilon|\le L/j$ and $\Upsilon(0)=0$. Since $|\Upsilon(u)|\le L|u|$, all integrals in Equation (6) for $\Upsilon(u)$ converge. For $\varphi\in\mathcal{D}_T$ with $\operatorname{supp}\varphi\subset(-\infty,T']\times B_R$, $T'<T$, the left sides of Equation (6) for $\Upsilon_j(u)$ (which vanish) and for $\Upsilon(u)$ differ by at most $\frac{2L}{j}\big(\int_{t_0}^{T'}\int_{B_R}(|\partial_t\varphi|+|w||\nabla\varphi|)+\int_{B_R}|\varphi(t_0)|\big)$, which tends to 0 because w is integrable on $(t_0,T')\times B_R$. This proves (i).

For (ii), let $\Upsilon\in\mathrm{Lip}_0$ have Lipschitz constant L, and let $\alpha\in C_c^\infty((-\infty,T))$. Since $\Upsilon(u)$ solves Equation (6) by (i), we test it against $\alpha(t)\varsigma_R(x)$ and get

$$\int_{t_0}^T\alpha'(t)\Big(\int\Upsilon(u(t))\varsigma_R\mathrm{d}x\Big)\mathrm{d}t+\alpha(t_0)\int\Upsilon(u_0)\varsigma_R\mathrm{d}x=-\int_{t_0}^T\alpha(t)\int\Upsilon(u)\,w\cdot\nabla\varsigma_R\mathrm{d}x\mathrm{d}t,$$

whose right side is at most $\|\alpha\|_{L^\infty(\mathbb{R})}L\|\nabla\varsigma\|_{L^\infty(\mathbb{R}^2)}R^{-1}\|uw\|_{L^1((0,T)\times\mathbb{R}^2)}$. As $R\to\infty$, dominated convergence, with $|\Upsilon(u)|\le L|u|$ and $u\in L^\infty(0,T;L^1(\mathbb{R}^2))$, gives $\int_{t_0}^T\alpha'B+\alpha(t_0)b_0=0$ with $B(t)=\int\Upsilon(u(t))$, a measurable function bounded by $L\|u\|_{L^\infty(0,T;L^1(\mathbb{R}^2))}$, and $b_0=\int\Upsilon(u_0)$. Lemma 3.1 gives (ii). □

Lemma 3.5. *Let $T>0$ and let ω be a renormalized weak solution of the Euler equations in $X_2(T)$ with datum ω^{in}. Then for every $t\in[0,T)$ and every $\Upsilon\in\mathrm{Lip}_0$,*

$$\int\Upsilon(\omega(t))\mathrm{d}x=\int\Upsilon(\omega^{\mathrm{in}})\mathrm{d}x. \tag{9}$$

Consequently, for every $t\in[0,T)$, $\|\omega(t)\|_{L^q(\mathbb{R}^2)}=\|\omega^{\mathrm{in}}\|_{L^q(\mathbb{R}^2)}$ for every $q\in[1,2]$, and $\omega(t)$ is equimeasurable with ω^{in}. Moreover $\omega\in C([0,T);L^q(\mathbb{R}^2))$ for every $q\in[1,2]$, and, with $v=K_{BS}*\omega$, for every $R>0$ and every $t\in[0,T)$,

$$\int_{\{|x|\ge 2R\}}|\omega(t)|\mathrm{d}x\le\int_{\{|x|\ge R\}}|\omega^{\mathrm{in}}|\mathrm{d}x+\frac{\|\nabla\varsigma\|_{L^\infty(\mathbb{R}^2)}}{R}\,\|\omega v\|_{L^1((0,T)\times\mathbb{R}^2)}, \tag{10}$$

where $\|\omega v\|_{L^1((0,T)\times\mathbb{R}^2)}\le C_{\mathrm{v}}TM^2$ with $M=\sup_{t\in[0,T]}\|\omega(t)\|_{(L^1\cap L^2)(\mathbb{R}^2)}$.

Proof. The number M is finite by Lemma 3.2(a), and $\|\omega(t)v(t)\|_{L^1(\mathbb{R}^2)}\le C_{\mathrm{v}}M^2$ for almost every t by Lemma 2.2(a); this gives the bound on $\|\omega v\|_{L^1((0,T)\times\mathbb{R}^2)}$. By the remark at the start of Section 3.2, $v\in L^\infty(0,T;W^{1,2}(B_R))$ for every $R>0$ and $\operatorname{div}v=0$.

We first prove Equation (9). Let $t_0\in[0,T)$ and $\Upsilon\in\mathrm{Lip}_0$. By Lemma 3.2(a) and (c), $\omega(t_0)\in(L^1\cap L^2)(\mathbb{R}^2)$, and ω satisfies Equation (6) on $[t_0,T)$ with field v and datum $\omega(t_0)$, as well as on $[0,T)$ with datum ω^{in}. Lemma 3.4(ii), applied on $[t_0,T)$ and on $[0,T)$ and compared at a time $t\in(t_0,T)$ that is exceptional for neither, gives $\int\Upsilon(\omega(t_0))=\int\Upsilon(\omega(t))=\int\Upsilon(\omega^{\mathrm{in}})$. For $s,\delta>0$ the functions $z\mapsto\min\{1,(\pm z-s)_+/\delta\}$ lie in Lip_0 and increase to $\mathbf{1}_{\{\pm z>s\}}$ as $\delta\downarrow 0$, so monotone convergence gives the equimeasurability. Integration of the distribution function, $\int|g|^q=\int_0^\infty qs^{q-1}\big(|\{g>s\}|+|\{g<-s\}|\big)\mathrm{d}s$, $1\le q<\infty$, gives the norms.

Next, $\omega\in C([0,T);L^2(\mathbb{R}^2))$: the norm $\|\omega(t)\|_{L^2(\mathbb{R}^2)}$ is constant on $[0,T)$, and ω is weakly continuous into $L^2(\mathbb{R}^2)$, so $\|\omega(t)-\omega(s)\|^2_{L^2(\mathbb{R}^2)}=\|\omega(t)\|^2_{L^2(\mathbb{R}^2)}+\|\omega(s)\|^2_{L^2(\mathbb{R}^2)}-2\int\omega(t)\omega(s)\to 0$ as $t\to s$. By Lemma 3.4(i) with $\Upsilon=|\cdot|$, $|\omega|$ satisfies Equation (6) on $[0,T)$ with field v and datum

$|\omega^{\rm in}|$. So Lemma 3.2 applies with $u = |\omega|$, $w = v$ and $u_0 = |\omega^{\rm in}|$; since $t \mapsto |\omega(t)|$ is continuous into $L^2(\mathbb{R}^2)$ on $[0,T)$, it agrees there with the representative of that lemma, because two weakly continuous functions that agree for almost every t agree for every t. Part (b) of that lemma with $\psi = \varsigma_R$, together with Equation (9) for $\Upsilon = |\cdot|$, gives for every $t \in [0,T)$

$$\int |\omega(t)|(1-\varsigma_R)\mathrm{d}x = \int |\omega^{\rm in}|(1-\varsigma_R)\mathrm{d}x - \int_0^t\!\!\int |\omega|\, v\cdot\nabla\varsigma_R \mathrm{d}x\mathrm{d}s.$$

Since $\mathbf{1}_{\{|x|\ge 2R\}} \le 1-\varsigma_R \le \mathbf{1}_{\{|x|\ge R\}}$ and $|\nabla\varsigma_R| \le \|\nabla\varsigma\|_{L^\infty(\mathbb{R}^2)}/R$, this gives Equation (10). Its right side does not depend on t and tends to 0 as $R\to\infty$. We split the $L^1(\mathbb{R}^2)$ norm into the parts on B_{2R} and outside it: for $s,t\in[0,T)$, by the Cauchy-Schwarz inequality on B_{2R} and Equation (10) for $\omega(t)$ and $\omega(s)$,

$$\|\omega(t)-\omega(s)\|_{L^1(\mathbb{R}^2)} \le |B_{2R}|^{1/2}\|\omega(t)-\omega(s)\|_{L^2(\mathbb{R}^2)} + 2\sup_{\tau\in[0,T)}\int_{\{|x|\ge 2R\}}|\omega(\tau)|\mathrm{d}x.$$

Given $\eta>0$, choose R so that the last term is less than $\eta/2$; then the $L^2(\mathbb{R}^2)$ continuity makes the first term less than $\eta/2$ for t close to s. Hence $\omega\in C([0,T);L^1(\mathbb{R}^2))$. Interpolation between $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ gives $\omega\in C([0,T);L^q(\mathbb{R}^2))$ for $q\in(1,2)$. □

3.4 Radial Data and the Criterion

Recall from Section 2 that, for $g\in L^1(\mathbb{R}^2)$, $\Gamma_g(r) = \int_{B_r} g$ is the integral of g over the disk B_r; for smooth g it is the circulation of $K_{BS}*g$ around the circle $\{|x|=r\}$.

Lemma 3.6. *Let $4/3<q<\infty$, $T>0$, and let $f\in(L^1\cap L^q)(\mathbb{R}^2)$ be radial. Then $K_{BS}*f(x) = \Gamma_f(|x|)\,x^\perp/(2\pi|x|^2)$ for almost every x, and $\omega(t)=f$, $t\in[0,T]$, is a renormalized weak solution in $X_q(T)$ with datum f.*

Proof. We first prove the formula for radial $h = h_0(|\cdot|)\in L^1(\mathbb{R}^2)$. We identify $\mathbb{R}^2$ with $\mathbb{C}$, so that $z^\perp = iz$ and $K_{BS}(z) = i/(2\pi\bar z)$. By Lemma 2.1(e), for almost every $x\neq 0$ the integral defining $K_{BS}*h(x)$ converges absolutely, and in polar coordinates $y = se^{i\vartheta}$,

$$K_{BS}*h(x) = \frac{i}{2\pi}\int_0^\infty h_0(s)\,s\int_0^{2\pi}\frac{\mathrm{d}\vartheta}{\bar x - se^{-i\vartheta}}\mathrm{d}s.$$

The inner integral equals $2\pi/\bar x$ for $s<|x|$, by the uniformly convergent expansion $(\bar x - se^{-i\vartheta})^{-1} = \bar x^{-1}\sum_{l\ge0}(s/\bar x)^l e^{-il\vartheta}$, and 0 for $s>|x|$, by the uniformly convergent expansion $(\bar x - se^{-i\vartheta})^{-1} = -s^{-1}e^{i\vartheta}\sum_{l\ge0}(\bar x/s)^l e^{il\vartheta}$, integrated term by term: $\int_0^{2\pi}e^{il\vartheta}\mathrm{d}\vartheta = 0$ for every integer $l\neq0$, so only the term $l=0$ of the first expansion survives, and no term of the second. Since $\Gamma_h(r) = 2\pi\int_0^r h_0(s)s\mathrm{d}s$, we get

$$K_{BS}*h(x) = \frac{i}{\bar x}\cdot\frac{\Gamma_h(|x|)}{2\pi} = \Gamma_h(|x|)\frac{x^\perp}{2\pi|x|^2}.$$

Now let $f = f_0(|\cdot|)$, $v_f = K_{BS}*f$, and $\Upsilon\in C^1(\mathbb{R})$ with $|\Upsilon'|\le L$ and $\Upsilon(0)=0$. The constant function $\omega(t)=f$ lies in $X_q(T)$ and is weakly continuous, $\Upsilon(f) = \Upsilon(f_0(|\cdot|))$ is radial, and $|\Upsilon(f)v_f|\le L|f||v_f|$ is integrable by Lemma 2.2(a). Let $\varphi\in\mathcal{D}_T$. In polar coordinates $x = r(\cos\theta,\sin\theta)$ we have $x^\perp\cdot\nabla = \partial_\theta$, so the formula for v_f gives $v_f\cdot\nabla\varphi(t) = \Gamma_f(r)(2\pi r^2)^{-1}\partial_\theta\varphi(t)$, and Fubini's theorem gives, for every t,

$$\int \Upsilon(f)\,v_f\cdot\nabla\varphi(t)\mathrm{d}x = \int_0^\infty \Upsilon(f_0(r))\frac{\Gamma_f(r)}{2\pi r}\Big(\int_0^{2\pi}\partial_\theta\varphi(t)\mathrm{d}\theta\Big)\mathrm{d}r = 0.$$

Since $\varphi(t) = 0$ for t near T, we also have $\int_0^T \int \Upsilon(f)\partial_t\varphi \mathrm{d}x\mathrm{d}t = -\int \Upsilon(f)\varphi(0)\mathrm{d}x$. Hence Equation (5) holds, and Equation (4) is the case $\Upsilon(s) = s$. $\square$

Definition 3.7. *Let $f \in (L^1 \cap L^2)(\mathbb{R}^2)$. A 2-admissible sequence for f on $[0,\infty)$ is a sequence of pairs $(f^{(n)}, \omega^{(n)})_{n\geq 1}$ such that $f^{(n)} \in (L^1 \cap L^2)(\mathbb{R}^2)$, $f^{(n)} \to f$ in $(L^1 \cap L^2)(\mathbb{R}^2)$, and $\omega^{(n)}$ is a renormalized weak solution on $[0,\infty)$ with datum $f^{(n)}$, in the sense of Definition 2.5 with $q = 2$.*

Theorem 3.8. *Let $f \in (L^1 \cap L^2)(\mathbb{R}^2)$ be radial, and let $(f^{(n)}, \omega^{(n)})$ be a 2-admissible sequence for f on $[0,\infty)$. Assume that for some $T > 0$*

$$c := \limsup_{n\to\infty} \sup_{0\leq t\leq T} \|\omega^{(n)}(t) - f\|_{L^1(\mathbb{R}^2)} > 0. \tag{11}$$

Then there exists a subsequence (n_k), together with a function $\omega_\infty \colon [0,\infty) \to (L^1 \cap L^2)(\mathbb{R}^2)$, with the following properties.

(i) *ω_∞ is a renormalized weak solution on $[0,\infty)$ with datum f, in the sense of Definition 2.5 with $q = 2$.*

(ii) *For every $t \in [0,\infty)$, $\omega_\infty(t)$ is equimeasurable with f. Hence $\|\omega_\infty(t)\|_{L^q(\mathbb{R}^2)} = \|f\|_{L^q(\mathbb{R}^2)}$ for every $q \in [1,\infty)$, and, whenever $f \in L^q(\mathbb{R}^2)$ with $q \in (4/3,\infty)$, $\omega_\infty \in X_q(T')$ for every $T' > 0$.*

(iii) *$\omega_\infty \in C([0,\infty); L^q(\mathbb{R}^2))$ whenever $f \in L^q(\mathbb{R}^2)$ with $q \in [1,\infty)$.*

(iv) $\sup_{0\leq t\leq T} \|\omega^{(n_k)}(t) - \omega_\infty(t)\|_{L^1(\mathbb{R}^2)} \to 0$.

(v) $\sup_{0\leq t\leq T} \|\omega_\infty(t) - f\|_{L^1(\mathbb{R}^2)} = c$. *The set $\mathcal{O} = \{t \in [0,\infty) : \omega_\infty(t) \neq f\}$ is open, $0 \notin \mathcal{O}$, and $\mathcal{O} \cap (0,T] \neq \emptyset$. With $t_0 = \inf \mathcal{O} \in [0,T)$, ω_∞ and the stationary solution f are distinct elements of $X_q(T')$ for every $T' > t_0$, whenever $f \in L^q(\mathbb{R}^2)$ with $q \in (4/3,\infty)$.*

(vi) *If $\omega^{(n)}(t)$ is 3-fold rotationally symmetric for all n and t, then so is $\omega_\infty(t)$ for every t.*

Proof. Write $\omega_n = \omega^{(n)}$ and $f_n = f^{(n)}$. We first construct the limit. Pass to a subsequence along which $\sup_{t\leq T} \|\omega_n(t) - f\|_{L^1(\mathbb{R}^2)} \to c$, and put $T_j = T + j$ for $j \geq 1$. For $t \geq 0$ choose j with $T_j > t$; by Definition 3.7, ω_n is a renormalized weak solution in $X_2(T_j)$, so Lemma 3.5 gives $\|\omega_n(t)\|_{L^r(\mathbb{R}^2)} = \|f_n\|_{L^r(\mathbb{R}^2)}$ for $r \in \{1,2\}$. Hence $M := \sup_n \sup_{t\geq 0} \|\omega_n(t)\|_{(L^1\cap L^2)(\mathbb{R}^2)} = \sup_n \|f_n\|_{(L^1\cap L^2)(\mathbb{R}^2)} < \infty$, since f_n converges in $(L^1 \cap L^2)(\mathbb{R}^2)$; in particular $c \leq M + \|f\|_{L^1(\mathbb{R}^2)} < \infty$.

Write $v_n = K_{BS} * \omega_n$, and let $\mathcal{F}$ be as in the proof of Lemma 3.2. Since $\|v_n(t)\|_{L^2(B_R)} \leq C(2,R)M$ by Lemma 2.2(a), Lemma 3.2(b) on $[0,T_j]$ shows that the functions $t \mapsto \int \omega_n(t)\psi$, $\psi \in \mathcal{F}$, are bounded by $M\|\psi\|_{L^2(\mathbb{R}^2)}$ and Lipschitz on every $[0,T_j]$ with a constant that depends neither on n nor on j. By the Arzelà-Ascoli theorem and a diagonal argument over $\psi \in \mathcal{F}$ and j, a subsequence (n_k) converges uniformly on every $[0,T_j]$ for every $\psi \in \mathcal{F}$; below, n runs through (n_k). Since $|\int(\omega_n(t) - \omega_m(t))\psi| \leq 2M\|\psi - \psi'\|_{L^2(\mathbb{R}^2)} + |\int(\omega_n(t) - \omega_m(t))\psi'|$ for $\psi \in L^2(\mathbb{R}^2)$ and $\psi' \in \mathcal{F}$, and $\mathcal{F}$ is dense in $L^2(\mathbb{R}^2)$, $\int \omega_n(t)\psi$ is uniformly Cauchy on every $[0,T_j]$ for every $\psi \in L^2(\mathbb{R}^2)$. For each t the limit is a linear functional of norm at most M, hence $\psi \mapsto \int \omega_\infty(t)\psi$ for a unique $\omega_\infty(t) \in L^2(\mathbb{R}^2)$. This gives the uniform weak convergence

$$\sup_{t\leq T_j} \Big| \int (\omega_n(t) - \omega_\infty(t))\psi \Big| \to 0 \qquad \text{for every } \psi \in L^2(\mathbb{R}^2) \text{ and every } j,$$

and ω_∞ is weakly continuous into $L^2(\mathbb{R}^2)$, as a locally uniform limit of the continuous functions $t \mapsto \int \omega_n(t)\psi$.

Thus $\omega_n(t) \rightharpoonup \omega_\infty(t)$ weakly in $L^2(\mathbb{R}^2)$ for every t. Fix $t \geq 0$. Weak lower semicontinuity of the $L^2(\mathbb{R}^2)$ norm gives $\|\omega_\infty(t)\|_{L^2(\mathbb{R}^2)} \leq \liminf_n \|\omega_n(t)\|_{L^2(\mathbb{R}^2)}$. For the $L^1(\mathbb{R}^2)$ norm, put $\lambda = \liminf_n \|\omega_n(t)\|_{L^1(\mathbb{R}^2)}$ and let $\varepsilon > 0$. There is a further subsequence $(\omega_{n_i}(t))_{i\geq 1}$ with $\|\omega_{n_i}(t)\|_{L^1(\mathbb{R}^2)} \leq \lambda + \varepsilon$ for every i; its terms are real, and it still converges weakly in $L^2(\mathbb{R}^2)$ to $\omega_\infty(t)$. The first conclusion of Lemma 2.2(b), with $p = 2$, applied to this subsequence with $\lambda + \varepsilon$ in the role of the bound M there, gives $\|\omega_\infty(t)\|_{L^1(\mathbb{R}^2)} \leq \lambda + \varepsilon$. As $\varepsilon > 0$ is arbitrary, $\|\omega_\infty(t)\|_{L^1(\mathbb{R}^2)} \leq \liminf_n \|\omega_n(t)\|_{L^1(\mathbb{R}^2)}$. Hence $\|\omega_\infty(t)\|_{L^1(\mathbb{R}^2)} + \|\omega_\infty(t)\|_{L^2(\mathbb{R}^2)} \leq \liminf_n \big(\|\omega_n(t)\|_{L^1(\mathbb{R}^2)} + \|\omega_n(t)\|_{L^2(\mathbb{R}^2)}\big) \leq M$, so $\sup_{t\geq 0} \|\omega_\infty(t)\|_{(L^1\cap L^2)(\mathbb{R}^2)} \leq M$. For $\phi \in L^\infty(\mathbb{R}^2)$, $t \mapsto \int \omega_\infty(t)\phi$ is the pointwise limit of the continuous functions $t \mapsto \int \omega_\infty(t)\phi \mathbf{1}_{B_\ell}$ as $\ell \to \infty$. So ω_∞ is weakly measurable into $L^1(\mathbb{R}^2)$ and into $L^2(\mathbb{R}^2)$, hence strongly measurable into both by Pettis's theorem (Section 2), as $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ are separable; with the bound by M and the convention on intersections in Section 2, $\omega_\infty \in X_2(T_j)$ for every j.

Lemma 2.2(b) with $p = 2$ and the uniform weak convergence give $v_n(t) \to v(t) := K_{BS} * \omega_\infty(t)$ in $L^2(B_R)$ for every t, and $\|v_n(t) - v(t)\|_{L^2(B_R)} \leq 2C(2,R)M$; so dominated convergence in t gives $\int_0^{T_j} \|v_n - v\|_{L^2(B_R)}\mathrm{d}t \to 0$ for every j and $R > 0$. Now let $\varphi \in \mathcal{D}_{T_j}$ with $\operatorname{supp}\varphi \subset \mathbb{R} \times B_R$. By the uniform weak convergence and dominated convergence in t, with the bound $2M\|\partial_t\varphi(t)\|_{L^2(\mathbb{R}^2)}$, $\int_0^{T_j}\int \omega_n \partial_t\varphi \to \int_0^{T_j}\int \omega_\infty \partial_t\varphi$; and $\int f_n\varphi(0) \to \int f\varphi(0)$, because $f_n \to f$ in $L^2(\mathbb{R}^2)$. In

$$\int_0^{T_j}\int (\omega_n v_n - \omega_\infty v)\cdot\nabla\varphi = \int_0^{T_j}\int \omega_n(v_n - v)\cdot\nabla\varphi + \int_0^{T_j}\int (\omega_n - \omega_\infty)\, v\cdot\nabla\varphi,$$

the first term on the right is at most $M\|\nabla\varphi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^2)}\int_0^{T_j}\|v_n - v\|_{L^2(B_R)}\mathrm{d}t \to 0$. In the second term on the right, the inner integral tends to 0 for each t by the uniform weak convergence, because $v(t)\cdot\nabla\varphi(t) \in L^2(\mathbb{R}^2)$, and is bounded by $2C(2,R)M^2\|\nabla\varphi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^2)}$; so dominated convergence applies. Thus Equation (4) for ω_n on $[0,T_j)$ passes to the limit, and ω_∞ is a weak solution in $X_2(T_j)$ with datum f for every j; being weakly continuous, it is its own representative in the sense of Lemma 3.2. For $T' > 0$ choose $T_j > T'$; since $\mathcal{D}_{T'} \subset \mathcal{D}_{T_j}$, the restriction of ω_∞ to $[0,T']$ is a weak solution in $X_2(T')$ with datum f, and it is renormalized: by the remark at the start of Section 3.2, Lemma 3.4(i) with $t_0 = 0$, $u = \omega_\infty$ and $w = K_{BS} * \omega_\infty$ gives Equation (6) on $[0,T')$ for $\Upsilon(\omega_\infty)$ with datum $\Upsilon(f)$, for every $\Upsilon \in \mathrm{Lip}_0$, in particular for every $\Upsilon \in C^1(\mathbb{R})$ with Υ' bounded and $\Upsilon(0) = 0$; this is Equation (5). This proves (i).

We next prove (ii) and (iii). By Lemma 3.5, applied to ω_∞ on each $[0,T_j)$, $\omega_\infty(t)$ and f are equimeasurable for every $t \geq 0$, and integration of the distribution function gives $\|\omega_\infty(t)\|_{L^q(\mathbb{R}^2)} = \|f\|_{L^q(\mathbb{R}^2)}$ for $q \in [1,\infty)$. For (iii), the case $q \in [1,2]$ is part of Lemma 3.5. Let $s_i \to t$ in $[0,\infty)$, and put $g_i = \omega_\infty(s_i)$ and $g = \omega_\infty(t)$. Then $g_i \rightharpoonup g$ weakly in $L^2(\mathbb{R}^2)$ by weak continuity of ω_∞, and $\|g_i\|_{L^2(\mathbb{R}^2)} = \|g\|_{L^2(\mathbb{R}^2)}$, so $\|g_i - g\|^2_{L^2(\mathbb{R}^2)} = 2\|g\|^2_{L^2(\mathbb{R}^2)} - 2\int g_i g \to 0$. Let $q \in (2,\infty)$ with $f \in L^q(\mathbb{R}^2)$, and let $K > 0$. Where $|g_i| \leq K$ and $|g| \leq K$ we have $|g_i - g|^q \leq (2K)^{q-2}|g_i - g|^2$. On the remaining set, where $|g_i| > K$ or $|g| > K$, we have $|g_i - g|^q \leq 2^q \max\{|g_i|,|g|\}^q \leq 2^q\big(|g_i|^q\mathbf{1}_{\{|g_i|>K\}} + |g|^q\mathbf{1}_{\{|g|>K\}}\big)$. Integrating these two bounds gives

$$\|g_i - g\|^q_{L^q(\mathbb{R}^2)} \leq (2K)^{q-2}\|g_i - g\|^2_{L^2(\mathbb{R}^2)} + 2^q\Big(\int_{\{|g_i|>K\}} |g_i|^q\mathrm{d}x + \int_{\{|g|>K\}} |g|^q\mathrm{d}x\Big).$$

The last two integrals depend only on distribution functions. For $h \in L^q(\mathbb{R}^2)$, write $|h|^q = K^q + \int_K^{|h|} qs^{q-1}\mathrm{d}s$ on $\{|h| > K\}$ and integrate in x; Tonelli's theorem gives

$$\int_{\{|h|>K\}} |h|^q\mathrm{d}x = K^q\,|\{|h| > K\}| + \int_K^\infty qs^{q-1}\,|\{|h| > s\}|\mathrm{d}s.$$

Since g_i and g are equimeasurable with f by (ii), the measure $|\{|h|>s\}| = |\{h>s\}| + |\{h<-s\}|$ is the same for $h = g_i$, g and f, for every $s>0$. Hence both integrals equal $\int_{\{|f|>K\}} |f|^q \mathrm{d}x$, and

$$\|g_i - g\|^q_{L^q(\mathbb{R}^2)} \le (2K)^{q-2}\|g_i - g\|^2_{L^2(\mathbb{R}^2)} + 2^{q+1}\int_{\{|f|>K\}} |f|^q \mathrm{d}x,$$

and letting $i\to\infty$ and then $K\to\infty$ gives $g_i \to g$ in $L^q(\mathbb{R}^2)$. This proves (iii). Now let $q\in(4/3,\infty)$ with $f\in L^q(\mathbb{R}^2)$ and $T'>0$. By (iii) and the norm identities, ω_∞ is continuous from $[0,T']$ into $(L^1\cap L^q)(\mathbb{R}^2)$, so $\omega_\infty \in X_q(T')$. This proves (ii).

We now prove (iv). Fix j with $T<T_j$. We claim: if (n_i) is a subsequence of (n_k) and $t_i\in[0,T]$ with $t_i\to t$, then $\omega_{n_i}(t_i)\to\omega_\infty(t)$ in $L^1(\mathbb{R}^2)$. Granting the claim, if (iv) failed, there would be $\epsilon>0$, a subsequence (n_i) and $t_i\in[0,T]$ with $\|\omega_{n_i}(t_i)-\omega_\infty(t_i)\|_{L^1(\mathbb{R}^2)}\ge\epsilon$; along a further subsequence $t_i\to t\in[0,T]$, and the claim and (iii) give $\|\omega_{n_i}(t_i)-\omega_\infty(t_i)\|_{L^1(\mathbb{R}^2)}\to 0$, a contradiction.

To prove the claim, write n for n_i. For $\psi\in L^2(\mathbb{R}^2)$,

$$\Big|\int(\omega_n(t_i)-\omega_\infty(t))\psi\Big| \le \sup_{s\le T_j}\Big|\int(\omega_n(s)-\omega_\infty(s))\psi\Big| + \Big|\int(\omega_\infty(t_i)-\omega_\infty(t))\psi\Big| \to 0,$$

by the uniform weak convergence on $[0,T_j]$ and weak continuity of ω_∞. By the norm identities for ω_n above and part (ii), $\|\omega_n(t_i)\|_{L^2(\mathbb{R}^2)} = \|f_n\|_{L^2(\mathbb{R}^2)} \to \|f\|_{L^2(\mathbb{R}^2)} = \|\omega_\infty(t)\|_{L^2(\mathbb{R}^2)}$; expanding $\|\omega_n(t_i)-\omega_\infty(t)\|^2_{L^2(\mathbb{R}^2)}$ shows that $\omega_n(t_i)\to\omega_\infty(t)$ in $L^2(\mathbb{R}^2)$, hence in $L^1(B_{2R})$ for every $R>0$. Equation (10) on $[0,T_j)$, applied to ω_n and to ω_∞, gives

$$\int_{\{|x|\ge 2R\}} \big(|\omega_n(t_i)| + |\omega_\infty(t)|\big)\mathrm{d}x \le \|f_n - f\|_{L^1(\mathbb{R}^2)} + 2\int_{\{|x|\ge R\}}|f|\mathrm{d}x + \frac{2\|\nabla\varsigma\|_{L^\infty(\mathbb{R}^2)}}{R}C_{\mathrm{v}}T_jM^2.$$

Letting $n\to\infty$ and then $R\to\infty$ proves the claim.

We now prove (v). By (iv),

$$\Big|\sup_{t\le T}\|\omega_n(t)-f\|_{L^1(\mathbb{R}^2)} - \sup_{t\le T}\|\omega_\infty(t)-f\|_{L^1(\mathbb{R}^2)}\Big| \le \sup_{t\le T}\|\omega_n(t)-\omega_\infty(t)\|_{L^1(\mathbb{R}^2)} \to 0,$$

so the second supremum equals $c>0$. By (iii) with $q=1$, $t\mapsto\|\omega_\infty(t)-f\|_{L^1(\mathbb{R}^2)}$ is continuous on $[0,\infty)$, so $\mathcal{O}$ is relatively open in $[0,\infty)$, and it meets $[0,T]$. Since $\omega_\infty(0)=f$ by Lemma 3.2(a), $0\notin\mathcal{O}$; so $\mathcal{O}$ is open in $\mathbb{R}$, it contains some $t_1\in(0,T]$, and $0\le t_0=\inf\mathcal{O}<t_1\le T$, because $\mathcal{O}$ contains a neighborhood of t_1. For $T'>t_0$ the set $\mathcal{O}\cap(0,T')$ is nonempty and open, hence of positive measure, and $\omega_\infty(t)\ne f$ on it. Thus, for every $q\in(4/3,\infty)$ with $f\in L^q(\mathbb{R}^2)$, ω_∞, which lies in $X_q(T')$ by (ii), and the constant function f, which lies in $X_q(T')$ and is a solution by Lemma 3.6, are distinct elements of $X_q(T')$.

Finally we prove (vi). Let $t\ge 0$ and $\psi\in C_c^\infty(\mathbb{R}^2)$. Since $|\det\mathsf{O}|=1$ and $\omega_n(t)\circ\mathsf{O}=\omega_n(t)$ almost everywhere, by the 3-fold rotational symmetry of $\omega_n(t)$, the uniform weak convergence gives

$$\begin{aligned}\int\omega_\infty(t,\mathsf{O}x)\psi(x)\mathrm{d}x &= \int\omega_\infty(t,y)\psi(\mathsf{O}^{-1}y)\mathrm{d}y = \lim_n\int\omega_n(t,y)\psi(\mathsf{O}^{-1}y)\mathrm{d}y\\ &= \lim_n\int\omega_n(t,\mathsf{O}x)\psi(x)\mathrm{d}x = \int\omega_\infty(t,x)\psi(x)\mathrm{d}x,\end{aligned}$$

so $\omega_\infty(t)\circ\mathsf{O}=\omega_\infty(t)$ almost everywhere. □

Corollary 3.9. *Let $f \in (L^1 \cap L^q)(\mathbb{R}^2)$ for every $q \in [1,\infty)$ be radial. Let $m \geq 1$ be an integer, $\chi \in C_c^\infty((0,\infty))$ real, and $\Psi(x) = \chi(|x|)\,(\bar{x})^m$ (with x written as a complex number). Let $(f^{(n)}, \omega^{(n)})$ be a 2-admissible sequence for f on $[0,\infty)$ such that $\omega^{(n)}(t)$ is 3-fold rotationally symmetric for all n and t. Assume that there are $c_0 > 0$, $\bar{T} \in (0,\infty)$ and times $T^{(n)} \in [0,\bar{T}]$ with*

$$\Big| \int \omega^{(n)}(T^{(n)})\,\Psi \Big| \geq c_0 \qquad \text{for all large } n.$$

Then there is a renormalized weak solution ω_∞ on $[0,\infty)$ with datum f, in the sense of Definition 2.5 with $q = 2$, such that $\omega_\infty(t)$ is equimeasurable with f and 3-fold rotationally symmetric for every $t \geq 0$, $\omega_\infty \in C([0,\infty); L^q(\mathbb{R}^2))$ for every $q \in [1,\infty)$, and, for some $t_0 \in [0,\bar{T})$, ω_∞ and f are distinct elements of $X_q(T')$ for every $T' > t_0$ and every $q \in (4/3,\infty)$.

Proof. Write $f = f_0(|\cdot|)$. Since χ has compact support in $(0,\infty)$, the function Ψ lies in $C_c^\infty(\mathbb{R}^2;\mathbb{C})$, and in polar coordinates $\int f\Psi = \int_0^\infty f_0(r)\chi(r)r^{m+1}\big(\int_0^{2\pi} e^{-im\theta}\mathrm{d}\theta\big)\mathrm{d}r = 0$, because $m \neq 0$. Hence, for all large n,

$$c_0 \leq \Big| \int (\omega^{(n)}(T^{(n)}) - f)\Psi \Big| \leq \|\Psi\|_{L^\infty(\mathbb{R}^2)} \sup_{t \leq \bar{T}} \|\omega^{(n)}(t) - f\|_{L^1(\mathbb{R}^2)}.$$

Thus $\|\Psi\|_{L^\infty(\mathbb{R}^2)} > 0$, and Equation (11) holds with $T = \bar{T}$, with $c \geq c_0/\|\Psi\|_{L^\infty(\mathbb{R}^2)}$. Theorem 3.8 gives a subsequence and ω_∞. By its part (i), ω_∞ is a renormalized weak solution on $[0,\infty)$ with datum f, with $q = 2$. Part (ii) gives equimeasurability, part (iii) gives $\omega_\infty \in C([0,\infty); L^q(\mathbb{R}^2))$ for every $q \in [1,\infty)$, and part (v) gives, with the set $\mathcal{O}$ defined there, $t_0 = \inf \mathcal{O} \in [0,\bar{T})$ and the distinctness in $X_q(T')$ for every $T' > t_0$ and $q \in (4/3,\infty)$. Part (vi) gives the 3-fold rotational symmetry of $\omega_\infty(t)$. □

Remark 3.10. Corollary 3.9 is a conditional non-uniqueness theorem. Let f be radial with $f \in (L^1 \cap L^q)(\mathbb{R}^2)$ for every $q \in [1,\infty)$. Suppose that one can produce data $f^{(n)} \to f$ in $(L^1 \cap L^2)(\mathbb{R}^2)$ and 3-fold rotationally symmetric renormalized weak solutions $\omega^{(n)}$ on $[0,\infty)$ with these data and vorticity in $L_t^\infty(L^1 \cap L^2)_x$, such that one fixed moment $\int \omega^{(n)}(T^{(n)})\Psi$ of the form above has modulus at least $c_0 > 0$ at times $T^{(n)} \leq \bar{T}$, with c_0 and $\bar{T}$ independent of n. Then the Euler equations with datum f have two distinct renormalized weak solutions: the stationary solution f and ω_∞, which differ in $X_q(T')$ for every $T' > t_0$ and every $q \in (4/3,\infty)$. Since $\int f\Psi = 0$ for radial f, the hypothesis says that the approximate solutions leave f by a fixed amount, in this moment, within a time bounded independently of n. Part II [8] produces such a sequence for the datum of Theorem 1.1.

4 The Ring and Its Linearization

We fix a certain exact radial steady state supported in an annulus, which we call the **Ring**, and study its linearization, that of the Euler equations at the Ring plus a point vortex of circulation $2\pi G$ at the origin.

4.1 The Ring

Definition 4.1. *Put*

$$a_0 = -7304210039150669 \cdot 2^{-54}, \qquad d = 5764607523034235 \cdot 2^{-58},$$
$$\xi_L = -4528212286209965 \cdot 2^{-52}, \qquad \xi_a = -4077852323472915 \cdot 2^{-52},$$
$$\xi_b = \tfrac{1}{2}, \qquad \xi_R = 5404319552844595 \cdot 2^{-53},$$

and $H(x) = (1+e^{-2x})^{-1}$ for $x \in \mathbb{R}$. Fix, once and for all, a function $\chi \in C_c^\infty(\mathbb{R})$ with values in $[0,1]$ that is nondecreasing on $(-\infty, \xi_a]$, equal to 1 on $[\xi_a, \xi_b]$, nonincreasing on $[\xi_b, \infty)$, and supported in $[\xi_L, \xi_R]$ (such functions exist by Lemma 4.2). Put

$$\varpi(\xi) = \chi(\xi)\, H\Big(\frac{\xi - a_0}{d}\Big) H\Big(-\frac{\xi}{d}\Big) \quad (\xi \in \mathbb{R}),$$
$$\overline{\omega}_{\mathrm{c}}(x) = \varpi(\log|x|) \quad (x \neq 0), \qquad \overline{\omega}_{\mathrm{c}}(0) = 0,$$

$$R_{\mathrm{c}} = e^{\xi_R}, \qquad \Phi(y) = \overline{\omega}_{\mathrm{c}}(R_{\mathrm{c}} y), \qquad a = e^{\xi_L - \xi_R}, \qquad \overline{G} = \tfrac{1}{50} R_{\mathrm{c}}^{-2}.$$

*To four significant figures, $a_0 \approx -0.4055$, $d \approx 0.02000$, $\xi_L \approx -1.005$, $\xi_a \approx -0.9055$, $\xi_R \approx 0.6000$, $R_{\mathrm{c}} \approx 1.822$, $a \approx 0.2008$ and $\overline{G} \approx 0.006024$. We call Φ **the ring**. The functions $\overline{\omega}_{\mathrm{c}}$ and Φ describe the same ring in the variables x and y, related by $x = R_{\mathrm{c}} y$; we say that $\overline{\omega}_{\mathrm{c}}$ is the ring in **reference units** and Φ the ring in **ring units**. From now on $m = 3$, except where an integer $m \geq 1$ is quantified explicitly.*

Figure 2 shows the ring, for the function χ constructed in the proof of Lemma 4.2.

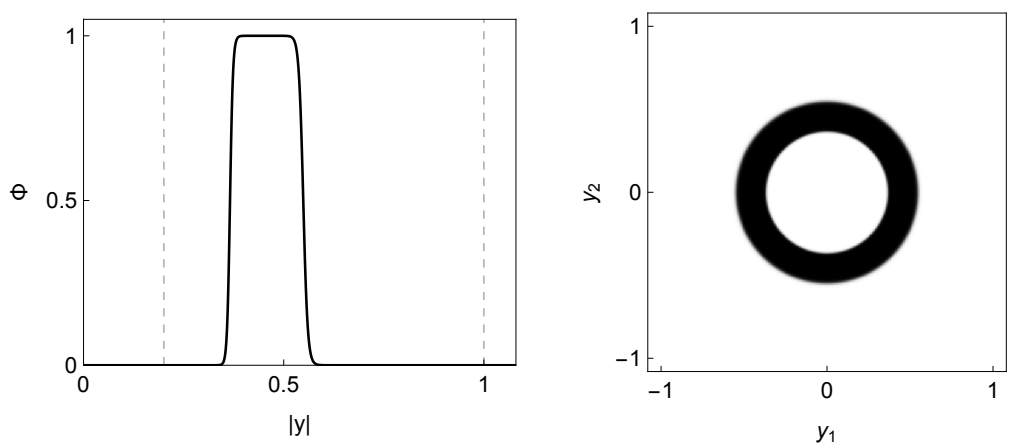

Figure 2: The ring Φ of Definition 4.1 in ring units, for the function χ constructed in the proof of Lemma 4.2. Left: the radial profile $\Phi(|y|)$; the dashed lines mark $|y| = a$ and $|y| = 1$, between which Φ is supported. Right: Φ on the plane, darker where Φ is larger (black on the plateau, where Φ is close to 1). Drawn with Mathematica from the exact formulas; the figure is an illustration and is not used in any proof.

Lemma 4.2. *A function χ as in Definition 4.1 exists. Moreover $a_0 < 0 < d$, $\xi_L < \xi_a \leq a_0/2 \leq \xi_b < \xi_R$, $\ell d < 1$ for every integer $1 \leq \ell \leq 9$, and $0 < a < 1$. The function $\overline{\omega}_{\mathrm{c}}$ lies in $C_c^\infty(\mathbb{R}^2)$, is radial, satisfies $0 \leq \overline{\omega}_{\mathrm{c}} < 1$, and $\operatorname{supp} \overline{\omega}_{\mathrm{c}} \subset \{e^{\xi_L} \leq |x| \leq e^{\xi_R}\}$. Consequently $\Phi \in C_c^\infty(\mathbb{R}^2)$ is radial, $0 \leq \Phi < 1$, $\Phi \not\equiv 0$, and $\operatorname{supp} \Phi \subset \{a \leq |y| \leq 1\}$.*

Proof. Let $e(x) = e^{-1/x}$ for $x > 0$ and $e(x) = 0$ for $x \leq 0$, and $S(x) = e(x)/(e(x) + e(1-x))$. Then $S \in C^\infty(\mathbb{R})$ is nondecreasing, $S = 0$ on $(-\infty, 0]$ and $S = 1$ on $[1, \infty)$. The function

$$\chi(\xi) = S\Big(\frac{\xi - \xi_L}{\xi_a - \xi_L}\Big)\, S\Big(\frac{\xi_R - \xi}{\xi_R - \xi_b}\Big)$$

is smooth with values in $[0,1]$. Its second factor is 1 for $\xi \leq \xi_b$ and its first factor is 1 for $\xi \geq \xi_a$; since $\xi_a \leq \xi_b$, χ is the first factor on $(-\infty, \xi_a]$, 1 on $[\xi_a, \xi_b]$, the second factor on $[\xi_b, \infty)$, and 0 outside $[\xi_L, \xi_R]$. It has the required properties once the order of the data is checked.

We compare the data as fractions with a common power of two. The signs of a_0 and d are those of their numerators; $4528212286209965 > 4077852323472915$ gives $\xi_L < \xi_a$; $a_0/2 = -7304210039150669 \cdot 2^{-55}$ and $\xi_a = -32622818587783320 \cdot 2^{-55}$ give $\xi_a < a_0/2 < 0 < \xi_b$; and $\xi_b = 4503599627370496 \cdot 2^{-53} < \xi_R$. For $\ell d < 1$ it suffices to take $\ell = 9$: $9 \cdot 5764607523034235 = 51881467707308115 < 2^{58} = 288230376151711744$. Since $\xi_L < \xi_R$, $0 < a < 1$.

Since ϖ is smooth and supported in $[\xi_L, \xi_R]$, $\overline{\omega}_{\mathrm{c}}$ is smooth on $\mathbb{R}^2 \setminus \{0\}$ and vanishes on $B_{e^{\xi_L}}$, so $\overline{\omega}_{\mathrm{c}} \in C_c^\infty(\mathbb{R}^2)$ with the stated support. Since $0 < H < 1$ and $0 \le \chi \le 1$, $0 \le \overline{\omega}_{\mathrm{c}} < 1$. The statements on Φ follow from $e^{\xi_L}/R_{\mathrm{c}} = a$, $e^{\xi_R}/R_{\mathrm{c}} = 1$, and $\Phi \not\equiv 0$ because $\varpi(\xi_a) = H((\xi_a - a_0)/d)H(-\xi_a/d) > 0$. □

In ring units we use the closed annuli

$$N = \{\tfrac{a}{2} \le |y| \le 2\}, \qquad N' = \{\tfrac{a}{4} \le |y| \le 3\};$$

so $\{a \le |y| \le 1\} \subset N \subset N'$, and the ring Φ is supported in N.

4.2 The Linearization and the Evans Function

Definition 4.3. *A **mode setting** consists of: a real radial function $\overline{\omega} \in C_c^\infty(\mathbb{R}^2)$, written also as a function of $r = |x|$; numbers $0 < r_1 \le r_2$ such that the radial derivative $\partial_r \overline{\omega}$ is supported in $[r_1, r_2]$, and $\xi_1 = \log r_1$, $\xi_2 = \log r_2$; a number $G \in \mathbb{R}$; and an annulus $N_* = \{\rho_- \le |x| \le \rho_+\}$ with $0 < \rho_- < r_1$ and $r_2 < \rho_+$. In a mode setting we put*

$$\Omega(r; G) = r^{-2}\Big(\frac{\Gamma_{\overline{\omega}}(r)}{2\pi} + G\Big), \qquad k_\ell(r) = \frac{i\ell\, \partial_r \overline{\omega}(r)}{r},$$
$$(\mathcal{G}_\ell g)(r) = -\frac{1}{2|\ell|}\int_0^\infty \Big(\frac{\min(r,\rho)}{\max(r,\rho)}\Big)^{|\ell|} g(\rho)\, \rho \mathrm{d}\rho$$

*for $\ell \in \mathbb{Z}\setminus\{0\}$. For an integer $s \ge 0$, Y^s is the space of $w \in H^s(\mathbb{R}^2; \mathbb{C})$ with $\operatorname{supp} w \subset N_*$, and Y_ℓ^s is the space of radial profiles g with $\operatorname{supp} g \subset [\rho_-, \rho_+]$ and $g(r)e^{i\ell\theta} \in H^s(\mathbb{R}^2)$, normed by $\|g(r)e^{i\ell\theta}\|_{H^s}$. As a set, $Y_\ell^0 = L^2([\rho_-, \rho_+], r\mathrm{d}r)$, and the two norms differ by the factor $\sqrt{2\pi}$. The **linearization** at $(\overline{\omega}, G)$, that is, the linearization of the Euler equations (1) at the steady state $\overline{\omega}$ plus a point vortex of circulation $2\pi G$ at the origin, is*

$$L(G)w = -\Omega(\cdot; G)\, \partial_\theta w - (K_{BS} * w) \cdot \nabla \overline{\omega}.$$

*The **mode operators** are $L_\ell(G)g = -i\ell\Omega g + k_\ell\, \mathcal{G}_\ell g$ for $\ell \ne 0$ and $L_0 = 0$. For $\ell \ge 1$, $\operatorname{Re}\lambda > 0$ and $\xi \in \mathbb{R}$, the **Rayleigh potential** is*

$$V_\ell(\xi; \lambda, G) = \ell^2 + \frac{i\ell\, \nu(\xi)}{\lambda + i\ell\, \Omega(e^\xi; G)}, \qquad \nu(\xi) = r\partial_r \overline{\omega}(r)\big|_{r=e^\xi},$$

and $M_\lambda = (\lambda + i\ell\Omega(\cdot; G))^{-1}$. This is the resolvent at λ of the transport part $-i\ell\Omega$ of $L_\ell(G)$, well defined because Ω is real; with it, the equation $(\lambda - L_\ell(G))g = h$ reads $g = M_\lambda(h + k_\ell \mathcal{G}_\ell g)$. The equation $\partial_\xi^2 \psi = V_\ell(\cdot; \lambda, G)\, \psi$ is the Rayleigh equation of the mode ℓ in the variable $\xi = \log r$.

Here $\Gamma_{\overline{\omega}}(r) = \int_{B_r} \overline{\omega} = 2\pi \int_0^r \overline{\omega}(\rho)\rho \mathrm{d}\rho$, so $\Omega(r; G)$, smooth on $(0, \infty)$, is the angular velocity of the flow of $\overline{\omega}$ and a point vortex of circulation $2\pi G$ at the origin. The **mode profiles** g_ℓ of $w \in Y^0$ are its angular modes (Section 2), $w = \sum_\ell g_\ell(r)e^{i\ell\theta}$ in $L^2(\mathbb{R}^2)$, and $P_\ell w = g_\ell(r)e^{i\ell\theta} = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\ell\phi}\, w \circ \mathsf{O}_\phi \mathrm{d}\phi$. Rotations preserve N_* and are bounded on $H^s(\mathbb{R}^2)$, uniformly in the angle: by the chain rule, for

$|\gamma| = j$ the derivative $\partial^\gamma(w \circ \mathsf{O}_\phi)$ is a sum of at most 2^j terms $c\,(\partial^\beta w) \circ \mathsf{O}_\phi$ with $|\beta| = j$ and $|c| \le 1$, so $\|w \circ \mathsf{O}_\phi\|_{H^s} \le C_s \|w\|_{H^s}$ with C_s depending only on s. (For $s \ge 2$ they are in general not isometries of the norm of Section 2.) By Minkowski's integral inequality, P_ℓ is therefore a bounded projection on every Y^s, with norm at most C_s, and $g_\ell \in Y^s_\ell$ for $w \in Y^s$.

We work in the variable $\xi = \log r$, the usual logarithmic change of variables, which maps the half-line $r > 0$ onto $\mathbb{R}$. Since $\partial_\xi = r\partial_r$, the radial part of the Laplacian on the mode ℓ satisfies

$$\partial_r^2\psi + r^{-1}\partial_r\psi - \ell^2 r^{-2}\psi = r^{-2}\big(\partial_\xi^2\psi - \ell^2\psi\big).$$

So r^2 times this operator is $\partial_\xi^2 - \ell^2$, which has constant coefficients and no first-order term, and the singular point $r = 0$ of the radial equation moves to $\xi = -\infty$. Since ν vanishes outside $[\xi_1, \xi_2]$, so does $V_\ell - \ell^2$; there the solutions of the Rayleigh equation are the combinations of $e^{\ell\xi} = r^\ell$ and $e^{-\ell\xi} = r^{-\ell}$. A dilation $x \mapsto cx$ becomes the translation $\xi \mapsto \xi + \log c$; this is how Equation (13) below compares the ring in reference units with the ring in ring units.

The Evans function of Definition 4.4 detects the eigenvalues of $L_\ell(G)$ in $\{\operatorname{Re}\lambda > 0\}$, for $\ell \ge 1$. If $L_\ell(G)g = \lambda g$ with $g \ne 0$, then $g = M_\lambda k_\ell \psi$ for the stream function $\psi = \mathcal{G}_\ell g$. By the proof of Lemma 4.7 below, ψ, as a function of ξ, is then a nonzero solution of the Rayleigh equation with $\psi = O(e^{\ell\xi})$ as $\xi \to -\infty$ and $\psi = O(e^{-\ell\xi})$ as $\xi \to +\infty$; these conditions say that $\psi(r)e^{i\ell\theta}$ is regular at the origin and decays at infinity. On $(-\infty, \xi_1]$ the first condition leaves only the multiples of $e^{\ell\xi}$, whose Cauchy data at ξ_1 are multiples of $(1, \ell)$; this is the normalization of ψ_ℓ in Definition 4.4. On $[\xi_2, \infty)$, $\psi_\ell = Ae^{\ell(\xi-\xi_2)} + Be^{-\ell(\xi-\xi_2)}$ with $2\ell A = \partial_\xi\psi_\ell(\xi_2) + \ell\psi_\ell(\xi_2)$, so ψ_ℓ satisfies the second condition exactly when $A = 0$. Thus $\mathcal{E}_\ell(\lambda; G) = 2\ell e^{-\ell(\xi_2-\xi_1)}A$ measures the growing part r^ℓ of ψ_ℓ at infinity, and Lemma 4.7 shows that its zeros in $\{\operatorname{Re}\lambda > 0\}$ are exactly the eigenvalues there.

The factor $e^{-\ell(\xi_2-\xi_1)}$ makes $\mathcal{E}_\ell$ equal to minus the Wronskian of the solutions that equal $e^{\ell\xi}$ near $-\infty$ and $e^{-\ell\xi}$ near $+\infty$ (Lemma 4.6(2) below). So $\mathcal{E}_\ell$ depends only on $\overline{\omega}$, G and λ, not on the choice of r_1 and r_2, and for $\overline{\omega} = 0$ it equals 2ℓ. Since $\mathcal{E}_\ell$ is holomorphic in λ (Lemma 4.6(1)) and is computed from an initial value problem on the compact interval $[\xi_1, \xi_2]$, its zeros can be counted with the argument principle. Part II verifies (N1) and (N2) of Hypothesis 4.17 below in this way, by certified winding numbers of $\mathcal{E}_3$.

Definition 4.4. *In a mode setting, for an integer $\ell \ge 1$ and $\operatorname{Re}\lambda > 0$, let $\psi_\ell(\cdot; \lambda, G)$ be the solution on $\mathbb{R}$ of $\partial_\xi^2\psi = V_\ell(\cdot; \lambda, G)\psi$ with $\psi(\xi_1) = 1$, $\partial_\xi\psi(\xi_1) = \ell$, and put*

$$\mathcal{E}_\ell(\lambda; G) = e^{-\ell(\xi_2-\xi_1)}\big(\partial_\xi\psi_\ell(\xi_2) + \ell\,\psi_\ell(\xi_2)\big),$$

*where $\psi_\ell = \psi_\ell(\cdot; \lambda, G)$. We call $\mathcal{E}_\ell$ the **Evans function**.*

Lemma 4.5. *In a mode setting the following hold.*

(1) *For $\ell \ne 0$ and $g \in Y^0_\ell$, $\mathcal{G}_\ell g$ is the unique $\psi \in C^1((0,\infty))$ with $\partial_r\psi$ locally absolutely continuous that solves $\partial_r^2\psi + r^{-1}\partial_r\psi - \ell^2 r^{-2}\psi = g$ almost everywhere on $(0,\infty)$ and is $O(r^{|\ell|})$ as $r \to 0$ and $O(r^{-|\ell|})$ as $r \to \infty$; equivalently, $(\mathcal{G}_\ell g)(r)e^{i\ell\theta} = \Delta^{-1}(g(r)e^{i\ell\theta})$, where Δ^{-1} is the convolution with $\frac{1}{2\pi}\log|x|$. The operator $\mathcal{G}_\ell$ is bounded from Y^0_ℓ to $C([\rho_-, \rho_+])$.*

(2) *$L(G)$ preserves angular modes: for an integer $s \ge 0$ and $w = \sum_\ell g_\ell(r)e^{i\ell\theta} \in Y^{s+1}$, $P_\ell L(G)w = (L_\ell(G)g_\ell)(r)e^{i\ell\theta}$ for every ℓ; for the coupling term this holds for every $w \in Y^0$: $P_\ell\big(-(K_{BS} * w)\cdot\nabla\overline{\omega}\big) = (k_\ell\mathcal{G}_\ell g_\ell)(r)e^{i\ell\theta}$. Moreover $L_{-\ell}(G)\bar{g} = \overline{L_\ell(G)g}$, and the coupling term commutes with rotations.*

(3) *For $\ell \ge 1$ and $\operatorname{Re}\lambda > 0$, $|\lambda + i\ell\Omega(r; G)| \ge \operatorname{Re}\lambda$ for every $r > 0$; V_ℓ is smooth in ξ, holomorphic in λ, and equal to ℓ^2 outside $[\xi_1, \xi_2]$.*

(4) *For $\ell \neq 0$, $L_\ell(G)$ is a bounded operator on Y^0_ℓ. Consequently the series $e^{tL_\ell(G)} = \sum_{n\geq 0} t^n L_\ell(G)^n/n!$ converges in operator norm for every $t \in \mathbb{R}$ and defines a group of bounded operators on Y^0_ℓ, continuous in t in operator norm, with $\frac{\mathrm{d}}{\mathrm{d}t} e^{tL_\ell(G)} = L_\ell(G)e^{tL_\ell(G)} = e^{tL_\ell(G)}L_\ell(G)$; in particular $L_\ell(G)$ generates the C_0-semigroup $(e^{tL_\ell(G)})_{t\geq 0}$.*

Proof. (1) Write $n = |\ell|$. For $g \in Y^0_\ell$ the integrals below converge absolutely by the Cauchy-Schwarz inequality, and

$$(\mathcal{G}_\ell g)(r) = -\frac{1}{2n}\Big[r^{-n}\int_0^r \rho^{n+1}g(\rho)\mathrm{d}\rho + r^n\int_r^\infty \rho^{1-n}g(\rho)\mathrm{d}\rho\Big]$$

is continuous in $r > 0$. The kernel has modulus at most $\frac{1}{2n}$ and g vanishes outside $[\rho_-, \rho_+]$, so by the Cauchy-Schwarz inequality in $L^2([\rho_-, \rho_+], \rho\mathrm{d}\rho)$, for every $r > 0$,

$$|(\mathcal{G}_\ell g)(r)| \leq \frac{1}{2n}\int_{\rho_-}^{\rho_+} |g|\rho\mathrm{d}\rho \leq \frac{1}{2n}\Big(\int_{\rho_-}^{\rho_+} \rho\mathrm{d}\rho\Big)^{1/2}\|g\|_{L^2(r\mathrm{d}r)} \leq \frac{\rho_+}{2n}\|g\|_{L^2(r\mathrm{d}r)},$$

as $\int_{\rho_-}^{\rho_+} \rho\mathrm{d}\rho = \frac{1}{2}(\rho_+^2 - \rho_-^2) \leq \rho_+^2$; this is the last assertion. Differentiating, the terms from the variable limits cancel, so $\psi = \mathcal{G}_\ell g \in C^1$ with $\partial_r\psi = \frac{1}{2}\big[r^{-n-1}\int_0^r \rho^{n+1}g - r^{n-1}\int_r^\infty \rho^{1-n}g\big]$ absolutely continuous, and $\partial_r^2\psi + r^{-1}\partial_r\psi - n^2r^{-2}\psi = g$ almost everywhere. Since g vanishes outside $[\rho_-, \rho_+]$, for $r \leq \rho_-$ the first integral in the formula for ψ vanishes and the second is $\int_{\rho_-}^{\rho_+} \rho^{1-n}g\mathrm{d}\rho$, and for $r \geq \rho_+$ the second vanishes and the first is $\int_{\rho_-}^{\rho_+} \rho^{n+1}g\mathrm{d}\rho$. So

$$\psi(r) = c_-r^n \ (r \leq \rho_-), \qquad \psi(r) = c_+r^{-n} \ (r \geq \rho_+), \qquad c_\mp = -\frac{1}{2n}\int_{\rho_-}^{\rho_+} \rho^{1\mp n}g(\rho)\mathrm{d}\rho,$$

and ψ satisfies the growth conditions. For uniqueness, let $\tilde{\psi}$ be a second solution with these properties and $D = \psi - \tilde{\psi}$. Then $D \in C^1$, $\partial_r D$ is locally absolutely continuous, and $\partial_r^2 D = -r^{-1}\partial_r D + n^2r^{-2}D$ almost everywhere; the right side is continuous, so $D \in C^2((0,\infty))$ solves the homogeneous equation classically. In the variable $\xi = \log r$ this equation reads $\partial_\xi^2 D = n^2 D$ (by the identity displayed before Definition 4.4), whose solutions are $Ae^{n\xi} + Be^{-n\xi}$; hence $D = Ar^n + Br^{-n}$ for constants A and B. As $r \to 0$, $D = O(r^n)$, so $B = r^n(D - Ar^n) = O(r^{2n})$ and $B = 0$; then, as $r \to \infty$, $Ar^n = D = O(r^{-n})$, so $A = 0$. Thus $D = 0$. For the equivalence, let first $g \in C_c^\infty((\rho_-, \rho_+))$, $w = g(r)e^{i\ell\theta}$, $\Psi = (\mathcal{G}_\ell g)(r)e^{i\ell\theta}$ and $\Psi_0 = \frac{1}{2\pi}\log|\cdot| * w$. Near the origin Ψ is a multiple of the harmonic polynomial x^n or $\bar{x}^n$ (x read as a complex number), so $\Psi \in C^\infty(\mathbb{R}^2)$, and $\Delta\Psi = w$ because $\Delta(\psi(r)e^{i\ell\theta}) = (\partial_r^2\psi + r^{-1}\partial_r\psi - \ell^2r^{-2}\psi)e^{i\ell\theta}$. Also $\Delta\Psi_0 = w$, and $\Psi_0(x) = O(|x|^{-1})$ as $|x| \to \infty$. Indeed, since $\ell \neq 0$, $\int w = \int_0^\infty g(r)r\mathrm{d}r\int_0^{2\pi} e^{i\ell\theta}\mathrm{d}\theta = 0$, so

$$\Psi_0(x) = \frac{1}{2\pi}\int \big(\log|x-y| - \log|x|\big)\, w(y)\mathrm{d}y.$$

Read x and y as complex numbers and let $|x| \geq 2\rho_+$. On $\operatorname{supp} w$, $|y| \leq \rho_+ \leq |x|/2$, so $s = |1 - y/x| \geq \frac{1}{2}$ and $|s-1| \leq |y|/|x|$; since $|\log s| \leq 2|s-1|$ for $s \geq \frac{1}{2}$, we get $|\log|x-y| - \log|x|| = |\log s| \leq 2|y|/|x|$ and $|\Psi_0(x)| \leq \frac{\rho_+}{\pi|x|}\|w\|_{L^1(\mathbb{R}^2)}$. So $\Psi - \Psi_0$ is harmonic and tends to 0 at infinity, and Liouville's theorem gives $\Psi = \Psi_0$. General $g \in Y^0_\ell$ follow by density, both sides being continuous from $L^2([\rho_-, \rho_+], r\mathrm{d}r)$ into $L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

(2) With the conventions of Section 2, $K_{BS} * w = \nabla^\perp\Delta^{-1}w$ and $(K_{BS} * w)\cdot\mathbf{e}_r = -r^{-1}\partial_\theta\Delta^{-1}w$. For $w = g(r)e^{i\ell\theta}$, $g \in C_c^\infty((\rho_-, \rho_+))$, $\ell \neq 0$, part (1) gives $-(K_{BS}*w)\cdot\nabla\overline{\omega} = -\partial_r\overline{\omega}(r)\,(K_{BS}*w)\cdot\mathbf{e}_r = \frac{i\ell\,\partial_r\overline{\omega}}{r}(\mathcal{G}_\ell g)e^{i\ell\theta} = (k_\ell\mathcal{G}_\ell g)e^{i\ell\theta}$; for $\ell = 0$, $\Delta^{-1}w$ is radial, so $K_{BS} * w = \nabla^\perp\Delta^{-1}w$ is a multiple of $\mathbf{e}_\theta$, while $\nabla\overline{\omega} = \partial_r\overline{\omega}\,\mathbf{e}_r$ is a multiple of $\mathbf{e}_r$, and the term vanishes. Both sides are continuous from Y^0_ℓ

into $L^2(\mathbb{R}^2)$. For the left side, w vanishes outside B_{ρ_+} and $\nabla\overline{\omega}$ is bounded and supported in B_{ρ_+}, so Lemma 2.1(c) with $\rho = R = \rho_+$ gives $\|(K_{BS} * w)\cdot\nabla\overline{\omega}\|_{L^2(\mathbb{R}^2)} \le 2\rho_+\|\nabla\overline{\omega}\|_{L^\infty(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)}$; for the right side, use (1). So the formula holds on Y^0_ℓ. The coupling term commutes with rotations, since $K_{BS}(\mathsf{O}_\phi x) = \mathsf{O}_\phi K_{BS}(x)$ and $\nabla\overline{\omega}$ is radial times $\mathbf{e}_r$; hence it commutes with P_ℓ, which gives the formula on Y^0. For $w \in Y^1$, $P_\ell(-\Omega\partial_\theta w) = -i\ell\Omega g_\ell e^{i\ell\theta}$ by integration by parts in θ. The conjugation relation holds because Ω, $\overline{\omega}$, K_{BS} and the kernel of $\mathcal{G}_\ell = \mathcal{G}_{-\ell}$ are real.

(3) $\mathrm{Re}(\lambda + i\ell\Omega) = \mathrm{Re}\,\lambda$ because Ω is real; ν is smooth and vanishes outside $[\xi_1,\xi_2]$, and $\Omega(e^\xi;G)$ is smooth in ξ.

(4) $\Omega(\cdot;G)$ and k_ℓ are bounded on $[\rho_-,\rho_+]$ and k_ℓ vanishes outside $[r_1,r_2]$; so multiplication by Ω, and $k_\ell\mathcal{G}_\ell$ by the last assertion of (1), are bounded on Y^0_ℓ. Write $L = L_\ell(G)$. The series converges absolutely in operator norm, since $\|t^nL^n/n!\| \le (|t|\,\|L\|)^n/n!$, uniformly for t in compact sets. The group law $e^{(t+t')L} = e^{tL}e^{t'L}$ follows from the Cauchy product of these absolutely convergent series, and termwise differentiation of the power series in t, which converges in operator norm, gives the derivative; so L is the generator. □

Recall from Definition 4.4 that, for an integer $\ell \ge 1$ and $\mathrm{Re}\,\lambda > 0$, $\psi_\ell(\cdot;\lambda,G)$ is the solution on $\mathbb{R}$ of the Rayleigh equation $\partial_\xi^2\psi = V_\ell(\cdot;\lambda,G)\,\psi$ with $\psi(\xi_1) = 1$ and $\partial_\xi\psi(\xi_1) = \ell$.

Lemma 4.6. *In a mode setting, let $\ell \ge 1$.*

(1) *$\psi_\ell(\xi;\lambda,G)$ and its ξ-derivative are holomorphic in $\lambda \in \{\mathrm{Re}\,\lambda > 0\}$, locally uniformly in ξ, and so is $\mathcal{E}_\ell(\cdot;G)$.*

(2) *Put $\psi_L = e^{\ell\xi_1}\psi_\ell(\cdot;\lambda,G)$, so that $\psi_L(\xi) = e^{\ell\xi}$ for $\xi \le \xi_1$, and let ψ_R be the solution of $\partial_\xi^2\psi = V_\ell\psi$ with $\psi_R(\xi) = e^{-\ell\xi}$ for $\xi \ge \xi_2$ (L and R for the left end $\xi \le \xi_1$ and the right end $\xi \ge \xi_2$, where these solutions are explicit and decay). The Wronskian $\mathcal{W}(\lambda) = \psi_L\,\partial_\xi\psi_R - \partial_\xi\psi_L\,\psi_R$ is independent of ξ, and $\mathcal{W}(\lambda) = -\mathcal{E}_\ell(\lambda;G)$.*

For the ring in reference units, $\overline{\omega} = \overline{\omega}_{\mathrm{c}}$, $(\xi_1,\xi_2) = (\xi_L,\xi_R)$, the Evans function of Definition 4.4 is the function $\mathcal{E}_\ell$ of Hypothesis 4.17 below.

Proof. (1) On $(-\infty,\xi_1]$, $V_\ell = \ell^2$, so $\psi_\ell = e^{\ell(\xi-\xi_1)}$ there, independently of λ. On $[\xi_1,\infty)$, $Z = (\psi,\partial_\xi\psi)$ solves $\partial_\xi Z = AZ$ with $A = \left(\begin{smallmatrix}0 & 1\\ V_\ell & 0\end{smallmatrix}\right)$; let $Z_0 = (1,\ell)$ and $Z_{n+1}(\xi) = Z_0 + \int_{\xi_1}^{\xi} AZ_n$. For compact $K \subset \{\mathrm{Re}\,\lambda > 0\}$ and $\bar\xi > \xi_1$, Lemma 4.5(3) gives $|A| \le C_A = 1 + \ell^2 + \ell\|\nu\|_{L^\infty(\mathbb{R})}/\min_K \mathrm{Re}\,\lambda$ on $[\xi_1,\bar\xi]\times K$, and by induction $|Z_{n+1} - Z_n| \le |Z_0|(C_A(\xi-\xi_1))^{n+1}/(n+1)!$. The iterates are continuous in (ξ,λ) and holomorphic in λ and converge uniformly on $[\xi_1,\bar\xi]\times K$, so their limit $(\psi_\ell,\partial_\xi\psi_\ell)$ is holomorphic in λ, uniformly in $\xi \in [\xi_1,\bar\xi]$. The same holds for ψ_R of (2), with data at ξ_2.

(2) The equation has no first-order term, so $\partial_\xi\mathcal{W} = 0$. At ξ_2, $\partial_\xi\psi_R = -\ell\psi_R = -\ell e^{-\ell\xi_2}$, so $\mathcal{W} = -e^{-\ell\xi_2}(\partial_\xi\psi_L + \ell\psi_L)(\xi_2) = -e^{-\ell(\xi_2-\xi_1)}(\partial_\xi\psi_\ell + \ell\psi_\ell)(\xi_2) = -\mathcal{E}_\ell(\lambda;G)$.

For the last assertion, $\overline{\omega}_{\mathrm{c}}$, $r_1 = e^{\xi_L}$, $r_2 = e^{\xi_R}$ and $N_* = \{\frac{1}{2}e^{\xi_L} \le |x| \le 2e^{\xi_R}\}$ form a mode setting by Lemma 4.2; $\nu = \partial_\xi\varpi$, and $\Gamma_{\overline{\omega}_{\mathrm{c}}}(e^\xi)/(2\pi) = \int_0^{e^\xi}\overline{\omega}_{\mathrm{c}}(\rho)\rho\mathrm{d}\rho = \int_{-\infty}^{\xi}\varpi(\xi')e^{2\xi'}\mathrm{d}\xi'$, so $\Omega(e^\xi;G) = \Omega_G(\xi)$, with Ω_G as in Hypothesis 4.17. So on $[\xi_L,\xi_R]$ the equation, the data and the formula for $\mathcal{E}_\ell$ are those of Hypothesis 4.17. □

For a closed operator A and an isolated point λ_0 of its spectrum, the **Riesz projection** at λ_0 is $\frac{1}{2\pi i}\oint_{|\lambda-\lambda_0|=\epsilon}(\lambda - A)^{-1}\mathrm{d}\lambda$, for any $\epsilon > 0$ such that the closed disk of radius ϵ meets the spectrum only at λ_0; its rank is the **algebraic multiplicity** of λ_0, and λ_0 is *algebraically simple* if this rank is 1.

Lemma 4.7. *In a mode setting, let $\ell \ge 1$ and $\mathrm{Re}\,\lambda > 0$.*

(1) *If $\mathcal{E}_\ell(\lambda;G)\neq 0$, then $\lambda - L_\ell(G)$ is invertible on Y^0_ℓ and on every Y^s_ℓ, and for $g\in Y^0_\ell$,*

$$\big((\lambda-L_\ell)^{-1}g\big)(r) = M_\lambda(r)\big(g(r)+k_\ell(r)\,\psi_g(\log r)\big),$$
$$\psi_g(\xi) = \frac{1}{\mathcal{W}(\lambda)}\Big[\psi_R(\xi)\int_{-\infty}^{\xi}\psi_L F\mathrm{d}\xi' + \psi_L(\xi)\int_{\xi}^{\infty}\psi_R F\mathrm{d}\xi'\Big],$$

where ψ_L, ψ_R and $\mathcal{W}$ are as in Lemma 4.6(2), and $F(\xi)=e^{2\xi}(M_\lambda g)(e^\xi)$.

(2) *If $\mathcal{E}_\ell(\lambda;G)=0$, then λ is an eigenvalue of $L_\ell(G)$ on Y^0_ℓ and on every Y^s_ℓ, with eigenfunction $g_\lambda = M_\lambda k_\ell\,\psi_\ell(\log r;\lambda,G)\neq 0$, which is smooth and supported in $[r_1,r_2]$.*

In particular, for each $\ell\geq 1$ the spectrum of $L_\ell(G)$ in $\{\operatorname{Re}\lambda>0\}$ is the zero set of $\mathcal{E}_\ell(\cdot;G)$ and does not depend on s; for $\ell\leq -1$ it is the complex conjugate of that of $L_{-\ell}(G)$.

Proof. For $\eta\in Y^0_\ell$ and $\psi=\mathcal{G}_\ell\eta$, the equation $(\lambda-L_\ell)\eta=g$ reads $\eta=M_\lambda(g+k_\ell\psi)$. Put $\tilde\psi(\xi)=\psi(e^\xi)$. By the identity displayed before Definition 4.4, the radial equation of Lemma 4.5(1) reads $\partial_\xi^2\tilde\psi-\ell^2\tilde\psi=e^{2\xi}\eta(e^\xi)$. At $r=e^\xi$, $e^{2\xi}k_\ell = r^2\cdot i\ell\,\partial_r\overline{\varpi}/r = i\ell\nu$, so $e^{2\xi}M_\lambda k_\ell = i\ell\nu M_\lambda = V_\ell-\ell^2$, all read at $r=e^\xi$; hence $u=\tilde\psi$ solves

$$\partial_\xi^2 u - V_\ell u = F,\qquad u=O(e^{\ell\xi})\ (\xi\to-\infty),\qquad u=O(e^{-\ell\xi})\ (\xi\to+\infty). \tag{12}$$

Conversely, if $\tilde\psi$ solves (12), put $\psi(r)=\tilde\psi(\log r)$ and $\eta=M_\lambda(g+k_\ell\psi)$. Then $\eta\in Y^0_\ell$, and $\partial_\xi^2\tilde\psi-\ell^2\tilde\psi=e^{2\xi}\eta(e^\xi)$, so ψ solves the radial equation of Lemma 4.5(1) with the growth conditions there; hence $\psi=\mathcal{G}_\ell\eta$ and $(\lambda-L_\ell)\eta=g$.

(1) By Lemma 4.6(2), $\mathcal{W}\neq 0$. Put $A(\xi)=\int_{-\infty}^{\xi}\psi_L F\mathrm{d}\xi'$ and $B(\xi)=\int_\xi^\infty\psi_R F\mathrm{d}\xi'$, so that $\mathcal{W}\psi_g=\psi_R A+\psi_L B$, $\partial_\xi A=\psi_L F$ and $\partial_\xi B=-\psi_R F$. Differentiating, the terms from the variable limits cancel: $\mathcal{W}\,\partial_\xi\psi_g=\partial_\xi\psi_R\,A+\partial_\xi\psi_L\,B+(\psi_R\psi_L-\psi_L\psi_R)F=\partial_\xi\psi_R\,A+\partial_\xi\psi_L\,B$. Differentiating again and using $\partial_\xi^2\psi_L=V_\ell\psi_L$ and $\partial_\xi^2\psi_R=V_\ell\psi_R$,

$$\mathcal{W}\,\partial_\xi^2\psi_g = V_\ell(\psi_R A+\psi_L B)+(\psi_L\,\partial_\xi\psi_R-\partial_\xi\psi_L\,\psi_R)F = \mathcal{W}\,(V_\ell\psi_g+F)$$

almost everywhere, by the definition of $\mathcal{W}$; so $\partial_\xi^2\psi_g=V_\ell\psi_g+F$. Since F vanishes outside $[\log\rho_-,\log\rho_+]$, ψ_g is a multiple of $\psi_L=e^{\ell\xi}$ below $\min(\xi_1,\log\rho_-)$ and of $\psi_R=e^{-\ell\xi}$ above $\max(\xi_2,\log\rho_+)$; so ψ_g solves (12). As shown at the start of the proof, this means that the function $r\mapsto M_\lambda(r)\big(g(r)+k_\ell(r)\psi_g(\log r)\big)$ solves the resolvent equation. If $(\lambda-L_\ell)\eta=0$, then $\xi\mapsto(\mathcal{G}_\ell\eta)(e^\xi)$ solves (12) with $F=0$, so it is $O(e^{\ell\xi})$ solving $\partial_\xi^2\psi=\ell^2\psi$ on $(-\infty,\xi_1]$, hence a multiple of ψ_L, and likewise of ψ_R; as $\mathcal{W}\neq 0$, it vanishes, and $\eta=M_\lambda k_\ell\mathcal{G}_\ell\eta=0$. For the bounds: M_λ is smooth on $(0,\infty)$ by Lemma 4.5(3), so it is a bounded multiplier on every Y^s_ℓ. Since $r\mathrm{d}r=e^{2\xi}\mathrm{d}\xi$, $|M_\lambda|\leq 1/\operatorname{Re}\lambda$ (Lemma 4.5(3)) and g vanishes outside $[\rho_-,\rho_+]$, the Cauchy-Schwarz inequality, as in the proof of Lemma 4.5(1), gives

$$\|F\|_{L^1(\mathbb{R})}=\int_0^\infty|M_\lambda g|\,r\mathrm{d}r\leq\frac{1}{\operatorname{Re}\lambda}\int_{\rho_-}^{\rho_+}|g|\,r\mathrm{d}r\leq\frac{\rho_+}{\operatorname{Re}\lambda}\|g\|_{L^2(r\mathrm{d}r)}=\frac{\rho_+}{\sqrt{2\pi}\,\operatorname{Re}\lambda}\|g\|_{Y^0_\ell},$$

Put $C_L=\sup_{\xi\leq\xi_2}(|\psi_L|+|\partial_\xi\psi_L|)$ and $C_R=\sup_{\xi\geq\xi_1}(|\psi_R|+|\partial_\xi\psi_R|)$; they are finite, as $\psi_L=e^{\ell\xi}$ for $\xi\leq\xi_1$, $\psi_R=e^{-\ell\xi}$ for $\xi\geq\xi_2$, and both are C^1 on $[\xi_1,\xi_2]$. For $\xi\in[\xi_1,\xi_2]$, $|A(\xi)|\leq C_L\|F\|_{L^1(\mathbb{R})}$ and $|B(\xi)|\leq C_R\|F\|_{L^1(\mathbb{R})}$, so the formulas for $\mathcal{W}\psi_g$ and $\mathcal{W}\,\partial_\xi\psi_g$ above give $|\psi_g|+|\partial_\xi\psi_g|\leq 2C_LC_R|\mathcal{W}|^{-1}\|F\|_{L^1(\mathbb{R})}\leq C\|g\|_{Y^0_\ell}$ on $[\xi_1,\xi_2]$, with C depending on λ. If $g\in Y^s_\ell$, then $g\in H^s((0,\infty))$, polar coordinates being smooth near N_*, so $\|F\|_{H^s([\xi_1,\xi_2])}\leq C\|g\|_{Y^s_\ell}$, and from

$\partial_\xi^2\psi_g = V_\ell\psi_g + F$ by induction $\|\psi_g\|_{H^{s+2}([\xi_1,\xi_2])} \le C\|g\|_{Y_\ell^s}$. As k_ℓ is smooth and supported in $[r_1, r_2]$, $\|k_\ell(r)\psi_g(\log r)e^{i\ell\theta}\|_{H^{s+2}} \le C\|g\|_{Y_\ell^s}$. Hence $(\lambda - L_\ell)^{-1}$ is bounded on Y_ℓ^0 and on every Y_ℓ^s.

(2) Now $\mathcal{W} = 0$. The functions ψ_L and ψ_R solve the same linear second-order equation, and their Wronskian vanishes, so they are linearly dependent; as $\psi_R \neq 0$, $\psi_L = c\,\psi_R$ for a constant c. Hence $\psi_\ell = e^{-\ell\xi_1}\psi_L = c\,e^{-\ell\xi_1}e^{-\ell\xi}$ for $\xi \ge \xi_2$, so $\psi_\ell = O(e^{-\ell\xi})$ as $\xi \to +\infty$: the vanishing of $\mathcal{W}$ is what gives the decay at $+\infty$. As $\psi_\ell = e^{\ell(\xi-\xi_1)}$ for $\xi \le \xi_1$, the condition at $-\infty$ holds as well, so ψ_ℓ solves (12) with $F = 0$, that is, with $g = 0$. As shown at the start of the proof, if u solves (12), then $\eta(r) = M_\lambda(r)\big(g(r) + k_\ell(r)\,u(\log r)\big)$ satisfies $(\lambda - L_\ell)\eta = g$ and $(\mathcal{G}_\ell\eta)(r) = u(\log r)$; with $g = 0$ and $u = \psi_\ell$ this gives $\eta = g_\lambda$, so $L_\ell g_\lambda = \lambda g_\lambda$ and $(\mathcal{G}_\ell g_\lambda)(r) = \psi_\ell(\log r; \lambda, G)$, and $g_\lambda \neq 0$ since $\psi_\ell(\xi_1) = 1$. As g_λ is smooth and supported in $[r_1, r_2]$, it lies in every Y_ℓ^s.

The spectral statement for $\ell \ge 1$ follows from (1) and (2). For $\ell \le -1$, complex conjugation is an isometry of $Y_{-\ell}^s$ onto Y_ℓ^s that conjugates $\lambda - L_\ell$ to $\bar\lambda - L_{-\ell}$, by Lemma 4.5(2). □

Theorem 4.8. *In a mode setting, let $\ell \ge 1$ and let λ_* be a zero of $\mathcal{E}_\ell(\cdot; G)$ with $\operatorname{Re}\lambda_* > 0$. Put $\psi_* = \psi_\ell(\cdot; \lambda_*, G)$, $g_* = M_{\lambda_*}k_\ell\,\psi_*(\log r)$ and*

$$J = \int_0^\infty g_*(r)\,\psi_*(\log r)\,M_{\lambda_*}(r)\,r\mathrm{d}r = i\ell\int_{\mathbb{R}} \frac{\nu(\xi)\,\psi_*(\xi)^2}{(\lambda_* + i\ell\Omega(e^\xi; G))^2}\mathrm{d}\xi.$$

(1) *$\psi_*(\xi_2) \neq 0$ and $e^{\ell(\xi_2-\xi_1)}\,\psi_*(\xi_2)\,\partial_\lambda\mathcal{E}_\ell(\lambda_*; G) = -J$.*

(2) *If $\partial_\lambda\mathcal{E}_\ell(\lambda_*; G) \neq 0$, then $J \neq 0$ by (1), and the Riesz projection of $L_\ell(G)$ at λ_*, on Y_ℓ^0 and on each Y_ℓ^s, is*

$$\Pi g = \ell_*(g)\,g_*, \qquad \ell_*(g) = \frac{1}{J}\int_0^\infty g(r)\,\psi_*(\log r)\,M_{\lambda_*}(r)\,r\mathrm{d}r;$$

it has rank one, so λ_* is an algebraically simple eigenvalue.

Proof. The two expressions for J agree because $r\mathrm{d}r = e^{2\xi}\mathrm{d}\xi$ and $e^{2\xi}g_* = i\ell\nu M_{\lambda_*}\psi_*$. Write $M_* = M_{\lambda_*}$.

(1) Put $\dot\psi = \partial_\lambda\psi_\ell(\cdot; \lambda, G)$ at $\lambda = \lambda_*$. It vanishes on $(-\infty, \xi_1]$, where $\psi_\ell(\xi; \lambda, G) = e^{\ell(\xi-\xi_1)}$ for every λ: there $V_\ell = \ell^2$, and the data $\psi(\xi_1) = 1$, $\partial_\xi\psi(\xi_1) = \ell$ do not depend on λ (proof of Lemma 4.6(1)). Moreover, since differentiation in λ and in ξ commute by Lemma 4.6(1) and the integral form of the equation, it solves $\partial_\xi^2\dot\psi = V_\ell\dot\psi + \partial_\lambda V_\ell\,\psi_*$ with $\partial_\lambda V_\ell = -i\ell\nu M_*^2$. Hence $\partial_\xi(\partial_\xi\dot\psi\,\psi_* - \dot\psi\,\partial_\xi\psi_*) = \partial_\lambda V_\ell\,\psi_*^2$. Since $\mathcal{E}_\ell(\lambda_*; G) = 0$, $\partial_\xi\psi_*(\xi_2) = -\ell\psi_*(\xi_2)$, so at ξ_2

$$\partial_\xi\dot\psi\,\psi_* - \dot\psi\,\partial_\xi\psi_* = \psi_*(\xi_2)(\partial_\xi\dot\psi + \ell\dot\psi)(\xi_2) = \psi_*(\xi_2)e^{\ell(\xi_2-\xi_1)}\partial_\lambda\mathcal{E}_\ell(\lambda_*; G),$$

while at $-\infty$ it vanishes. Integrating the identity $\partial_\xi(\partial_\xi\dot\psi\,\psi_* - \dot\psi\,\partial_\xi\psi_*) = \partial_\lambda V_\ell\,\psi_*^2$ in ξ from $-\infty$ to ξ_2, as $\operatorname{supp}\nu \subset [\xi_1, \xi_2]$, gives $\psi_*(\xi_2)e^{\ell(\xi_2-\xi_1)}\partial_\lambda\mathcal{E}_\ell(\lambda_*; G) = -i\ell\int\nu M_*^2\psi_*^2\mathrm{d}\xi = -J$. If $\psi_*(\xi_2) = 0$, then $\partial_\xi\psi_*(\xi_2) = 0$ and $\psi_* \equiv 0$ by uniqueness, contradicting $\psi_*(\xi_1) = 1$.

(2) Let D be a closed disk about λ_* in $\{\operatorname{Re}\lambda > 0\}$ containing no other zero of $\mathcal{E}_\ell(\cdot; G)$; it exists because $\partial_\lambda\mathcal{E}_\ell(\lambda_*; G) \neq 0$. By Lemma 4.7, λ_* is isolated in the spectrum on Y_ℓ^0 and on each Y_ℓ^s. For $g \in Y_\ell^0$ and $\operatorname{Re}\lambda > 0$ let $\mathcal{N}(\lambda)g$ be the bracket in the formula for ψ_g in Lemma 4.7(1), that is,

$$(\mathcal{N}(\lambda)g)(\xi) = \psi_R(\xi)\int_{-\infty}^\xi \psi_L F\mathrm{d}\xi' + \psi_L(\xi)\int_\xi^\infty \psi_R F\mathrm{d}\xi', \qquad F(\xi) = e^{2\xi}(M_\lambda g)(e^\xi),$$

with ψ_L, ψ_R taken at λ. Then $\mathcal{N}(\lambda)g = \mathcal{W}(\lambda)\psi_g = -\mathcal{E}_\ell(\lambda; G)\psi_g$ wherever $\mathcal{E}_\ell(\lambda; G) \neq 0$, so for $\lambda \in D \setminus \{\lambda_*\}$

$$\big((\lambda - L_\ell)^{-1}g\big)(r) = M_\lambda(r)g(r) - \frac{M_\lambda(r)k_\ell(r)\,(\mathcal{N}(\lambda)g)(\log r)}{\mathcal{E}_\ell(\lambda; G)}.$$

Unlike ψ_g, $\mathcal{N}(\lambda)g$ involves no division by $\mathcal{E}_\ell$: for fixed r, $M_\lambda(r)g(r)$ and $(\mathcal{N}(\lambda)g)(\log r)$ are holomorphic in λ on all of $\{\operatorname{Re}\lambda > 0\}$, across λ_* too, because ψ_L, ψ_R and F are, with integrable bounds. So the only singularity of the resolvent at λ_* is the factor $1/\mathcal{E}_\ell$. Since D meets the spectrum only at λ_*, the Riesz projection is $\Pi g = \frac{1}{2\pi i}\oint_{\partial D}(\lambda - L_\ell)^{-1} g \mathrm{d}\lambda$, the Y^0_ℓ-valued integral of a continuous function of $\lambda \in \partial D$. For each λ we use the representative of $(\lambda - L_\ell)^{-1}g$ given by the formula above, which is defined at every r and, for fixed r, continuous in $\lambda \in \partial D$. The Riemann sums of the Y^0_ℓ-valued integral converge to Πg in Y^0_ℓ, hence almost everywhere along a subsequence; at each r they are Riemann sums of the scalar integral $\frac{1}{2\pi i}\oint_{\partial D}((\lambda - L_\ell)^{-1}g)(r)\mathrm{d}\lambda$ and converge to it. So $(\Pi g)(r)$ equals this scalar integral for almost every r. In it, $M_\lambda(r)g(r)$ is holomorphic on D and contributes 0 by Cauchy's theorem. In the other term, the numerator $h_r(\lambda) = M_\lambda(r)k_\ell(r)(\mathcal{N}(\lambda)g)(\log r)$ is holomorphic on D, and $\mathcal{E}_\ell(\cdot; G)$ is holomorphic on D with the single zero λ_*, which is simple; so the integrand has at most a simple pole at λ_*, with residue $-h_r(\lambda_*)/\partial_\lambda \mathcal{E}_\ell(\lambda_*; G)$. By the residue theorem, for almost every r,

$$(\Pi g)(r) = -\frac{M_*(r)k_\ell(r)\,(\mathcal{N}(\lambda_*)g)(\log r)}{\partial_\lambda \mathcal{E}_\ell(\lambda_*; G)}.$$

At λ_*, $\mathcal{W} = 0$, so $\psi_R = \kappa\psi_L$ with $\kappa = e^{-\ell\xi_2}/(e^{\ell\xi_1}\psi_*(\xi_2))$, and

$$\mathcal{N}(\lambda_*)g = \kappa\psi_L \int_{\mathbb{R}} \psi_L F \mathrm{d}\xi' = \kappa e^{2\ell\xi_1}\psi_* \int_0^\infty g(\rho)\,\psi_*(\log\rho) M_*(\rho)\,\rho\mathrm{d}\rho.$$

Therefore

$$\Pi g = -\frac{e^{-\ell(\xi_2-\xi_1)}}{\psi_*(\xi_2)\,\partial_\lambda \mathcal{E}_\ell(\lambda_*; G)}\Big(\int_0^\infty g\,\psi_* M_*\,\rho\mathrm{d}\rho\Big) g_* = \ell_*(g)\,g_*,$$

since $J = -e^{\ell(\xi_2-\xi_1)}\psi_*(\xi_2)\,\partial_\lambda\mathcal{E}_\ell(\lambda_*; G)$ by (1). Since $\ell_*(g_*) = J/J = 1$, Π has rank one. The resolvents on Y^s_ℓ are restrictions of those on Y^0_ℓ, so the Riesz projection on Y^s_ℓ is the restriction of Π, which maps into $\operatorname{span}\{g_*\} \subset Y^s_\ell$. □

Lemma 4.9. *Under the assumptions of Theorem 4.8, assume that λ_* is a simple zero, and put $p^{\mathrm{out}}_* = \int_0^\infty \rho^{\ell+1} g_*(\rho)\mathrm{d}\rho$ and $p^{\mathrm{in}}_* = \int_0^\infty \rho^{1-\ell} g_*(\rho)\mathrm{d}\rho$, so that $p^{\mathrm{out}}_* = p^{\mathrm{out}}_\ell[g_* e^{i\ell\theta}]$ and $p^{\mathrm{in}}_* = p^{\mathrm{in}}_\ell[g_* e^{i\ell\theta}]$.*

(1) *$\ell_*(g_*) = 1$, $\ell_*(L_\ell g) = \lambda_*\ell_*(g)$ for $g \in Y^0_\ell$, and $\ell_*(e^{tL_\ell}g) = e^{\lambda_* t}\ell_*(g)$ for $t \geq 0$, where e^{tL_ℓ} is the semigroup generated by $L_\ell(G)$ on Y^0_ℓ (Lemma 4.5(4)).*

(2) *$p^{\mathrm{in}}_* = -2\ell e^{-\ell\xi_1}$ and $p^{\mathrm{out}}_* = -2\ell e^{\ell\xi_2}\psi_*(\xi_2)$.*

(3) *Put $b(r) = -\frac{i}{2} r^{-\ell-1}\partial_r\overline{\varpi}(r)$ and $\beta = \ell_*(b)$. Then $\beta = -p^{\mathrm{in}}_*/(2\ell J)$.*

(4) *$p^{\mathrm{out}}_* \neq 0$, $p^{\mathrm{in}}_* \neq 0$ and $\beta \neq 0$.*

Proof. (1) Write $M_* = M_{\lambda_*}$ and put $X(r) = \psi_*(\log r)M_*(r)$, so that $k_\ell X = g_*$ and $(\lambda_* + i\ell\Omega)X = \psi_* = \mathcal{G}_\ell g_*$ (see the proof of Lemma 4.7(2)), and $\langle f, g\rangle = \int f\bar{g}\, r\mathrm{d}r$, which is $(2\pi)^{-1}$ times the inner product of Y^0_ℓ. By Definition 4.3, $(\mathcal{G}_\ell g)(r) = \int_0^\infty K_\ell(r,\rho)\,g(\rho)\,\rho\mathrm{d}\rho$ with $K_\ell(r,\rho) = -\frac{1}{2|\ell|}\big(\min(r,\rho)/\max(r,\rho)\big)^{|\ell|}$. This integral kernel, taken with respect to the measure $\rho\mathrm{d}\rho$, is real and symmetric, $K_\ell(r,\rho) = K_\ell(\rho, r)$. As K_ℓ is bounded, Fubini's theorem gives, for the operator $\mathcal{G}_\ell$, $\langle \mathcal{G}_\ell f, h\rangle = \langle f, \mathcal{G}_\ell h\rangle$ for $f, h \in L^2([\rho_-, \rho_+], r\mathrm{d}r)$, and $\overline{\mathcal{G}_\ell h} = \mathcal{G}_\ell\bar{h}$. The operator L_ℓ is bounded on Y^0_ℓ (Lemma 4.5(4)); we compute its adjoint L^*_ℓ for $\langle\cdot,\cdot\rangle$. For $f, h \in Y^0_\ell$, $\langle -i\ell\Omega f, h\rangle = \langle f, i\ell\Omega h\rangle$ as Ω is real, and $\langle k_\ell\mathcal{G}_\ell f, h\rangle = \langle \mathcal{G}_\ell f, \bar{k}_\ell h\rangle = \langle f, \mathcal{G}_\ell(\bar{k}_\ell h)\rangle$. The function $\mathcal{G}_\ell(\bar{k}_\ell h)$ is defined on all of $(0,\infty)$, but f vanishes outside $[\rho_-, \rho_+]$, so only its restriction to $[\rho_-, \rho_+]$ enters, and that restriction lies in

Y^0_ℓ. Hence $L^*_\ell h = i\ell\Omega h + \mathbf{1}_{[\rho_-,\rho_+]}\mathcal{G}_\ell(\bar{k}_\ell h)$. Conjugating $(\lambda_* + i\ell\Omega)X = \mathcal{G}_\ell(k_\ell X)$ gives $L^*_\ell \bar{X} = \overline{\lambda_*}\bar{X}$ on $[\rho_-,\rho_+]$. Since $\ell_*(g) = J^{-1}\langle g, \bar{X}\rangle$, we get $\ell_*(g_*) = 1$ and $\ell_*(L_\ell g) = J^{-1}\langle g, \overline{\lambda_*}\bar{X}\rangle = \lambda_*\ell_*(g)$. The functional ℓ_* is bounded on Y^0_ℓ, that is, a continuous linear map $Y^0_\ell \to \mathbb{C}$: X is continuous on $[\rho_-,\rho_+]$, as ψ_* is continuous and $|M_*| \le 1/\operatorname{Re}\lambda_*$ (Lemma 4.5(3)), so the Cauchy-Schwarz inequality gives $|\ell_*(g)| \le |J|^{-1}\|X\|_{L^2([\rho_-,\rho_+],r\mathrm{d}r)}\|g\|_{L^2(r\mathrm{d}r)}$. By Lemma 4.5(4), $t \mapsto e^{tL_\ell}g$ is differentiable in Y^0_ℓ with derivative $L_\ell e^{tL_\ell}g$, so the function $t \mapsto \ell_*(e^{tL_\ell}g)$ has derivative $\ell_*(L_\ell e^{tL_\ell}g) = \lambda_*\ell_*(e^{tL_\ell}g)$, which gives the last identity.

(2) As in the proof of (1), $\mathcal{G}_\ell g_* = \psi_*(\log r)$, and g_* vanishes outside $[r_1, r_2]$. For $r \ge r_2$ the definition of $\mathcal{G}_\ell$ gives $\mathcal{G}_\ell g_*(r) = -\frac{1}{2\ell}r^{-\ell}\int \rho^{\ell+1}g_*\mathrm{d}\rho$, and for $r \le r_1$ it gives $-\frac{1}{2\ell}r^{\ell}\int \rho^{1-\ell}g_*\mathrm{d}\rho$. Compare with $\psi_*(\xi) = e^{\ell(\xi-\xi_1)}$ for $\xi \le \xi_1$ and $\psi_*(\xi) = \psi_*(\xi_2)e^{-\ell(\xi-\xi_2)}$ for $\xi \ge \xi_2$. The identities $p^{\mathrm{out}}_* = p^{\mathrm{out}}_\ell[g_* e^{i\ell\theta}]$ and $p^{\mathrm{in}}_* = p^{\mathrm{in}}_\ell[g_* e^{i\ell\theta}]$ follow from Definition 2.3 by integrating in θ.

(3) From $g_* = M_* k_\ell \psi_*$, $(\partial_r\overline{\omega})\psi_* M_* = r g_*/(i\ell)$, so

$$\beta = \frac{1}{J}\int b\,\psi_* M_*\, r\mathrm{d}r = -\frac{i}{2J}\int r^{-\ell}(\partial_r\overline{\omega})\psi_* M_*\mathrm{d}r = -\frac{1}{2\ell J}\int r^{1-\ell}g_*\mathrm{d}r = -\frac{p^{\mathrm{in}}_*}{2\ell J}.$$

(4) $p^{\mathrm{in}}_* \ne 0$ by (2). If $p^{\mathrm{out}}_* = 0$, then $\psi_*(\xi_2) = 0$ by (2), contradicting Theorem 4.8(1). As λ_* is a simple zero, $J \ne 0$ by Theorem 4.8(1), and $\beta \ne 0$ by (3). □

4.3 An A Priori Bound for Unstable Eigenvalues

Lemma 4.10. *In a mode setting, let $\mathcal{U}_+ = \{r : \partial_r\overline{\omega}(r) > 0\}$ and $\mathcal{U}_- = \{r : \partial_r\overline{\omega}(r) < 0\}$, and assume that $r < \rho$ for all $r \in \mathcal{U}_+$ and $\rho \in \mathcal{U}_-$. For $\ell \ge 1$ put*

$$Z_\ell = \int_{\mathcal{U}_+}\int_{\mathcal{U}_-}\left(\frac{r}{\rho}\right)^{2\ell}|\partial_r\overline{\omega}(r)|\,|\partial_r\overline{\omega}(\rho)|\mathrm{d}\rho\mathrm{d}r.$$

If $g \in Y^0_{\pm\ell}$, $g \ne 0$, $L_{\pm\ell}(G)g = \lambda g$ and $\operatorname{Re}\lambda > 0$, then $\operatorname{Re}\lambda \le \frac{1}{2}Z_\ell^{1/2}$, and Z_ℓ is nonincreasing in ℓ.

Proof. By Lemma 4.7, the eigenvalues of $L_{-\ell}$ in $\{\operatorname{Re}\lambda > 0\}$ are the conjugates of those of L_ℓ, so we may take the sign $+$. Let $\psi = \mathcal{G}_\ell g$, which is continuous, and $\langle f, h\rangle = \int f\bar{h}\, r\mathrm{d}r$. Since $\operatorname{Re}\lambda > 0$, $\lambda + i\ell\Omega \ne 0$ (Lemma 4.5(3)), so $g = M_\lambda k_\ell\psi$ vanishes outside $\{\partial_r\overline{\omega} \ne 0\}$; there put $W = r/|\partial_r\overline{\omega}|$ and $\|g\|^2_W = \int |g|^2 W r\mathrm{d}r = \int \ell^2|\partial_r\overline{\omega}|\,|\psi|^2|\lambda + i\ell\Omega|^{-2}\mathrm{d}r$, which is finite, and positive as $g \ne 0$. Multiplying $\lambda g = -i\ell\Omega g + k_\ell\psi$ by $\bar{g}Wr$, integrating, and using $k_\ell W r = i\ell\,(\operatorname{sgn}\partial_r\overline{\omega})\,r$, we get $\lambda\|g\|^2_W = -i\ell\int\Omega|g|^2 W r\mathrm{d}r + i\ell\int(\operatorname{sgn}\partial_r\overline{\omega})\,\psi\bar{g}\,r\mathrm{d}r$, and the first term is imaginary. With $g_a = g\mathbf{1}_{\mathcal{U}_+}$, $g_b = g\mathbf{1}_{\mathcal{U}_-}$,

$$\int(\operatorname{sgn}\partial_r\overline{\omega})\,\psi\bar{g}\,r\mathrm{d}r = \langle\mathcal{G}_\ell g_a, g_a\rangle - \langle\mathcal{G}_\ell g_b, g_b\rangle + \langle\mathcal{G}_\ell g_b, g_a\rangle - \langle\mathcal{G}_\ell g_a, g_b\rangle.$$

The kernel K_ℓ of $\mathcal{G}_\ell$ is real and symmetric (proof of Lemma 4.9(1)), so the first two terms are real and the last is the conjugate of $X_{ab} = \langle\mathcal{G}_\ell g_b, g_a\rangle$; hence $\operatorname{Re}\lambda\,\|g\|^2_W = -2\ell\operatorname{Im}X_{ab}$. For $r \in \mathcal{U}_+$, $\rho \in \mathcal{U}_-$ we have $r < \rho$, so $|K_\ell(r,\rho)| = \frac{1}{2\ell}(r/\rho)^\ell$. Writing $|g_a|r = (|g_a|^2 W r)^{1/2}|\partial_r\overline{\omega}|^{1/2}$, likewise for g_b, the Cauchy-Schwarz inequality on $\mathcal{U}_+ \times \mathcal{U}_-$ gives $|X_{ab}| \le \frac{1}{2\ell}Z_\ell^{1/2}\|g_a\|_W\|g_b\|_W \le \frac{1}{4\ell}Z_\ell^{1/2}\|g\|^2_W$, by the AM-GM inequality $\|g_a\|_W\|g_b\|_W \le \frac{1}{2}(\|g_a\|^2_W + \|g_b\|^2_W) = \frac{1}{2}\|g\|^2_W$ (g_a and g_b have disjoint supports and $g = g_a + g_b$); so $\operatorname{Re}\lambda \le \frac{1}{2}Z_\ell^{1/2}$. Since $r/\rho < 1$ on $\mathcal{U}_+ \times \mathcal{U}_-$, the integrand of Z_ℓ decreases in ℓ. □

Lemma 4.11. *Let χ, ϖ and $\overline{\omega}_{\mathrm{c}}$ be as in Definition 4.1. Then:*

(1) *$\partial_\xi \varpi \geq 0$ on $(-\infty, a_0/2)$ and $\partial_\xi \varpi \leq 0$ on $(a_0/2, \infty)$; so $\overline{\omega}_{\mathrm{c}}$ satisfies the hypothesis of Lemma 4.10;*

(2) *for every integer $\ell \geq 1$ with $\ell d < 1$, the quantity Z_ℓ of Lemma 4.10 for $\overline{\omega} = \overline{\omega}_{\mathrm{c}}$ is $Z_\ell = Z_\ell^a Z_\ell^b$ with*

$$Z_\ell^a = \int_{\{\partial_\xi \varpi > 0\}} e^{2\ell(\xi - a_0/2)} \partial_\xi \varpi(\xi) \mathrm{d}\xi, \qquad Z_\ell^b = \int_{\{\partial_\xi \varpi < 0\}} e^{-2\ell(\xi - a_0/2)} |\partial_\xi \varpi(\xi)| \mathrm{d}\xi,$$

and $Z_\ell^a, Z_\ell^b \leq e^{\ell a_0} \Theta_\ell + \epsilon_*$, where $\Theta_\ell = \pi \ell d / \sin(\pi \ell d)$ and $\epsilon_* = \max\{H((\xi_a - a_0)/d),\, H(-\xi_b/d)\}$.

Proof. By Lemma 4.2, $a_0 < 0 < d$ and $\xi_L < \xi_a \leq a_0/2 \leq \xi_b < \xi_R$. Put $\xi_c = a_0/2$ and $P(\xi) = H(\frac{\xi - a_0}{d}) H(-\frac{\xi}{d})$, so $\varpi = \chi P$. Since $H(x) = \frac{1}{2}(1 + \tanh x)$, $\partial_x H(x) = 2H(x)H(-x) = \frac{1}{2}\operatorname{sech}^2 x$ and $\partial_\xi \log P(\xi) = \frac{2}{d}\big[H(-\frac{\xi - a_0}{d}) - H(\frac{\xi}{d})\big]$, which is positive if and only if $-(\xi - a_0) > \xi$, i.e. $\xi < \xi_c$, as H is increasing. For $\xi < \xi_c$, $\partial_\xi \chi \geq 0$ (by monotonicity on $(-\infty, \xi_a]$, and $\partial_\xi \chi = 0$ on $[\xi_a, \xi_c]$), $\chi \geq 0$ and $\partial_\xi P > 0$, so $\partial_\xi \varpi = (\partial_\xi \chi) P + \chi\, \partial_\xi P \geq 0$; for $\xi > \xi_c$ likewise $\partial_\xi \varpi \leq 0$. As $\partial_r \overline{\omega}_{\mathrm{c}}(r) = (\partial_\xi \varpi)(\log r)/r$, $\{\partial_r \overline{\omega}_{\mathrm{c}} > 0\} \subset (0, e^{\xi_c}]$ and $\{\partial_r \overline{\omega}_{\mathrm{c}} < 0\} \subset [e^{\xi_c}, \infty)$ are disjoint; this proves (1).

(2) In the double integral defining Z_ℓ (Lemma 4.10), with $r \in \mathcal{U}_+$ and $\rho \in \mathcal{U}_-$, substitute $\xi = \log r$ and $\xi' = \log \rho$. As $\partial_r \overline{\omega}_{\mathrm{c}}(r) = (\partial_\xi \varpi)(\log r)/r$, the sets $\mathcal{U}_\pm$ become $\{\pm \partial_\xi \varpi > 0\}$, $|\partial_r \overline{\omega}_{\mathrm{c}}(r)| \mathrm{d}r = |\partial_\xi \varpi(\xi)| \mathrm{d}\xi$, $|\partial_r \overline{\omega}_{\mathrm{c}}(\rho)| \mathrm{d}\rho = |\partial_\xi \varpi(\xi')| \mathrm{d}\xi'$ and $r/\rho = e^{\xi - \xi'}$, so

$$Z_\ell = \int_{\{\partial_\xi \varpi > 0\}} \int_{\{\partial_\xi \varpi < 0\}} e^{2\ell(\xi - \xi')}\, |\partial_\xi \varpi(\xi)|\, |\partial_\xi \varpi(\xi')| \mathrm{d}\xi' \mathrm{d}\xi,$$

and the factorization follows from $e^{2\ell(\xi - \xi')} = e^{2\ell(\xi - \xi_c)} e^{-2\ell(\xi' - \xi_c)}$. For the bound on Z_ℓ^a, write $\partial_\xi \varpi = (\partial_\xi \chi) P + \chi\, \partial_\xi P$ and bound the two terms on $(-\infty, \xi_c]$, which contains $\{\partial_\xi \varpi > 0\}$. First, $\partial_\xi P = \frac{1}{d}(\partial_x H)(\frac{\xi - a_0}{d}) H(-\frac{\xi}{d}) - \frac{1}{d} H(\frac{\xi - a_0}{d})(\partial_x H)(-\frac{\xi}{d})$. The second term is ≤ 0, as $H, \partial_x H \geq 0$, and we drop it; in the first, $H(-\frac{\xi}{d}) \leq 1$ and $\partial_x H = \frac{1}{2}\operatorname{sech}^2$. So $\partial_\xi P \leq \frac{1}{2d}\operatorname{sech}^2\big(\frac{\xi - a_0}{d}\big)$ for every ξ, and, as $0 \leq \chi \leq 1$, also $\chi\, \partial_\xi P \leq \frac{1}{2d}\operatorname{sech}^2\big(\frac{\xi - a_0}{d}\big)$. Second, by Definition 4.1, χ is nondecreasing on $(-\infty, \xi_a]$ and equal to 1 on $[\xi_a, \xi_b]$, which contains $[\xi_a, \xi_c]$; so on $(-\infty, \xi_c]$ we have $\partial_\xi \chi \geq 0$, and $\partial_\xi \chi \neq 0$ only for $\xi < \xi_a$. For $\xi \leq \xi_a$, $P(\xi) \leq H(\frac{\xi - a_0}{d}) \leq H(\frac{\xi_a - a_0}{d})$, as $H \leq 1$ and H is increasing; so $(\partial_\xi \chi) P \leq H(\frac{\xi_a - a_0}{d})\, \partial_\xi \chi$ on $(-\infty, \xi_c]$. Both bounds are nonnegative. We integrate the first, times $e^{2\ell(\xi - \xi_c)}$, over all of $\mathbb{R}$, and for the second we use $e^{2\ell(\xi - \xi_c)} \leq 1$ on $(-\infty, \xi_c]$ and $\int_{-\infty}^{\xi_c} \partial_\xi \chi \mathrm{d}\xi = \chi(\xi_c) = 1$. The substitution $\xi = a_0 + dx$ gives

$$Z_\ell^a \leq e^{2\ell(a_0 - \xi_c)} \int_{\mathbb{R}} e^{2\ell d x}\, \tfrac{1}{2}\operatorname{sech}^2 x \mathrm{d}x + H\Big(\frac{\xi_a - a_0}{d}\Big) = e^{\ell a_0} \Theta_\ell + H\Big(\frac{\xi_a - a_0}{d}\Big),$$

by $\int_{\mathbb{R}} e^{\alpha x} \operatorname{sech}^2 x \mathrm{d}x = \pi\alpha / \sin(\pi\alpha/2)$ for $0 < |\alpha| < 2$, with $\alpha = 2\ell d$. As sech^2 is even, $\int_{\mathbb{R}} e^{\alpha x} \operatorname{sech}^2 x \mathrm{d}x = 2\int_0^\infty \cosh(\alpha x)(\cosh x)^{-2} \mathrm{d}x$, which is $2B(1 + \frac{\alpha}{2}, 1 - \frac{\alpha}{2})$ by [16, (5.12.7)] with $a = 1$ and $b = \frac{\alpha}{2}$ (admissible as $|\alpha| < 2$). By [16, (5.12.1)], $2B(1 + \frac{\alpha}{2}, 1 - \frac{\alpha}{2}) = 2\Gamma(1 + \frac{\alpha}{2})\Gamma(1 - \frac{\alpha}{2}) = \alpha\Gamma(\frac{\alpha}{2})\Gamma(1 - \frac{\alpha}{2})$, and the reflection formula $\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z)$ [16, (5.5.3)] gives $\pi\alpha / \sin(\pi\alpha/2)$. The bound for Z_ℓ^b is the mirror image: on $[\xi_c, \infty)$, which contains $\{\partial_\xi \varpi < 0\}$, $-\partial_\xi P \leq \frac{1}{2d}\operatorname{sech}^2\big(\frac{\xi}{d}\big)$ (now the first term of $\partial_\xi P$ is ≥ 0 and is dropped, and $H(\frac{\xi - a_0}{d}) \leq 1$); $\partial_\xi \chi \leq 0$ there, and $\partial_\xi \chi \neq 0$ only for $\xi > \xi_b$, where $P \leq H(-\xi_b/d)$; and with $\xi = dy$, $e^{-2\ell(dy - \xi_c)} = e^{\ell a_0} e^{-2\ell d y}$. □

4.4 Mode Bounds and the Semigroup

For $\ell \in \mathbb{Z}$ and a profile g supported in $[\rho_-, \rho_+]$ put

$$\|g\|_{k,\ell}^2 = \sum_{j=0}^{k} (1 + \ell^2)^{k-j} \|\partial_r^j g\|_{L^2([\rho_-, \rho_+], r\mathrm{d}r)}^2,$$

and let $\|g\|_{k,\ell;[\rho_-,\rho_+]}$ be the same expression for a profile g on $(0,\infty)$ that need not be supported in $[\rho_-,\rho_+]$.

Lemma 4.12. *In a mode setting, for every integer $k \geq 0$ there are $c_k^{\mathrm{eq}}, C_k^{\mathrm{eq}} > 0$, depending only on k, ρ_- and ρ_+, such that every $w = \sum_\ell g_\ell(r)e^{i\ell\theta} \in Y^k$ satisfies $c_k^{\mathrm{eq}} \sum_\ell \|g_\ell\|_{k,\ell}^2 \leq \|w\|_{H^k}^2 \leq C_k^{\mathrm{eq}} \sum_\ell \|g_\ell\|_{k,\ell}^2$.*

Proof. Near N_*, $\partial_r = \cos\theta\,\partial_1 + \sin\theta\,\partial_2$, $\partial_\theta = -x_2\partial_1 + x_1\partial_2$, $\partial_1 = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta$ and $\partial_2 = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta$. Iterating, each ∂^α, $|\alpha| = j$, is a combination of the $\partial_r^i\partial_\theta^n$, $i+n \leq j$, with coefficients bounded on N_* in terms of j, $\rho_\pm$, and conversely. As $\operatorname{supp} w \subset N_*$, $\|w\|_{H^k}^2$ is comparable to $\sum_{i,n\geq 0,\ i+n\leq k} \|\partial_r^i\partial_\theta^n w\|_{L^2(\mathbb{R}^2)}^2 = 2\pi\sum_\ell\sum_{i,n\geq 0,\ i+n\leq k} \ell^{2n}\|\partial_r^i g_\ell\|_{L^2(r\mathrm{d}r)}^2$ (Parseval in θ). Finally $2^{-p}(1+x)^p \leq \sum_{n=0}^p x^n \leq (1+x)^p$ for $x \geq 0$, by the binomial theorem, as $1 \leq \binom{p}{n} \leq 2^p$; take $x = \ell^2$, $p = k-i$. $\square$

Corollary 4.13. *In a mode setting, let $s \geq 0$ be an integer. There is $C_K = C_K(s,\rho_-,\rho_+,\|\partial_r\overline{\omega}\|_{C^s})$ such that $\|k_\ell\mathcal{G}_\ell g\|_{s,\ell} \leq C_K(1+\ell^2)^{-1/2}\|g\|_{s,\ell}$ for every $\ell \neq 0$, every G, and every $g \in H^s((0,\infty))$ supported in $[\rho_-,\rho_+]$.*

Proof. Let $\ell \neq 0$ and let $g \in H^s((0,\infty))$ be supported in $[\rho_-,\rho_+]$. We first bound $\mathcal{G}_\ell g$. Put $\xi = \log r$, $\tilde{g}(\xi) = g(e^\xi)$, $\tilde{\psi}(\xi) = (\mathcal{G}_\ell g)(e^\xi)$ and $\Xi = e^{2\xi}\tilde{g} \in H^s(\mathbb{R})$, compactly supported. For $r = e^\xi$ and $\rho = e^\eta$, the kernel of $\mathcal{G}_\ell$ (proof of Lemma 4.9(1)) is $K_\ell(r,\rho) = -\frac{1}{2|\ell|}\big(\min(r,\rho)/\max(r,\rho)\big)^{|\ell|} = -\frac{1}{2|\ell|}e^{-|\ell||\xi-\eta|}$, since $\min(r,\rho)/\max(r,\rho) = e^{-|\xi-\eta|}$, and $\rho\mathrm{d}\rho = e^{2\eta}\mathrm{d}\eta$. So $\tilde{\psi} = -\frac{1}{2|\ell|}e^{-|\ell||\cdot|} * \Xi$. We use the Fourier transform $\hat{u}(\zeta) = \int_\mathbb{R} e^{-i\xi\zeta}u(\xi)\mathrm{d}\xi$ on $\mathbb{R}$, for which $\frac{1}{2|\ell|}e^{-|\ell||\xi|}$ has Fourier transform $(\ell^2+\zeta^2)^{-1}$; hence the Fourier transform of $\tilde{\psi}$ is $\zeta \mapsto -\hat{\Xi}(\zeta)/(\ell^2+\zeta^2)$. With this normalization, Plancherel's theorem reads $\|u\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi}\int_\mathbb{R}|\hat{u}|^2\mathrm{d}\zeta$, which is where the single factor $\frac{1}{2\pi}$ below comes from, and $\partial_\xi^j u$ has Fourier transform $(i\zeta)^j\hat{u}$. For u on $\mathbb{R}$ and an integer $k \geq 0$ put $|u|_{k,\ell}^2 = \sum_{j=0}^k(1+\ell^2)^{k-j}\|\partial_\xi^j u\|_{L^2(\mathbb{R})}^2$; then

$$|u|_{k,\ell}^2 = \frac{1}{2\pi}\int_\mathbb{R}\sum_{j=0}^k(1+\ell^2)^{k-j}\zeta^{2j}\,|\hat{u}(\zeta)|^2\mathrm{d}\zeta.$$

Multiplying the inequality $2^{-k}(1+x)^k \leq \sum_{n=0}^k x^n \leq (1+x)^k$ of the proof of Lemma 4.12 by A^k, with $x = B/A$, gives $\sum_{j=0}^k A^{k-j}B^j \leq (A+B)^k \leq 2^k\sum_{j=0}^k A^{k-j}B^j$ for $A > 0$, $B \geq 0$; we take $A = 1+\ell^2$ and $B = \zeta^2$. The first of these inequalities, with $k = s+2$, gives the first step below; the formula for the Fourier transform of $\tilde{\psi}$ and $1+\ell^2+\zeta^2 \leq 2(\ell^2+\zeta^2)$ (as $\ell \neq 0$) give the second; and the second inequality, with $k = s$, gives the third:

$$|\tilde{\psi}|_{s+2,\ell}^2 \leq \frac{1}{2\pi}\int(1+\ell^2+\zeta^2)^{s+2}\frac{|\hat{\Xi}|^2}{(\ell^2+\zeta^2)^2}\mathrm{d}\zeta \leq \frac{4}{2\pi}\int(1+\ell^2+\zeta^2)^s|\hat{\Xi}|^2\mathrm{d}\zeta \leq 4\cdot 2^s|\Xi|_{s,\ell}^2.$$

On $\tilde{I} = [\log\rho_-,\log\rho_+]$, $\partial_r = e^{-\xi}\partial_\xi$ and $\partial_\xi = r\partial_r$, so ∂_r^j is a combination of ∂_ξ^i, $i \leq j$, with coefficients bounded on $\tilde{I}$, and conversely; with $r\mathrm{d}r = e^{2\xi}\mathrm{d}\xi$ and $(1+\ell^2)^{k-j} \leq (1+\ell^2)^{k-i}$ for $i \leq j$, for every integer $k \geq 0$ the norms $\|\cdot\|_{k,\ell;[\rho_-,\rho_+]}$ and $|\cdot|_{k,\ell}$ restricted to $\tilde{I}$ are equivalent, with constants depending on k, ρ_- and ρ_+. Multiplication by $e^{2\xi}$ is bounded for $|\cdot|_{s,\ell}$ on functions supported in $\tilde{I}$, by the Leibniz rule and the same monotonicity of the weights. Therefore $\|\mathcal{G}_\ell g\|_{s+2,\ell;[\rho_-,\rho_+]} \leq C|\tilde{\psi}|_{s+2,\ell} \leq C|\Xi|_{s,\ell} \leq C|\tilde{g}|_{s,\ell} \leq C\|g\|_{s,\ell}$, with $C = C(s,\rho_-,\rho_+)$.

The operator $k_\ell \mathcal{G}_\ell$ does not involve G, and $k_\ell \mathcal{G}_\ell g$ is supported in $I_A = [r_1, r_2]$. Since $1/r \le 1/\rho_-$ on I_A, the Leibniz rule gives

$$\|\partial_r^j (r^{-1}(\partial_r \overline{\omega})\psi)\|_{L^2(I_A, r\mathrm{d}r)} \le C \sum_{i=0}^{j} \|\partial_r^i \psi\|_{L^2(I_A, r\mathrm{d}r)}$$

with C depending on s, ρ_- and $\|\partial_r \overline{\omega}\|_{C^s}$. Using $(1+\ell^2)^{s-j} \le (1+\ell^2)^{s-i}$ for $i \le j$, and then $(1+\ell^2)^{s+2-i} = (1+\ell^2)^2 (1+\ell^2)^{s-i}$,

$$\begin{aligned}\|k_\ell \mathcal{G}_\ell g\|_{s,\ell}^2 &\le C\ell^2 \sum_{i=0}^{s} (1+\ell^2)^{s-i} \|\partial_r^i \mathcal{G}_\ell g\|_{L^2(I_A, r\mathrm{d}r)}^2 \\ &\le C\ell^2 (1+\ell^2)^{-2} \|\mathcal{G}_\ell g\|_{s+2,\ell;[\rho_-,\rho_+]}^2 \le C(1+\ell^2)^{-1} \|g\|_{s,\ell}^2,\end{aligned}$$

since $I_A \subset [\rho_-, \rho_+]$, and by the bound above. □

Lemma 4.14. *In a mode setting, let $s \ge 0$ be an integer and $G_+ \ge 0$, and let $|G| \le G_+$. Put $L^{\mathrm{tr}} = -\Omega \partial_\theta$ and $\mathcal{K}w = -(K_{BS} * w) \cdot \nabla \overline{\omega}$.*

(1) *$(e^{tL^{\mathrm{tr}}} w)(r,\theta) = w(r, \theta - \Omega(r;G)t)$ is a C_0-group on Y^s, and there is C_s, depending only on s, $\overline{\omega}$, $\rho_\pm$ and G_+, with $\|e^{tL^{\mathrm{tr}}}\|_{Y^s \to Y^s} \le C_s(1+|t|)^s$ and $\|e^{-i\ell\Omega t} g\|_{s,\ell} \le C_s(1+|t|)^s \|g\|_{s,\ell}$ for all $\ell \in \mathbb{Z}$, $|G| \le G_+$ and $t \in \mathbb{R}$; $\mathcal{K}$ is bounded on Y^s.*

(2) *The Dyson-Phillips series $e^{tL} = \sum_{n \ge 0} S_n(t)$, $S_0(t) = e^{tL^{\mathrm{tr}}}$, $S_{n+1}(t) = \int_0^t S_0(t-\tau)\mathcal{K} S_n(\tau)\mathrm{d}\tau$, converges in operator norm uniformly on compact t-sets and defines a C_0-semigroup on Y^s that satisfies both Duhamel formulas and preserves each mode. Its generator $\hat{L}$ satisfies $Y^{s+1} \subset D(\hat{L})$ and $\hat{L}w = L(G)w$ for $w \in Y^{s+1}$, with $\|L(G)w\|_{Y^s} \le C\|w\|_{Y^{s+1}}$.*

(3) *On each mode, $e^{tL}|_{Y_\ell^s} = \exp(tL_\ell(G))$, the norm-continuous semigroup generated by the bounded operator $L_\ell(G)$ on Y_ℓ^s.*

In (3), Y_ℓ^s is identified with $P_\ell Y^s$ through $g \mapsto g e^{i\ell\theta}$; $L_\ell(G)$ is bounded on Y_ℓ^s because multiplication by Ω is, and $k_\ell \mathcal{G}_\ell$ is by Corollary 4.13 and Lemma 4.12; so, by the argument in the proof of Lemma 4.5(4), $L_\ell(G)$ generates a norm-continuous group $\exp(tL_\ell(G))$ on Y_ℓ^s, with derivative $L_\ell(G)\exp(tL_\ell(G)) = \exp(tL_\ell(G))L_\ell(G)$.

Proof. (1) On Y_ℓ^s, $e^{tL^{\mathrm{tr}}}$ is multiplication by $e^{i\varphi}$, $\varphi = -\ell\Omega(r;G)t$. By the Leibniz and Faà di Bruno formulas, $\partial_r^j(e^{i\varphi} g) = e^{i\varphi} \sum_{i=0}^{j} \binom{j}{i} B_i \, \partial_r^{j-i} g$, where the complete Bell polynomial $B_i = B_i(i\partial_r\varphi, \dots, i\partial_r^i \varphi)$ is a sum of at most C_i products of at most i factors $\partial_r^q \varphi = -\ell t \, \partial_r^q \Omega$. On $[\rho_-, \rho_+]$, $\|\Omega(\cdot;G)\|_{C^s} \le \|\Omega(\cdot;0)\|_{C^s} + G_+ \|r^{-2}\|_{C^s}$, so $|B_i| \le C(1+|\ell||t|)^i \le C 2^{i/2}(1+|t|)^i (1+\ell^2)^{i/2}$ and

$$(1+\ell^2)^{s-j} \|\partial_r^j (e^{i\varphi} g)\|^2 \le C(1+|t|)^{2s} \sum_{i=0}^{j} (1+\ell^2)^{s-(j-i)} \|\partial_r^{j-i} g\|^2;$$

summing over $j \le s$ gives the mode bound. The map $(r,\theta) \mapsto (r, \theta - \Omega(r;G)t)$ is a smooth diffeomorphism near N_* preserving N_*, so $e^{tL^{\mathrm{tr}}}$ maps Y^s to itself, and Lemma 4.12 gives the bound on Y^s after enlarging C_s by the factor $(C_s^{\mathrm{eq}}/c_s^{\mathrm{eq}})^{1/2}$. For $w \in Y^s$ and each ℓ, $\|(e^{-i\ell\Omega t} - 1) g_\ell\|_{s,\ell} \to 0$ as $t \to 0$ by dominated convergence in the formula above, with the summable bound $(C_s(1+|t|)^s + 1)^2 \|g_\ell\|_{s,\ell}^2$; so $e^{tL^{\mathrm{tr}}} w \to w$ in H^s by Lemma 4.12. The group law holds because Ω does not depend on θ. By Lemma 4.5(2), $\mathcal{K}$ acts on mode profiles as $k_\ell \mathcal{G}_\ell$, so $\|\mathcal{K}w\|_{H^s} \le (C_s^{\mathrm{eq}}/c_s^{\mathrm{eq}})^{1/2} C_K \|w\|_{H^s}$ by Corollary 4.13 and Lemma 4.12, and $\operatorname{supp} \mathcal{K}w \subset \operatorname{supp} \partial_r \overline{\omega} \subset N_*$.

(2) By induction $\|S_n(t)\| \le C_s^{n+1}(1+t)^{s(n+1)}\|\mathcal{K}\|^n t^n/n!$, so the series converges in operator norm uniformly on compact sets, and its sum $T(t)$ is strongly continuous, as each S_n is. Summing the recursion, $T(t) = S_0(t) + \int_0^t S_0(t-\tau)\mathcal{K}T(\tau)\mathrm{d}\tau$. Fix $t' \ge 0$; we show that both $T(\cdot + t')$ and $T(\cdot)T(t')$ solve

$$\Psi(t) = S_0(t)T(t') + \int_0^t S_0(t-\tau)\mathcal{K}\Psi(\tau)\mathrm{d}\tau \qquad (t \ge 0).$$

For $T(\cdot)T(t')$, multiply the identity for $T(t)$ on the right by $T(t')$. For $T(\cdot + t')$, write the identity at time $t+t'$ and split the integral at $\tau = t'$. On $[0, t']$, $S_0(t+t'-\tau) = S_0(t)S_0(t'-\tau)$ by the group law of S_0, and the bounded operator $S_0(t)$ can be taken out of the integral; on $[t', t+t']$, substitute $\tau = t' + \sigma$. This gives

$$T(t+t') = S_0(t)\Big(S_0(t') + \int_0^{t'} S_0(t'-\tau)\mathcal{K}T(\tau)\mathrm{d}\tau\Big) + \int_0^t S_0(t-\sigma)\mathcal{K}T(\sigma+t')\mathrm{d}\sigma,$$

and the bracket is $T(t')$, which is the equation for $\Psi = T(\cdot + t')$. The difference D of two strongly continuous, locally bounded solutions satisfies $\|D(t)w\| \le C\int_0^t \|D(\tau)w\|\mathrm{d}\tau$, so $D = 0$ by iteration; hence T is a C_0-semigroup. By induction and Fubini's theorem, $S_n(t)$ is the integral of $S_0(t-\tau_n)\mathcal{K}S_0(\tau_n - \tau_{n-1})\cdots\mathcal{K}S_0(\tau_1)$ over $0 < \tau_1 < \cdots < \tau_n < t$, so also $S_{n+1}(t) = \int_0^t S_n(t-\tau)\mathcal{K}S_0(\tau)\mathrm{d}\tau$, which gives the second Duhamel formula $T(t) = S_0(t) + \int_0^t T(t-\tau)\mathcal{K}S_0(\tau)\mathrm{d}\tau$. The P_ℓ commute with S_0 and $\mathcal{K}$, which commute with rotations (S_0 because Ω is radial, $\mathcal{K}$ by Lemma 4.5(2)), hence with $T(t)$. For $w \in Y^{s+1}$, $\|L^{\mathrm{tr}}w\|_{H^s} \le C\|w\|_{H^{s+1}}$, as Ω and the coefficients of ∂_θ are smooth near N_*, and $(e^{hL^{\mathrm{tr}}}w - w)/h = \frac{1}{h}\int_0^h e^{\tau L^{\mathrm{tr}}}L^{\mathrm{tr}}w\mathrm{d}\tau \to L^{\mathrm{tr}}w$ in Y^s, since $\Omega\partial_\theta$ commutes with $e^{\tau L^{\mathrm{tr}}}$. By the second Duhamel formula, $(T(h)w - w)/h = (e^{hL^{\mathrm{tr}}}w - w)/h + \frac{1}{h}\int_0^h T(h-\tau)\mathcal{K}e^{\tau L^{\mathrm{tr}}}w\mathrm{d}\tau \to L^{\mathrm{tr}}w + \mathcal{K}w = L(G)w$.

(3) On Y_ℓ^s, $S_0(t) = \exp(-i\ell\Omega t)$ and $\mathcal{K} = k_\ell\mathcal{G}_\ell$, so $T(t)|_{Y_\ell^s}$ solves the first Duhamel equation for these bounded operators. So does $\exp(tL_\ell)$, since

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big[\exp(-(t-\tau)i\ell\Omega)\exp(\tau L_\ell)\big] = \exp(-(t-\tau)i\ell\Omega)k_\ell\mathcal{G}_\ell\exp(\tau L_\ell).$$

By uniqueness they coincide. □

Proposition 4.15. *In a mode setting, let $s \ge 0$ be an integer, $\gamma > 0$ and $G_+ \ge 0$. Let C_s be as in Lemma 4.14(1), C_K as in Corollary 4.13, $c_s(\gamma) = \sup_{t\ge 0}(1+t)^s e^{-\gamma t}$, and let $\mathcal{M}_\gamma \ge 1$ satisfy $C_s C_K(1+\mathcal{M}_\gamma^2)^{-1/2} I_s(\gamma) \le \frac{1}{2}$, with $I_s(\gamma)$ as in Equation (3). Then for every $\ell \in \mathbb{Z}$ with $|\ell| \ge \mathcal{M}_\gamma$, every G with $|G| \le G_+$ and every $t \ge 0$, $\|e^{tL_\ell(G)}\|_{s,\ell\to s,\ell} \le 2C_s\, c_s(\gamma)\, e^{\gamma t}$.*

Proof. Fix such ℓ and G. Since L_ℓ is bounded for $\|\cdot\|_{s,\ell}$ (remark after Lemma 4.14), $Y_T := \sup_{0\le t\le T} e^{-\gamma t}\|e^{tL_\ell}\|_{s,\ell\to s,\ell}$ is finite for $T < \infty$. By Lemma 4.14(3) and the first Duhamel formula, $e^{tL_\ell} = e^{-i\ell\Omega t} + \int_0^t e^{-i\ell\Omega(t-\tau)}k_\ell\mathcal{G}_\ell e^{\tau L_\ell}\mathrm{d}\tau$, so for $t \le T$, by Lemma 4.14(1) and Corollary 4.13,

$$\begin{aligned} e^{-\gamma t}\|e^{tL_\ell}\| &\le C_s(1+t)^s e^{-\gamma t} + \int_0^t C_s(1+t-\tau)^s e^{-\gamma(t-\tau)}\,\frac{C_K}{(1+\ell^2)^{1/2}}\,e^{-\gamma\tau}\|e^{\tau L_\ell}\|\mathrm{d}\tau \\ &\le C_s c_s(\gamma) + \tfrac{1}{2}Y_T, \end{aligned}$$

by the choice of $\mathcal{M}_\gamma$ and $\int_0^t (1+u)^s e^{-\gamma u}\mathrm{d}u \le I_s(\gamma)$. Taking the supremum over $t \le T$ gives $Y_T \le 2C_s c_s(\gamma)$, for all T. □

For $\mathcal{Z} \subset \mathbb{Z}$ let $Y_{\mathcal{Z}}^s$ be the closed subspace of $w \in Y^s$ with $g_\ell = 0$ for $\ell \notin \mathcal{Z}$; it is invariant under e^{tL} by Lemma 4.14(2).

Lemma 4.16. *In a mode setting, let $s \geq 0$ be an integer, $G_+ \geq 0$, $|G| \leq G_+$, $\mathcal{Z} \subset \mathbb{Z}$, and let $\hat{L}_{\mathcal{Z}}$ be the generator of the restriction of e^{tL} to $Y^s_{\mathcal{Z}}$. Then:*

(1) *$w \in D(\hat{L}_{\mathcal{Z}})$ if and only if $w \in Y^s_{\mathcal{Z}}$ and $\sum_{\ell \in \mathcal{Z}} \|L_\ell(G) g_\ell\|^2_{s,\ell} < \infty$, and then $P_\ell \hat{L}_{\mathcal{Z}} w = (L_\ell(G) g_\ell) e^{i\ell\theta}$ for every ℓ;*

(2) *let $\gamma > 0$, let $\mathcal{M}_\gamma$ be as in Proposition 4.15, and let $\operatorname{Re}\lambda > \gamma$. For $|\ell| \geq \mathcal{M}_\gamma$, λ lies in the resolvent set of $L_\ell(G)$ on Y^s_ℓ, and $\|(\lambda - L_\ell(G))^{-1}\|_{s,\ell \to s,\ell} \leq 2C_s c_s(\gamma)/(\operatorname{Re}\lambda - \gamma)$;*

(3) *for $\operatorname{Re}\lambda > \gamma > 0$: λ lies in the resolvent set of $\hat{L}_{\mathcal{Z}}$ if and only if λ lies in the resolvent set of $L_\ell(G)$ on Y^s_ℓ for every $\ell \in \mathcal{Z}$ with $|\ell| < \mathcal{M}_\gamma$; and then $P_\ell(\lambda - \hat{L}_{\mathcal{Z}})^{-1} w = ((\lambda - L_\ell(G))^{-1} g_\ell) e^{i\ell\theta}$ for every $\ell \in \mathcal{Z}$;*

(4) *if $\sigma_+ \subset \{\operatorname{Re}\lambda > \gamma\}$ is a compact set that is open and closed in the spectrum of $\hat{L}_{\mathcal{Z}}$, then the Riesz projection Π_{σ_+} of $\hat{L}_{\mathcal{Z}}$ for σ_+ satisfies $P_\ell \Pi_{\sigma_+} w = (\Pi_\ell g_\ell) e^{i\ell\theta}$, where Π_ℓ is the Riesz projection of $L_\ell(G)$ on Y^s_ℓ for $\sigma_+ \cap \operatorname{spec} L_\ell(G)$ (and $\Pi_\ell = 0$ if this set is empty).*

Proof. (1) Since P_ℓ is bounded and commutes with e^{tL}, for $w \in D(\hat{L}_{\mathcal{Z}})$ Lemma 4.14(3) gives $P_\ell \hat{L}_{\mathcal{Z}} w = \lim_{h \to 0} h^{-1}(\exp(hL_\ell) g_\ell - g_\ell) e^{i\ell\theta} = (L_\ell g_\ell) e^{i\ell\theta}$, and $\sum_\ell \|L_\ell g_\ell\|^2_{s,\ell} \leq (c_s^{\mathrm{eq}})^{-1} \|\hat{L}_{\mathcal{Z}} w\|^2_{H^s}$ by Lemma 4.12. Conversely, if the sum is finite, $v := \sum_\ell (L_\ell g_\ell) e^{i\ell\theta}$ converges in $Y^s_{\mathcal{Z}}$ by Lemma 4.12, and for $t > 0$

$$e^{tL} w - w = \sum_\ell \Big(\int_0^t \exp(\tau L_\ell) L_\ell g_\ell \mathrm{d}\tau \Big) e^{i\ell\theta} = \int_0^t e^{\tau L} v \mathrm{d}\tau;$$

dividing by t and letting $t \to 0$ shows $w \in D(\hat{L}_{\mathcal{Z}})$.

(2) We use the semigroup formula for the resolvent. Let A be a bounded operator with $\|e^{tA}\| \leq K e^{\gamma t}$ for $t \geq 0$, and let $\operatorname{Re}\lambda > \gamma$. Then $R = \int_0^\infty e^{-\lambda t} e^{tA} \mathrm{d}t$ converges in operator norm, $\|R\| \leq K/(\operatorname{Re}\lambda - \gamma)$, and $R = (\lambda - A)^{-1}$: integrating $\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-\lambda t} e^{tA}\big) = -e^{-\lambda t} e^{tA}(\lambda - A) = -(\lambda - A) e^{-\lambda t} e^{tA}$ over $[0, \infty)$ gives $R(\lambda - A) = (\lambda - A) R = 1$. By Proposition 4.15, $A = L_\ell(G)$ on Y^s_ℓ satisfies this with $K = 2C_s c_s(\gamma)$ when $|\ell| \geq \mathcal{M}_\gamma$, which gives (2).

(3) If λ is in the resolvent set of $\hat{L}_{\mathcal{Z}}$, then by (1) P_ℓ maps $D(\hat{L}_{\mathcal{Z}})$ into itself and commutes with $\hat{L}_{\mathcal{Z}}$ there, so $(\lambda - \hat{L}_{\mathcal{Z}})^{-1}$ commutes with P_ℓ and its restriction to the mode ℓ inverts $\lambda - L_\ell$. Conversely, under the condition, (2) and the finiteness of $\{\ell \in \mathcal{Z} : |\ell| < \mathcal{M}_\gamma\}$ give $K := \sup_{\ell \in \mathcal{Z}} \|(\lambda - L_\ell)^{-1}\|_{s,\ell \to s,\ell} < \infty$. For $w \in Y^s_{\mathcal{Z}}$, $u = \sum_\ell ((\lambda - L_\ell)^{-1} g_\ell) e^{i\ell\theta}$ converges in $Y^s_{\mathcal{Z}}$, with $\|u\|_{H^s} \leq (C_s^{\mathrm{eq}}/c_s^{\mathrm{eq}})^{1/2} K \|w\|_{H^s}$ (Lemma 4.12). Since $L_\ell u_\ell = \lambda u_\ell - g_\ell$, the sum in (1) is finite, so $u \in D(\hat{L}_{\mathcal{Z}})$ and $(\lambda - \hat{L}_{\mathcal{Z}}) u = w$; and $(\lambda - \hat{L}_{\mathcal{Z}}) u = 0$ forces $(\lambda - L_\ell) u_\ell = 0$ for all ℓ by (1), so $u = 0$.

(4) Let $\Gamma \subset \{\operatorname{Re}\lambda > \gamma\}$ be a contour in the resolvent set of $\hat{L}_{\mathcal{Z}}$ winding once around σ_+ and not around the rest of the spectrum. Since P_ℓ is bounded, $P_\ell \Pi_{\sigma_+} w = \frac{1}{2\pi i} \oint_\Gamma P_\ell (\lambda - \hat{L}_{\mathcal{Z}})^{-1} w \mathrm{d}\lambda = \big(\frac{1}{2\pi i} \oint_\Gamma (\lambda - L_\ell)^{-1} g_\ell \mathrm{d}\lambda\big) e^{i\ell\theta}$ by (3). The spectrum of L_ℓ inside Γ is $\sigma_+ \cap \operatorname{spec} L_\ell$, by (3), which gives the claim. □

4.5 The Numerical Hypothesis

The rest of the paper assumes the following statements, which are proved in Appendix A of Part II [8] (Theorem 1.3 there).

Hypothesis 4.17. *Let $a_0, d, \xi_L, \xi_a, \xi_b, \xi_R$, H, χ and ϖ be as in Definition 4.1. For $G \geq 0$ and $\xi \in \mathbb{R}$ put*

$$\Omega_G(\xi) = e^{-2\xi} \Big(G + \int_{-\infty}^{\xi} \varpi(\xi') e^{2\xi'} \mathrm{d}\xi' \Big).$$

For an integer $\ell \geq 1$, $G \geq 0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, let $\psi = \psi_\ell(\cdot;\lambda,G)$ be the solution on $[\xi_L,\xi_R]$ of

$$\partial_\xi^2\psi = \Big(\ell^2 + \frac{i\ell\,\partial_\xi\varpi(\xi)}{\lambda + i\ell\,\Omega_G(\xi)}\Big)\psi, \qquad \psi(\xi_L) = 1, \quad \partial_\xi\psi(\xi_L) = \ell,$$

and put

$$\mathcal{E}_\ell(\lambda;G) = e^{-\ell(\xi_R-\xi_L)}\big(\partial_\xi\psi(\xi_R) + \ell\psi(\xi_R)\big),$$

a holomorphic function of λ in $\{\operatorname{Re}\lambda > 0\}$ (Lemma 4.6). Put $\Theta_\ell = \pi\ell d/\sin(\pi\ell d)$ and $\epsilon_* = \max\{H((\xi_a - a_0)/d),\, H(-\xi_b/d)\}$. Then:

(N1) *for every $G \in [0,\frac{1}{50}]$, the function $\mathcal{E}_3(\cdot;G)$ has exactly one zero, counted with multiplicity, in the closed half-plane $\{\operatorname{Re}\lambda \geq \frac{9}{200}\}$;*

(N2) *for every $G \in [0,\frac{1}{50}]$, the function $\mathcal{E}_3(\cdot;G)$ has no zero in the closed strip $\{\frac{9}{200} \leq \operatorname{Re}\lambda \leq \frac{211}{2500}\}$;*

(N3) *for $\ell \in \{6,9\}$, $e^{\ell a_0}\Theta_\ell + \epsilon_* < \frac{9}{100}$.*

By Lemma 4.2, $6d < 1$ and $9d < 1$, so Θ_6 and Θ_9 are finite positive numbers.

4.6 Consequences of the Numerical Hypothesis

We use the ring in both normalizations of Definition 4.1. In reference units the ring is $\overline{\omega}_{\mathrm{c}}(x) = \varpi(\log|x|)$, given directly by the explicit profile ϖ, and Hypothesis 4.17, which Part II verifies with computer assistance, is stated in these units, for $G \in [0,\frac{1}{50}]$. In ring units the ring Φ is supported in $\{a \leq |y| \leq 1\}$, with outer radius 1; this is the normalization of the building block of the datum (2), whose rings are the rescaled copies $A_k\Phi(\cdot/r_k)$, and of the spaces X^s and the operator $L(G)$ that Part II uses. Equation (13) below translates between the two.

In each case we fix a mode setting (Definition 4.3); recall that $(\xi_1,\xi_2) = (\log r_1, \log r_2)$, where $[r_1,r_2]$ contains the support of $\partial_r\overline{\omega}$. In reference units, $\overline{\omega} = \overline{\omega}_{\mathrm{c}}$, whose profile ϖ is supported in $[\xi_L,\xi_R]$; we take $(\xi_1,\xi_2) = (\xi_L,\xi_R)$ and $N_* = \{\frac{1}{2}e^{\xi_L} \leq |x| \leq 2e^{\xi_R}\}$. In ring units, $\overline{\omega} = \Phi$; since $\log|x| = \log|y| + \xi_R$, the profile of Φ in $\xi = \log|y|$ is $\varpi(\xi + \xi_R)$, supported in $[\xi_L - \xi_R, 0]$, so we take $(\xi_1,\xi_2) = (\xi_L - \xi_R, 0)$, that is $(r_1,r_2) = (a,1)$, and $N_* = N$. Both are mode settings by Lemma 4.2. We write $\mathcal{E}_\ell$ and $\mathcal{E}_\ell^\Phi$ for the two Evans functions; likewise, superscripts $\overline{\omega}_{\mathrm{c}}$ and Φ on Ω and ν indicate the profile. Since $\Phi = \overline{\omega}_{\mathrm{c}}(R_{\mathrm{c}}\,\cdot)$, $\Gamma_\Phi(r) = R_{\mathrm{c}}^{-2}\Gamma_{\overline{\omega}_{\mathrm{c}}}(R_{\mathrm{c}}r)$, so

$$\Omega^\Phi(r;G) = r^{-2}\Big(\frac{R_{\mathrm{c}}^{-2}\Gamma_{\overline{\omega}_{\mathrm{c}}}(R_{\mathrm{c}}r)}{2\pi} + G\Big) = (R_{\mathrm{c}}r)^{-2}\Big(\frac{\Gamma_{\overline{\omega}_{\mathrm{c}}}(R_{\mathrm{c}}r)}{2\pi} + R_{\mathrm{c}}^2G\Big) = \Omega^{\overline{\omega}_{\mathrm{c}}}(R_{\mathrm{c}}r;R_{\mathrm{c}}^2G),$$

and $\nu^\Phi(\xi) = \nu^{\overline{\omega}_{\mathrm{c}}}(\xi + \xi_R)$, as $R_{\mathrm{c}} = e^{\xi_R}$. Hence the Rayleigh potential for (Φ,G) at ξ is the one for $(\overline{\omega}_{\mathrm{c}}, R_{\mathrm{c}}^2G)$ at $\xi + \xi_R$. So if $\psi_\ell(\cdot;\lambda,R_{\mathrm{c}}^2G)$ is the solution of Definition 4.4 for $(\overline{\omega}_{\mathrm{c}}, R_{\mathrm{c}}^2G)$, normalized at ξ_L, then $\psi_\ell(\cdot + \xi_R;\lambda,R_{\mathrm{c}}^2G)$ is the one for (Φ,G), normalized at $\xi_L - \xi_R$, and the formula for the Evans function gives

$$\mathcal{E}_\ell^\Phi(\lambda;G) = \mathcal{E}_\ell(\lambda;R_{\mathrm{c}}^2G) \qquad (\ell \geq 1,\ \operatorname{Re}\lambda > 0,\ G \in \mathbb{R}). \tag{13}$$

Corollary 4.18. *Assume* (N1) *and* (N2). *For every $G \in [0,\frac{1}{50}]$, the function $\mathcal{E}_3(\cdot;G)$ of Hypothesis 4.17 has exactly one zero $\lambda_*^{\mathrm{c}}(G)$ in $\{\operatorname{Re}\lambda \geq \frac{9}{200}\}$; it is simple and $\operatorname{Re}\lambda_*^{\mathrm{c}}(G) > \frac{211}{2500}$.*

Proof. By (N1), $\mathcal{E}_3(\cdot;G)$ has exactly one zero $\lambda_*^{\mathrm{c}}(G)$ in $\{\operatorname{Re}\lambda \geq \frac{9}{200}\}$, counted with multiplicity, so this zero is simple; by (N2), $\operatorname{Re}\lambda_*^{\mathrm{c}}(G) > \frac{211}{2500}$. □

For the rest of the paper, X^s denotes the space of 3-fold rotationally symmetric $w \in H^s(\mathbb{R}^2;\mathbb{C})$ with $\operatorname{supp} w \subset N = \{\frac{a}{2} \le |y| \le 2\}$; thus $X^s = Y^s_{3\mathbb{Z}}$ for the ring in ring units, and $X^s_\ell = Y^s_\ell$ for $\ell \in 3\mathbb{Z}$. On X^s, $L(G)$ denotes the linearization at (Φ, G) in ring units, that is, the generator of the restriction of e^{tL} to X^s; by Lemma 4.14(2) it extends the differential operator on X^{s+1}, and Lemma 4.16 describes it mode by mode.

Proposition 4.19. *Assume* (N1)–(N3)*, and for* $G \in [0, \overline{G}]$ *let* $\lambda_*(G) = \lambda^{\mathrm{c}}_*(R^2_{\mathrm{c}}G)$ *be as in Corollary 4.18. Then* $\operatorname{Re}\lambda_*(G) > \frac{211}{2500}$*, and for every* $G \in [0, \overline{G}]$ *and every integer* $s \ge 0$*:* $\operatorname{spec}(L_3(G)|_{X^s_3}) \cap \{\operatorname{Re}\lambda > \frac{9}{200}\} = \{\lambda_*(G)\}$*, and* $\lambda_*(G)$ *is an algebraically simple eigenvalue of* $L_3(G)$ *on* X^s_3*;* $\operatorname{spec}(L_{-3}(G)|_{X^s_{-3}}) \cap \{\operatorname{Re}\lambda > \frac{9}{200}\} = \{\overline{\lambda_*(G)}\}$*;* $\operatorname{spec}(L_\ell(G)|_{X^s_\ell}) \cap \{\operatorname{Re}\lambda > \frac{9}{200}\} = \emptyset$ *for* $\ell \in 3\mathbb{Z} \setminus \{3, -3\}$*;*

$$\operatorname{spec}(L(G)|_{X^s}) \cap \{\operatorname{Re}\lambda > \tfrac{9}{200}\} = \{\lambda_*(G), \overline{\lambda_*(G)}\};$$

and the Riesz projection of $L(G)$ *on* X^s *for this set acts as the Riesz projection of* $L_3(G)$ *at* $\lambda_*(G)$ *on the mode* 3*, as the Riesz projection of* $L_{-3}(G)$ *at* $\overline{\lambda_*(G)}$ *on the mode* -3*, and as* 0 *on the other modes.*

Proof. Fix $G \in [0, \overline{G}]$ and $s \ge 0$; then $R^2_{\mathrm{c}}G \in [0, \frac{1}{50}]$. For $\ell = \pm 3$: by Lemma 4.7, run in the mode setting of Φ, the spectrum of $L_{\pm 3}(G)$ on $X^s_{\pm 3}$ in $\{\operatorname{Re}\lambda > 0\}$ consists of the zeros of $\mathcal{E}^\Phi_3(\cdot; G)$ and their conjugates respectively, all eigenvalues. By (13) and the last assertion of Lemma 4.6, $\mathcal{E}^\Phi_3(\cdot; G)$ is the function $\mathcal{E}_3(\cdot; R^2_{\mathrm{c}}G)$ of Hypothesis 4.17. So by Corollary 4.18 the part of this spectrum in $\{\operatorname{Re}\lambda \ge \frac{9}{200}\}$ is $\{\lambda_*(G)\}$ for $L_3(G)$ and $\{\overline{\lambda_*(G)}\}$ for $L_{-3}(G)$, where $\lambda_*(G) = \lambda^{\mathrm{c}}_*(R^2_{\mathrm{c}}G)$ is a simple zero of $\mathcal{E}^\Phi_3(\cdot; G)$ and $\operatorname{Re}\lambda_*(G) > \frac{211}{2500}$; and Theorem 4.8(2) shows that $\lambda_*(G)$ is algebraically simple on X^s_3. Let $|\ell| = 6$ or $|\ell| \ge 9$, and let $\ell' = 6$ if $|\ell| = 6$ and $\ell' = 9$ if $|\ell| \ge 9$. By Lemma 4.7, every point λ of $\operatorname{spec}(L_\ell(G)|_{X^s_\ell})$ in $\{\operatorname{Re}\lambda > 0\}$ is an eigenvalue, with an eigenfunction in $X^s_\ell = Y^s_\ell \subset Y^0_\ell$. By Lemma 4.11(1), $\overline{\omega}_{\mathrm{c}}$ satisfies the hypothesis of Lemma 4.10, and so does Φ, as $\partial_r\Phi(r) = R_{\mathrm{c}}\,(\partial_r\overline{\omega}_{\mathrm{c}})(R_{\mathrm{c}}r)$; the substitution $u = R_{\mathrm{c}}r$ shows that Z_ℓ is the same for both. By Lemma 4.10, the monotonicity of Z_ℓ (as $\ell' \le |\ell|$), Lemma 4.11(2) with ℓ' in place of ℓ (admissible since $\ell' d < 1$, Lemma 4.2) and (N3), $\operatorname{Re}\lambda \le \frac{1}{2}Z^{1/2}_{|\ell|} \le \frac{1}{2}Z^{1/2}_{\ell'} = \frac{1}{2}(Z^a_{\ell'}Z^b_{\ell'})^{1/2} \le \frac{1}{2}\big(e^{\ell' a_0}\Theta_{\ell'} + \epsilon_*\big) < \frac{1}{2}\cdot\frac{9}{100}$; and $L_0 = 0$. This proves the mode statements. Apply Lemma 4.16 with $\mathcal{Z} = 3\mathbb{Z}$, $G_+ = \overline{G}$, $\gamma = \frac{9}{200}$. Every λ with $\operatorname{Re}\lambda > \frac{9}{200}$ other than $\lambda_*(G)$, $\overline{\lambda_*(G)}$ is in the resolvent set of every $L_\ell(G)$, hence of $L(G)$ by Lemma 4.16(3), while $\lambda_*(G)$ and $\overline{\lambda_*(G)}$ are eigenvalues with eigenfunctions $g_*e^{3i\theta}$ and $\overline{g_*}e^{-3i\theta}$, where $g_* = M_{\lambda_*(G)}k_3\,\psi_3(\log r; \lambda_*(G), G)$ is the eigenfunction of Lemma 4.7(2) in the mode setting of Φ. This finite set is open and closed in the spectrum, and Lemma 4.16(4) gives its Riesz projection. □

The next definition collects the eigendata that Part II uses. Two of them, $\beta(G)$ and $p^{\mathrm{out}}_*(G)$, describe how a ring interacts with the other rings of the datum. Let g be a real vorticity supported in a disk $\{|y| \le \rho\}$ with $\rho < a$, inside the ring. By the expansion following Definition 2.3, the part of its stream function in the angular modes ± 3 is $-\operatorname{Re}\big(\frac{1}{3}p^{\mathrm{out}}_3[g]\,(\bar{y})^{-3}\big)$ on the ring, and since $(\bar{y})^{-3} = r^{-3}e^{3i\theta}$, the corresponding velocity h has radial component $h \cdot \mathbf{e}_r = -r^{-1}\partial_\theta\big(-\operatorname{Re}(\frac{1}{3}p^{\mathrm{out}}_3[g]r^{-3}e^{3i\theta})\big) = \operatorname{Re}\big(i\,p^{\mathrm{out}}_3[g]\,r^{-4}e^{3i\theta}\big)$. The forcing $-h \cdot \nabla\Phi = -(h \cdot \mathbf{e}_r)\,\partial_r\Phi$ that this field exerts on the ring therefore has mode-3 part $p^{\mathrm{out}}_3[g]\,b(r)e^{3i\theta}$, with $b(r) = -\frac{i}{2}r^{-4}\partial_r\Phi(r)$. So $be^{3i\theta}$ is the forcing of the ring, in the mode 3, by an interior field of unit outer moment, and $\beta(G)$ below is its component along the unstable eigenfunction. Conversely, $p^{\mathrm{out}}_*(G)$ is the outer moment of the unstable eigenfunction, which determines the field that a perturbed ring exerts on the rings outside it. In the relay of Part II, the kick that a ring gives to the next ring outward has unstable component $p^{\mathrm{out}}_3[g]\,\beta(G)$, where g is the inner ring rescaled to the units of the outer one; near saturation the

perturbation of the inner ring is a multiple of its unstable eigenfunction, so $p_3^{\mathrm{out}}[g]$ is a multiple of p_*^{out}. This is why Lemma 4.23 bounds $|\beta|\,|p_*^{\mathrm{out}}|$ from below.

Definition 4.20. *Assume* (N1)–(N3)*, and let* $G \in [0, \overline{G}]$*. With the mode setting of* Φ *in ring units and* $\ell = 3$*, let* $\lambda_*(G)$ *be as in Proposition 4.19,* $g_* = g_*(G)$ *and* $J(G)$ *as in Theorem 4.8,* $n_*(G) = \|g_* e^{3i\theta}\|_{L^2(\mathbb{R}^2)}$*, and*

$$\phi_*(G) = n_*(G)^{-1} g_*(r) e^{3i\theta}, \qquad \sigma(G) = \operatorname{Re}\lambda_*(G).$$

Let $\ell_*(G)$ *be the functional on* X^0 *given by* $\ell_*(G)(w) = n_*(G)\,\ell_*(w_3)$*, where* w_3 *is the mode-3 profile of* w *and* ℓ_* *is the functional of Theorem 4.8; so* $\ell_*(G)(\phi_*(G)) = 1$ *and* $\ell_*(G)(\overline{\phi_*(G)}) = 0$*. The same formula defines* $\ell_*(G)$ *on* $L^2(N)$*. Put* $\beta(G) = \ell_*(G)\big(b\, e^{3i\theta}\big)$ *with* $b(r) = -\frac{i}{2} r^{-4} \partial_r \Phi(r)$*,* $p_*^{\mathrm{out}}(G) = p_3^{\mathrm{out}}[\phi_*(G)]$ *(Definition 2.3), let* $\Pi(G)$ *be the Riesz projection of Proposition 4.19, so that* $\Pi(G)w = 2\operatorname{Re}\big(\ell_*(G)(w)\phi_*(G)\big)$ *for real* w*, and put*

$$z = \lambda_*(0), \qquad \sigma = \operatorname{Re} z = \sigma(0), \qquad \sigma_\flat = \min_{G \in [0,\overline{G}]} \sigma(G).$$

The formula for $\Pi(G)$ follows from Proposition 4.19 and Theorem 4.8(2): on the mode 3 the Riesz projection is $w_3 \mapsto \ell_*(w_3) g_* = \ell_*(G)(w)\, n_*^{-1} g_*$, and for real w the mode -3 part is the conjugate of the mode 3 part (Lemma 4.5(2)). By the linearity of p_3^{out} and ℓ_*, $p_*^{\mathrm{out}}(G) = p_*^{\mathrm{out}}/n_*(G)$ and $\beta(G) = n_*(G)\beta$, with p_*^{out} and β of Lemma 4.9 for Φ, $\ell = 3$ and $\lambda_* = \lambda_*(G)$. The minimum defining $\sigma_\flat$ exists by Proposition 4.21(a) below. With $M_* = M_{\lambda_*(G)}$, the functional $\ell_*(G)$ on $L^2(N)$ is represented by the restriction to N of $x \mapsto \frac{n_*}{2\pi \bar{J}}\, \overline{\psi_*(\log|x|) M_*(|x|)}\, e^{3i\theta}$, which lies in X^0, so its norms on $L^2(N)$ and on X^0 agree.

Proposition 4.21. *Assume* (N1)–(N3)*. Then:*

(a) $G \mapsto \lambda_*(G)$ *is continuous on* $[0, \overline{G}]$*; in particular* $G \mapsto \sigma(G)$ *is continuous,* $\sigma_\flat$ *is attained, and* $\sigma_\flat > \frac{211}{2500}$*;*

(b) *the functions* $G \mapsto |p_*^{\mathrm{out}}(G)|$*,* $|\beta(G)|$*,* $\|\ell_*(G)\|_{(X^0)^*}$ *and* $\|\phi_*(G)\|_{H^k}$ *(*$k \geq 0$*) are continuous on* $[0, \overline{G}]$*, and the first two are positive.*

Proof. (a) In reference units put $E(\lambda, G) = \mathcal{E}_3(\lambda; G)$, the function of Hypothesis 4.17 by Lemma 4.6. In the Picard iteration of Lemma 4.6(1), with $G \in [0, \frac{1}{50}]$ as a parameter, the iterates are continuous in (ξ, λ, G) and holomorphic in λ, and the bound C_A does not depend on G, by Lemma 4.5(3). So E is continuous on $\{\operatorname{Re}\lambda > 0\} \times [0, \frac{1}{50}]$ and holomorphic in λ, and $\partial_\lambda E$ is continuous there by Cauchy's formula. Fix $G_0 \in [0, \frac{1}{50}]$ and the simple zero $\lambda_0 = \lambda_*^{\mathrm{c}}(G_0)$ (the eigenvalue in reference units of Corollary 4.18, the superscript c referring to $\overline{\omega}_{\mathrm{c}}$; in ring units $\lambda_*(G) = \lambda_*^{\mathrm{c}}(R_{\mathrm{c}}^2 G)$) of $E(\cdot, G_0)$, let $\delta_1 > 0$ be such that $\{|\lambda - \lambda_0| \leq \delta_1\} \subset \{\operatorname{Re}\lambda > \frac{9}{200}\}$ contains no other zero of $E(\cdot, G_0)$, and let $0 < \delta < \delta_1$. By continuity and compactness there is $\epsilon > 0$ with $E \neq 0$ on $\{|\lambda - \lambda_0| = \delta\} \times I_\epsilon$, where $I_\epsilon = \{G \in [0, \frac{1}{50}] : |G - G_0| < \epsilon\}$. On the interval I_ϵ the number of zeros $\frac{1}{2\pi i} \oint_{|\lambda - \lambda_0| = \delta} \partial_\lambda E / E \mathrm{d}\lambda$ of $E(\cdot, G)$ in the disk is continuous, integer-valued and 1 at G_0, hence 1. This zero lies in $\{\operatorname{Re}\lambda > \frac{9}{200}\}$, so it is $\lambda_*^{\mathrm{c}}(G)$ by (N1), and $|\lambda_*^{\mathrm{c}}(G) - \lambda_0| < \delta$ for $G \in I_\epsilon$. So λ_*^{c} is continuous on $[0, \frac{1}{50}]$, and (a) follows from $\lambda_*(G) = \lambda_*^{\mathrm{c}}(R_{\mathrm{c}}^2 G)$, $\sigma(G) > \frac{211}{2500}$ (Proposition 4.19) and the compactness of $[0, \overline{G}]$.

(b) In ring units, let $\psi(\xi; \lambda, G)$ be the solution of Definition 4.4 for Φ, normalized at $\xi_1 = \xi_L - \xi_R$. By the Picard argument and the equation, ψ and all its ξ-derivatives are continuous in (ξ, λ, G). Hence $\psi_*(G) := \psi(\cdot; \lambda_*(G), G)$ and $g_*(G) = M_{\lambda_*(G)} k_3 \psi_*(G)$, supported in $[r_1, r_2] = [a, 1]$, are continuous in G in every $C^k([a, 1])$. So $n_*(G)$ and $J(G) = \int g_* \psi_* M_* r \mathrm{d}r$ are continuous, $n_* > 0$

by Lemma 4.7(2), and $J \neq 0$ by Theorem 4.8(1), as $\lambda_*(G)$ is a simple zero of $\mathcal{E}_3^{\Phi}(\cdot;G)$ (proof of Proposition 4.19). By Lemma 4.9(2) and (3) and the linearity of p_3^{out} and ℓ_*, with $\xi_2 = 0$,

$$p_*^{\mathrm{out}}(G) = -\frac{6\psi_*(\xi_2)}{n_*}, \quad \beta(G) = -\frac{n_*\, p_3^{\mathrm{in}}[g_* e^{3i\theta}]}{6J} = \frac{e^{-3\xi_1} n_*}{J};$$

these are continuous, and nonzero by Lemma 4.9(4). Also $\|\phi_*(G)\|_{H^k}$ is continuous, as $g_*(G)$ is continuous in $C^k([a,1])$ and $n_* > 0$, and so is $\|\ell_*(G)\|_{(X^0)^*} = \frac{n_*}{2\pi|J|}\|\psi_*(\log|x|)M_*(|x|)\|_{L^2(N)}$, by the representation after Definition 4.20. □

Proposition 4.22. *Assume* (N1)–(N3). *Let $s \geq 0$ be an integer and $\sigma' \in (\frac{9}{200}, \sigma_\flat)$. There is $C_S = C_S(s,\sigma')$, depending only on s, σ' and the ring, such that for every $G \in [0,\overline{G}]$ and $t \geq 0$:*

$$\|e^{tL(G)}(1-\Pi(G))\|_{X^s\to X^s} \leq C_S e^{\sigma' t}, \qquad \|\Pi(G)\|_{X^s\to X^s} \leq C_S,$$
$$\|e^{tL(G)}\|_{X^s\to X^s} \leq C_S e^{\sigma(G)t},$$

and, for real $w \in X^s$ and for $w \in X^0$ respectively,

$$e^{tL(G)}\Pi(G)w = 2\,\mathrm{Re}\,\big(\ell_*(G)(w)e^{\lambda_*(G)t}\phi_*(G)\big), \qquad \ell_*(G)(e^{tL(G)}w) = e^{\lambda_*(G)t}\ell_*(G)(w).$$

Proof. We use the setting of Φ in ring units, with $G_+ = \overline{G}$. By Lemma 4.14(3), Proposition 4.19 and Lemma 4.12 (which costs the factor $(C_s^{\mathrm{eq}}/c_s^{\mathrm{eq}})^{1/2}$), it suffices to bound $\|\exp(tL_\ell)(1-\Pi_\ell)\|_{s,\ell\to s,\ell}$ uniformly in $\ell \in 3\mathbb{Z}$ and G, where $\Pi_{\pm 3}$ are the mode Riesz projections and $\Pi_\ell = 0$ otherwise. For the fixed σ', let $\mathcal{M}_{\sigma'}$ be the number $\mathcal{M}_\gamma$ of Proposition 4.15 with $\gamma = \sigma'$, and let $\mathcal{M}_0 = \max\{\mathcal{M}_{\sigma'}, 4\}$ be the larger of the two numbers $\mathcal{M}_{\sigma'}$ and 4; the 4 ensures that $|\ell| \geq \mathcal{M}_0$ excludes the modes ± 3. For $|\ell| \geq \mathcal{M}_0$ (so $\Pi_\ell = 0$) that proposition gives the bound $2C_s c_s(\sigma')e^{\sigma' t}$, and $\exp(tL_0) = 1$. For the finitely many $\ell \in 3\mathbb{Z}$ with $0 < |\ell| < \mathcal{M}_0$, $L_\ell(G) = L_\ell(0) - i\ell G r^{-2}$ is bounded and affine in G, so $R_* := 1 + \max_{0<|\ell|<\mathcal{M}_0}\sup_{G\in[0,\overline{G}]}\|L_\ell(G)\|_{s,\ell} < \infty$. Fix $\sigma'' \in (\frac{9}{200}, \sigma')$ and let Γ be the boundary of $[-R_*, \sigma''] \times [-R_*, R_*]$. By Proposition 4.19, the spectrum of $L_\ell(G)$ in $\{\mathrm{Re}\,\lambda \geq \sigma''\}$ is empty, or $\{\lambda_*(G)\}$ for $\ell = 3$, or $\{\overline{\lambda_*(G)}\}$ for $\ell = -3$, with $\mathrm{Re}\,\lambda_*(G) \geq \sigma_\flat > \sigma''$; the rest lies in $\{|\lambda| < R_*,\ \mathrm{Re}\,\lambda < \sigma''\}$, inside Γ. So Γ lies in the resolvent set. For a bounded operator A and $\varrho > \|A\|$, $\frac{1}{2\pi i}\oint_{|\lambda|=\varrho} e^{t\lambda}(\lambda - A)^{-1}\mathrm{d}\lambda = e^{tA}$ by the Neumann series; take $A = L_\ell(G)$, $\varrho = 2R_*$, so that the circle encloses Γ. By Cauchy's theorem the circle may be replaced by Γ and a small circle around the eigenvalue outside Γ, if any; on the range of the rank-one projection $\Pi_{\pm 3}$ the resolvent is $(\lambda - \lambda_*)^{-1}$ (or its conjugate), and on the range of $1 - \Pi_{\pm 3}$ it is holomorphic near λ_*. Hence

$$\exp(tL_\ell)(1-\Pi_\ell) = \frac{1}{2\pi i}\oint_\Gamma e^{t\lambda}(\lambda - L_\ell(G))^{-1}\mathrm{d}\lambda,$$

of norm at most $\frac{|\Gamma|}{2\pi}e^{\sigma'' t}\sup_\Gamma\|(\lambda - L_\ell(G))^{-1}\|_{s,\ell}$, and the supremum over $\Gamma \times [0,\overline{G}]$ and $0 < |\ell| < \mathcal{M}_0$ is finite because inversion is continuous on invertible operators and $\Gamma \times [0,\overline{G}]$ is compact. This gives the first bound.

Since $\Pi_3 g = \ell_*(g)g_*$, the norm $\|\Pi(G)\|_{X^s\to X^s}$ is at most $C\|\ell_*(G)\|_{(X^0)^*}\|\phi_*(G)\|_{H^s}$, which is bounded on $[0,\overline{G}]$ by Proposition 4.21(b). The mode-3 part of $e^{tL}\Pi w$ is $\exp(tL_3)\ell_*(w_3)g_* = e^{\lambda_* t}\ell_*(w_3)g_*$, and for real w the mode -3 part is its conjugate (Lemma 4.5(2)); this gives the formula, and for complex w the bound $\|e^{tL}\Pi\|_{X^s\to X^s} \leq Ce^{\sigma(G)t}$, so the third bound follows as $\sigma' < \sigma_\flat \leq \sigma(G)$. Finally $\ell_*(G)(e^{tL}w) = n_*\ell_*(\exp(tL_3)w_3) = e^{\lambda_* t}\ell_*(G)(w)$ by Lemma 4.14(3) and Lemma 4.9(1). The constants depend only on s, σ' and the ring. □

Lemma 4.23. *Assume* (N1)–(N3). *Here Φ is the ring in ring units (Definition 4.1), N and N' are the annuli of Section 4.1, and all maxima and minima are taken over $G \in [0, \overline{G}]$; they are attained, since the functions involved are continuous on this compact interval (Proposition 4.21).*

(1) *The constants*

$$c_N = \min|\beta| \cdot \min|p_*^{\mathrm{out}}|, \qquad C_N = \max|\beta| \cdot \max|p_*^{\mathrm{out}}|, \qquad c_{\mathrm{out}} = 2\pi \min|p_*^{\mathrm{out}}|$$

satisfy $0 < c_N \le C_N < \infty$ and $c_{\mathrm{out}} > 0$.

(2) *The number $\|\ell_*\| := \max_G \|\ell_*(G)\|_{(X^0)^*}$ is finite, and so is $\max_G \|\phi_*(G)\|_{H^k}$ for every integer $k \ge 0$.*

(3) *Put*

$$C_{\ell_*} = \|\ell_*\| \, \|\nabla\Phi\|_{L^2(\mathbb{R}^2)}, \qquad C_{\mathrm{p}} = \frac{4}{\sqrt{\pi}},$$
$$C_\nabla(k) = 2^k \Big(\frac{(k+1)(k+2)}{2}\Big)^{1/2} \|\Phi\|_{H^{k+1}} \qquad (k \ge 0),$$

with the norms of $C^k(N')$ and H^k fixed in Section 2. Then for every $G \in [0, \overline{G}]$, every integer $k \ge 0$, every vector field $h \in C^k(N'; \mathbb{R}^2)$ and every $g \in L^2(N)$,

$$|\ell_*(G)(-h \cdot \nabla\Phi)| \le C_{\ell_*} \|h\|_{C^0(N')}, \qquad |p_3^{\mathrm{out}}[g]| \le C_{\mathrm{p}} \|g\|_{L^2(N)},$$
$$\|h \cdot \nabla\Phi\|_{H^k} \le C_\nabla(k) \|h\|_{C^k(N')}.$$

None of these constants depends on G; C_{ℓ_*} and C_{p} do not depend on k, while $C_\nabla(k)$ does.

Proof. (1), (2) By Proposition 4.21(b), $|\beta|$, $|p_*^{\mathrm{out}}|$, $\|\ell_*(G)\|_{(X^0)^*}$ and $\|\phi_*(G)\|_{H^k}$ are continuous on $[0, \overline{G}]$, and $|\beta|$, $|p_*^{\mathrm{out}}|$ are positive. Continuous functions on the compact interval $[0, \overline{G}]$ attain their extrema; this gives $c_N > 0$, $C_N < \infty$, $c_{\mathrm{out}} > 0$, $\|\ell_*\| < \infty$ and the bound on $\|\phi_*\|_{H^k}$.

(3) Since $\operatorname{supp}\Phi \subset \{a \le |y| \le 1\} \subset N$,

$$|\ell_*(G)(-h \cdot \nabla\Phi)| \le \|\ell_*\| \, \|h \cdot \nabla\Phi\|_{L^2(\mathbb{R}^2)} \le \|\ell_*\| \, \|\nabla\Phi\|_{L^2(\mathbb{R}^2)} \|h\|_{C^0(N')}.$$

Next, $|p_3^{\mathrm{out}}[g]| \le \frac{1}{2\pi}\|g\|_{L^2(N)} \||y|^3\|_{L^2(N)}$ and $\||y|^3\|_{L^2(N)}^2 = 2\pi \int_{a/2}^2 r^7 \mathrm{d}r \le 2\pi \cdot 2^8/8 = 64\pi$, so $|p_3^{\mathrm{out}}[g]| \le \frac{8\sqrt{\pi}}{2\pi}\|g\|_{L^2(N)} = \frac{4}{\sqrt{\pi}}\|g\|_{L^2(N)}$. Finally, for $|\alpha| \le k$, since $\nabla\Phi$ is supported in N' and $|\partial^\beta h| \le \|h\|_{C^k(N')}$ there for $|\beta| \le k$, the Leibniz rule and the Cauchy-Schwarz inequality in $\mathbb{R}^2$ give

$$\|\partial^\alpha(h \cdot \nabla\Phi)\|_{L^2(\mathbb{R}^2)} \le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \sup_{N'} |\partial^\beta h| \, \big\||\partial^{\alpha-\beta}\nabla\Phi|\big\|_{L^2(\mathbb{R}^2)} \le 2^{|\alpha|} \|h\|_{C^k(N')} \|\Phi\|_{H^{k+1}};$$

summing the squares over the $(k+1)(k+2)/2$ multi-indices gives the last inequality. □

Statements and Declarations

The author has no competing interests to declare.

References

[1] Albritton, D., Bruè, E., Colombo, M., De Lellis, C., Giri, V., Janisch, M., Kwon, H. *Instability and Non-uniqueness for the 2D Euler Equations, after M. Vishik*. Annals of Mathematics Studies. Princeton University Press. (2024). arXiv:2112.04943. We use the numbering of the book.

[2] Bernicot, F., Hmidi, T. *On the Global Well-Posedness for Euler Equations with Unbounded Vorticity*. Dyn. Partial Differ. Equ. Vol. 12. No. 2. pp. 127-155. (2015).

[3] Bernicot, F., Keraani, S. *On the Global Well-Posedness of the 2D Euler Equations for a Large Class of Yudovich Type Data*. Ann. Sci. Éc. Norm. Supér. (4). Vol. 47. No. 3. pp. 559-576. (2014).

[4] Bruè, E., Colombo, M. *Nonuniqueness of Solutions to the Euler Equations with Vorticity in a Lorentz Space*. Comm. Math. Phys. Vol. 403. No. 2. pp. 1171-1192. (2023).

[5] Bruè, E., Colombo, M., Kumar, A. *Flexibility of Two-Dimensional Euler Flows with Integrable Vorticity*. Duke Math. J. Vol. 175. No. 9. pp. 1593-1647. (2026). arXiv:2408.07934. We use the numbering of the arXiv version.

[6] Buck, M., Modena, S. *Non-Uniqueness and Energy Dissipation for 2D Euler Equations with Vorticity in Hardy Spaces*. J. Math. Fluid Mech. Vol. 26. No. 2. Paper No. 26. (2024).

[7] Chen, Q., Miao, C., Zheng, X. *The Two-Dimensional Euler Equation in Yudovich and bmo-Type Spaces*. Rev. Mat. Iberoam. Vol. 35. No. 1. pp. 195-240. (2019).

[8] Coiculescu, M.P. *Non-Uniqueness of Weak Solutions to the Unforced Two-Dimensional Euler Equations with Vorticity in $L^1 \cap L^p$, II: The Nonlinear Construction*. Companion paper.

[9] Crippa, G., Stefani, G. *An Elementary Proof of Existence and Uniqueness for the Euler Flow in Localized Yudovich Spaces*. Calc. Var. Partial Differential Equations. Vol. 63. No. 7. Paper No. 168. (2024).

[10] DiPerna, R.J., Lions, P.-L. *Ordinary Differential Equations, Transport Theory and Sobolev Spaces*. Invent. Math. Vol. 98. No. 3. pp. 511-547. (1989).

[11] DiPerna, R.J., Majda, A.J. *Concentrations in Regularizations for 2-D Incompressible Flow*. Comm. Pure Appl. Math. Vol. 40. No. 3. pp. 301-345. (1987).

[12] Elgindi, T.M., Murray, R.W., Said, A.R. *Well-Posedness and Singularity Formation beyond the Yudovich Class*. J. Eur. Math. Soc. Published online. (2026). DOI 10.4171/JEMS/1817. arXiv:2312.17610.

[13] Hölder, E. *Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit*. Math. Z. Vol. 37. No. 1. pp. 727-738. (1933).

[14] Kato, T. *On Classical Solutions of the Two-Dimensional Non-Stationary Euler Equation*. Arch. Rational Mech. Anal. Vol. 25. No. 3. pp. 188-200. (1967).

[15] Mengual, F. *Non-Uniqueness of Admissible Solutions for the 2D Euler Equation with L^p Vortex Data*. Comm. Math. Phys. Vol. 405. No. 9. Paper No. 207. (2024).

[16] Olver, F.W.J., Olde Daalhuis, A.B., Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V., Cohl, H.S., McClain, M.A. (eds.). *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.2.8 of 2026-09-15.

[17] Pettis, B.J. *On Integration in Vector Spaces*. Trans. Amer. Math. Soc. Vol. 44. No. 2. pp. 277-304. (1938).

[18] Taniuchi, Y. *Uniformly Local L^p Estimate for 2-D Vorticity Equation and Its Application to Euler Equations with Initial Vorticity in bmo*. Comm. Math. Phys. Vol. 248. No. 1. pp. 169-186. (2004).

[19] Vishik, M. *Incompressible Flows of an Ideal Fluid with Vorticity in Borderline Spaces of Besov Type*. Ann. Sci. École Norm. Sup. (4). Vol. 32. No. 6. pp. 769-812. (1999).

[20] Vishik, M. *Instability and Non-Uniqueness in the Cauchy Problem for the Euler Equations of an Ideal Incompressible Fluid. Part I*. arXiv:1805.09426. (2018).

[21] Vishik, M. *Instability and Non-Uniqueness in the Cauchy Problem for the Euler Equations of an Ideal Incompressible Fluid. Part II*. arXiv:1805.09440. (2018).

[22] Wolibner, W. *Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long*. Math. Z. Vol. 37. No. 1. pp. 698-726. (1933).

[23] Yudovich, V.I. *Non-Stationary Flow of an Ideal Incompressible Liquid*. USSR Comput. Math. Math. Phys. Vol. 3. No. 6. pp. 1407-1456. (1963).

[24] Yudovich, V.I. *Uniqueness Theorem for the Basic Nonstationary Problem in the Dynamics of an Ideal Incompressible Fluid*. Math. Res. Lett. Vol. 2. No. 1. pp. 27-38. (1995).