

PARTIAL REGULARITY, UNIQUENESS, AND
NON-UNIQUENESS FOR THE NAVIER-STOKES
EQUATIONS

Matei P. Coiculescu

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Abstract

In this dissertation, we prove several results about the incompressible Navier-Stokes and Euler equations.

First, we prove that there is a family of pseudodifferential equations, containing the hyperdissipative incompressible Navier-Stokes equations, that simultaneously admits a partial regularity theory and the possibility of singularity formation. In particular, our results strongly suggest that a proof of global regularity for the incompressible Navier-Stokes equations cannot rely solely on a development of partial regularity theory.

Second, we present a novel approach to the Liouville problem for the stationary Navier-Stokes equations. As an application of our method, we prove conditional Liouville theorems with assumptions on the antiderivative of the velocity that represent substantial improvements on what was heretofore known. In particular, under some conditions, the solution $u \equiv 0$ is the unique solution to the stationary Navier-Stokes equations.

Third, we construct initial data in the critical space BMO^{-1} from which there exist two distinct global solutions to the incompressible Navier-Stokes equations, both smooth for all $t > 0$. This result is the first example of non-uniqueness for the Navier-Stokes equations with data at the critical regularity.

The last chapter contains miscellaneous results on self-similar solutions to the incompressible Euler equations.

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Mistakes are an important and instructive part of mathematics, perhaps as important a part as proofs. Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works consist of proofs, just as poems consist of characters.

-Vladimir Igorevich Arnold

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Chapter 1

Introduction

1.1 The Navier-Stokes Equations

The principal object of our study is the incompressible Navier-Stokes equations in a three dimensional domain without boundary (i.e. $\mathbb{R}^3$ or $\mathbb{T}^3$). This mathematical object is a system of nonlinear partial differential equations that models the motion of an incompressible, viscous, Newtonian fluid. In particular, if $u(x, t)$ is the velocity field of a fluid in $\mathbb{R}^3 \times [0, T)$ dynamically changing in time t , with initial vector field u_0 , external force f and fluid pressure p , then the Navier-Stokes equations state that u satisfies:

$$\begin{aligned}\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f \\ \operatorname{div} u &= 0 \\ u(t = 0) &= u_0\end{aligned}\tag{1.1}$$

The parameter ν determines the viscosity of the fluid. Since its value has no influence on our work, we may always assume that the viscosity $\nu = 1$. We also assume that there is no external force, i.e. $f = 0$, and that the density of the fluid is everywhere equal to 1.

The Navier-Stokes equations are phenomenological, in contrast with the incompressible Euler

equations that model incompressible ideal fluids. The derivation of the Navier-Stokes equations relies on the first principles of the conservation of mass and the conservation of momentum as well as an assumption on the constitutive law of the viscous stresses in a fluid. The assumption placed on the viscous stress tensor is that it depends in a linear and isotropic way on the strain tensor (the symmetric part of ∇u , where u is the velocity of the fluid), and this assumption more or less leads to the presence and precise form of the “dissipation term” $\nu \Delta u$ in the equations.

For differential equations derived from physical principles that admit an initial value problem, one expects well-posedness in the sense of Hadamard (due perhaps to the apparently deterministic nature of macroscopic physics). In particular, given a reasonable initial condition, one expects the existence of a unique solution in-time of the differential equation. In addition, one expects some sort of continuity of the solution in-time with respect to the initial condition. Due to the phenomenological nature of the incompressible Navier-Stokes equations, the question of whether the equations are well-posed becomes fundamental to our understanding of fluid dynamics. Indeed, if the equations are *not* well-posed, then the assumption made in their derivation can be put into question: the assumed model may not function in all physically reasonable scenarios. For example, the motion of turbulent fluids (or the onset of turbulence) may not be adequately described (or predicted) by the equations.

A fluid occupying some portion of $\mathbb{R}^3$ with smooth, compactly supported velocity $u(x, t)$ (i.e. the fluid is at rest far from the origin and smooth in both space and time variables) is in addition a continuous medium that has total kinetic energy

$$\int_{\mathbb{R}^3} |u|^2 dx.$$

The total kinetic energy of a fluid happens to be, for the incompressible Navier-Stokes equations without force, one of only two quantities known to have an a priori estimate. Indeed, if we test Equation (1.1) with u itself and integrate by parts using the divergence free condition, we obtain:

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 dx + 2 \int_{\mathbb{R}^3} |\nabla u|^2 dx = 0.$$

If we further integrate in time, we get

$$\int_{\mathbb{R}^3} |u(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, t)|^2 dx dt = \int_{\mathbb{R}^3} |u(x, 0)|^2 dx. \quad (1.2)$$

Therefore, the kinetic energy and the *enstrophy* $\iint |\nabla u|^2 dx dt$ are formally bounded for all time by the kinetic energy of the initial condition.

Jean Leray, in his seminal work [64], introduced a notion of weak solution to the Navier-Stokes equations. His notion of weak solution is essentially based on the previous observation that the kinetic energy satisfies an a priori estimate under the evolution of the equations. Leray then proved the global-in-time existence of weak solutions that satisfy the global energy inequality (i.e. Equation (1.2) with “=” replaced with “ $\leq$ ”) and strong convergence in L^2 to the initial data. In fact, Leray’s notion of solution is stronger than what is necessary for the previously mentioned properties and is more akin to the notion of suitable weak solution introduced in [16], which we shall discuss shortly. After Leray, Serrin proved two striking facts in [84]. First, he proved that if a weak solution u on $\mathbb{R}^3 \times (0, T)$ is in $L_t^s L_x^q$ for $2/s + 3/q < 1$ and $q > 3$, then $u \in C_{t,x}^\infty(\mathbb{R}^3 \times (0, T])$. Second, he proved that if a weak solution is in $L_t^s L_x^q(B_1(0) \times (-1, 0))$ for $2/s + 3/q < 1$ and $q > 3$ then the weak solution is in fact in $L_{t,x}^\infty(B_{1/2}(0) \times (-1/4, 0))$. The first statement is a global regularity result that provides a sufficient condition for the existence of a global-in-time smooth solution. The second result is a local regularity result that has as a consequence that if singularity formation occurs for the Navier-Stokes equations, then the velocity field becomes unbounded in the L^∞ norm. The latter statement means that singularity formation for the Navier-Stokes equations is inherently unphysical (the speed of light is one such bound on the velocity for real fluids). We also remark that the critical case (when $2/s + 3/q = 1$) of the results described above was proved independently by Sohr in [85] and Giga in [48].

The importance of the question of well-posedness for the Navier-Stokes equations is also highlighted by its proposal as one of the Clay Problems [42]. For various well-justified technical reasons, the Clay problem formulates the initial value problem with physically reasonable, Schwartz initial data, for which local-in-time existence of a smooth solution is known. It is a striking fact, consid-

ering that they have been studied for several hundred years, that the questions of global existence and uniqueness questions for the three-dimensional incompressible Navier-Stokes equations are open problems.

In this dissertation, we contribute some results to our knowledge of the Navier-Stokes equations. First, we prove that there is a family of pseudodifferential equations, containing the hyperdissipative incompressible Navier-Stokes equations, that simultaneously admits a partial regularity theory and the possible of singularity formation. In particular, our results strongly suggest that a proof of global regularity for the incompressible Navier-Stokes equations cannot simply rely on a development of the partial regularity theory. Second, we present a novel approach to the Liouville problem for the stationary Navier-Stokes equations. As an application of our method, we prove conditional Liouville theorems with assumptions on the antiderivative of the velocity that represent substantial improvements on what was heretofore known. Third, we construct initial data in the critical space BMO^{-1} from which there exist two distinct global solutions to the incompressible Navier-Stokes equations, both smooth for all $t > 0$. One consequence of this construction is the sharpness of the small data global well-posedness result of Koch and Tataru.

We also include, in the final chapter of this dissertation, some miscellaneous results on self-similar solutions of the incompressible Euler equations. These results complement our work in [9] on the stability of the inviscid power-law vortex. Our work in the first section of this chapter shows that the linearization around any homogeneous steady state of the Euler equations cannot have an unstable eigenvalue if the operator also generates a semigroup. Our work in the second section proves that bounded self-similar solutions of the two-dimensional Euler equations are necessarily equal to zero (and a bit more).

Before delving into the proofs, we introduce the context behind each problem regarding the Navier-Stokes equations in the following sections.

1.2 Partial Regularity and Blowup

This chapter presents our work in [28].

We may consider the Navier-Stokes equations with “fractional dissipation”. In particular let $\alpha \geq 0$ and consider the pseudodifferential operator $(-\Delta)^\alpha$ whose Fourier symbol is $|\xi|^{2\alpha}$. Whenever α equals a positive integer k , the operator coincides, up to a sign, with the composition of the classical Laplacian $(-\Delta)$ with itself k times. The unforced α -dissipative Navier-Stokes system in $\mathbb{R}^3$ is the following system of pseudodifferential equations:

$$\begin{aligned}\partial_t u + (-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \\ u(t=0) &= u_0\end{aligned}\tag{1.3}$$

The α -dissipative Navier-Stokes system is typically called hyperdissipative if $\alpha > 1$ and hypodissipative if $\alpha < 1$. The case when $\alpha = 1$ coincides with the classical Navier-Stokes equations. For $\alpha \geq 5/4$ and for any smooth function u_0 which decays sufficiently fast at infinity, it is known that the system in Equation (1.3) has a classical global-in-time solution. In the case of Equation (1.3) on $\mathbb{R}^n$, classical global-in-time solutions are guaranteed whenever $\alpha > (n+2)/4$ and the initial data is smooth and has sufficient decay. We shall offer a proof of the latter fact. Global regularity in $\mathbb{R}^3$ is also known for an open set of initial data if α is slightly less than $5/4$, see [33]. Generally, when $\alpha < 5/4$, the global existence of classical solutions is still a major open question [42].

One approach towards proving global regularity is to show that the set where singularities of the vector field develop is small in some sense. In particular, one might want to find a bound on the Hausdorff dimension of such a set. Finding such bounds is within the scope of partial regularity theory. We now provide a short summary of the partial regularity theory for the classical Navier-Stokes equations.

Leray showed in [63] that the set of singular times has Lebesgue measure equal to zero. Without too much additional effort, one can improve this to the statement that the $1/2$ -dimensional Hausdorff measure of the set of singular times is equal to zero (cf. [92]). Scheffer wrote several papers, beginning with [76], bounding the Hausdorff dimension of the singular set in space at the time of first blowup. The best known partial regularity result was obtained by Caffarelli, Kohn, and

Nirenberg in [16] for a particular class of weak solutions called suitable weak solutions. Let u be a suitable weak solution on any open subset of spacetime $\mathbb{R}^3 \times \mathbb{R}$. If we define the singular set (in space-time) as

$$S = \{(x, t) : u \notin L^\infty \text{ in any neighborhood of } (x, t)\}$$

then the 1-dimensional Hausdorff measure of S is equal to zero. Away from the set S , the vector field u is bounded, which implies that u is smooth in the spatial variable by the Serrin higher regularity theorem, see [84] for more details. The bound on the Hausdorff dimension obtained by Caffarelli, Kohn, and Nirenberg in 1982 has not been improved.

The work of Colombo, De Lellis, and Massaccesi in [32] provides the optimal extension of the Caffarelli-Kohn-Nirenberg theory, dealing definitively with the hyperdissipative case. Tang and Yu in [87] and the erratum [88] were the first to analyze the hypodissipative case $3/4 < \alpha < 1$. Kwon and Ożański in [60] further sharpened the work done in [87]. Additionally, the only partial regularity results that we know of for non-stationary higher-dimensional Navier-Stokes equations are found in [40], where Dong and Gu prove a result analogous to the one in [16] in four dimensions, and in the paper [98] of Wu. However, we cannot see a way to transfer these techniques to the setting of our interest.

Our present work is inspired by the paper [57] of Katz and Pavlovic. The authors of [57] prove a partial regularity theorem like Scheffer's in [76] by using Fourier analysis to reduce the problem to the study of an infinite-dimensional system of ordinary differential equations. The fluid vector field is decomposed into wavelet energy packets by localizing in both physical and frequency space in a way compatible with the Heisenberg Uncertainty Principle, and the dynamics of the wavelet coefficients can be considered a system of ordinary differential equations. One of the advantages of their approach when compared to the work of Scheffer or Caffarelli, Kohn, and Nirenberg is that the local energy inequality is unnecessary for the proof. Ożański's work in [73], which clarifies and extends the Fourier analytic method of Katz and Pavlovic, gets a partial regularity result for Leray-Hopf weak solutions of the hyperdissipative Navier-Stokes equations in $\mathbb{R}^3$ as well as a bound on the box-counting dimension of the singular set.

We now briefly discuss some work done in the “negative” direction of the global regularity problem. Besides numerical studies like [51] that suggest the development of a finite-time singularity for the Navier-Stokes equations, the paper [89] by Tao provides evidence that at the very least, attacking the problem of global regularity using abstract methods depending on basic L^p bounds for the nonlinearity is a direction certain to fail. Let $\mathbb{P}$ denote the Leray projection on divergence-free vector fields (see Section 1.6 in [92]). Tao constructs a nonlinearity $C(u, v)$ that is an averaged version of the Euler bilinear operator

$$\mathcal{E}(u, v) = \frac{1}{2} \mathbb{P} \left((u \cdot \nabla) v + (v \cdot \nabla) u \right),$$

satisfies the cancellation identity and various “harmonic-type” estimates such as

$$\|C(u, v)\|_p \leq K (\|u\|_q \|\nabla v\|_r + \|v\|_q \|\nabla u\|_r)$$

where $1/p = 1/q + 1/r$ and $1 < p, q, r < \infty$, and whose associated partial differential equation admits a finite-time singularity. Here and in the sequel, that an operator C is an “averaged version of $\mathcal{E}$ ” means that:

$$\begin{aligned} \langle C(u, v), w \rangle &= \\ &= \int_{\Omega} \langle \mathcal{E}(m_{1,\omega} \text{Rot}_{R_{1,\omega}} \text{Dil}_{\lambda_{1,\omega}} u, m_{2,\omega} \text{Rot}_{R_{2,\omega}} \text{Dil}_{\lambda_{2,\omega}} v), m_{3,\omega} \text{Rot}_{R_{3,\omega}} \text{Dil}_{\lambda_{3,\omega}} w \rangle d\mu(\omega) \end{aligned} \quad (1.4)$$

for all Schwartz vector fields u, v, w . Here, ω is a variable in the probability space (Ω, μ) , and the maps $R_{i,\omega} : \Omega \rightarrow SO(3)$, $\lambda_{i,\omega} : \Omega \rightarrow [C^{-1}, C]$, and $m_{i,\omega} : \Omega \rightarrow \mathfrak{M}_0$ are measurable maps. We use the notation $\mathfrak{M}_0$ for the space of all Fourier multipliers m of order 0 with Schwartz symbols equipped with the topology (hence also the σ -algebra) generated by the Schwartz seminorms. The operators Dil and Rot are respectively the dilation and rotation operator in $\mathbb{R}^3$. See [89] for more details. Similarly, Scheffer proves in [77] that suitable weak solutions of the Navier-Stokes inequality admit interior singularities. Later in [78], Scheffer shows that singularity formation can occur on a fractal set of any dimension less than one, thereby proving that the method used in [16] to achieve partial

regularity is sharp.

Our main result in this chapter suggests that improving partial regularity results for the Navier-Stokes equations is not a path to answering the global regularity question, unless one uses more specific structural properties of the Navier-Stokes nonlinearity. We denote the parabolic cylinder of radius r at (x, t) by:

$$\mathcal{P}_r(x, t) := \{(y, s) : |y - x| < r \text{ and } t - r^2 < s < t\}$$

Here and in the sequel we denote the closed singular set at blowup time T by

$$S_T := \{x \in \mathbb{R}^n : \forall r > 0, u \notin L_t^\infty C_x^\infty(\mathcal{P}_r(x, T))\}.$$

Our main theorem provides an upper bound on the Hausdorff dimension of S_T for blowup solutions of an averaged three-dimensional α -dissipative Navier-Stokes Equation. In particular, we prove:

Theorem 1.2.1. *Let $1 < \alpha < 5/4$ be arbitrary. There exists a symmetric bilinear operator C that is an averaged version of $\mathcal{E}$, that obeys the cancellation identity*

$$\langle C(u, u), u \rangle = 0$$

for all $u \in H^{10}(\mathbb{R}^3)$ that are divergence-free in the distributional sense, and such that, for some Schwartz divergence-free initial vector field u_0 , there is no global-in-time solution to

$$\partial_t u + (-\Delta)^\alpha u + C(u, u) + \nabla p = 0$$

$$u(t = 0) = u_0 \quad \text{div } u = 0.$$

Moreover, if T is the time of first blowup, then the closed set S_T has Hausdorff dimension at most $5 - 4\alpha$.

The partial regularity portion of Theorem 1.2.1 follows from a more general partial regularity theorem for a family of pseudodifferential equations of “Navier-Stokes type”. Then, proving The-

orem 1.2.1 amounts to finding an equation in the family admitting a finite-time singularity. First we define a certain class $\mathfrak{B}$ of bilinear operators.

Definition 1.2.2. *We call a symmetric bilinear operator B **amenable** if and only if:*

1. B is defined for Schwartz vector fields
2. B satisfies the Cancellation Identity:

$$\langle \mathbb{P}B(u, u), u \rangle = 0 \quad \text{for divergence-free } u \quad (1.5)$$

3. The linear maps $B_v^1(u) := \mathbb{P}B(u, v)$, $B_v^2(u) := \mathbb{P}B(v, u)$ are pseudodifferential operators in the class $OPS_{1,1}^m$ for some real number m (we define this class in the last section of Chapter 2)
4. For all $0 < \gamma < 3$ and for all $1 \leq p_1, p_2, r \leq \infty$ satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{\gamma}{3} = \frac{1}{r}$$

we have for some constant K depending only on γ , p_1 , p_2 , and r :

$$\|\mathbb{P}B(u, v)\|_{L^r} \leq K \|u\|_{L^{p_1}} (\|v\|_{L^{p_2}} + \|\nabla v\|_{L^{p_2}}) \quad (1.6)$$

for all $u \in L^{p_1}$ and $v \in W^{1,p_2}$ with support of diameter at most 2^{100} (note that the endpoint cases of integrability are included).

5. *Scaling:* There exists a number $\lambda > 1$ such that for any $k \in \mathbb{Z}$:

$$B(u(\lambda^k y), u(\lambda^k y))(x) = \lambda^k B(u(y), u(y))(\lambda^k x)$$

The class $\mathfrak{B}$ is the collection of all amenable operators. Throughout the rest of our work, we may alternately say that $B \in \mathfrak{B}$ or that B is amenable. Our partial regularity theorem is:

Theorem 1.2.3. *Let $n \geq 3$ and let $(n+1)/4 < \alpha < (n+2)/4$. Let B be an amenable bilinear operator, and let u_0 be a Schwartz vector field. Suppose $u(t)$ is a smooth vector field on $\mathbb{R}^n \times [0, T) \rightarrow \mathbb{R}^n$ and suppose there exists a smooth function p on $\mathbb{R}^n \times [0, T)$ such that:*

$$\partial_t u + (-\Delta)^\alpha u + B(u, u) + \nabla p = 0$$

$$\operatorname{div} u = 0$$

$$u(t=0) = u_0$$

If T is the time of first blowup, then the closed set S_T has Hausdorff dimension at most $n+2-4\alpha$.

One corollary of Theorem 1.2.3 is a partial regularity result for the α -dissipative Navier-Stokes equations for any dimension $n \geq 3$ whenever $\alpha > (n+1)/4$, which, since it is an improvement on what is already known, we state separately:

Corollary 1.2.4. *Let $n \geq 3$ and let $(n+1)/4 < \alpha < (n+2)/4$. Let u_0 be a Schwartz vector field. Suppose $u(t)$ is a smooth vector field on $\mathbb{R}^n \times [0, T) \rightarrow \mathbb{R}^n$ and suppose there exists a smooth function p on $\mathbb{R}^n \times [0, T)$ such that:*

$$\partial_t u + (-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla p = 0$$

$$\operatorname{div} u = 0$$

$$u(t=0) = u_0$$

If T is the time of first blowup, then the closed set S_T has Hausdorff dimension at most $n+2-4\alpha$.

Previously, there was no known partial regularity result for the non-stationary α -dissipative Navier-Stokes equations in $\mathbb{R}^n$ with spatial dimension $n \geq 5$. By performing a highly technical analysis of pseudodifferential operators, we get a partial regularity result that also holds for a large family of pseudodifferential equations in addition to the fractionally dissipative Navier-Stokes equations. In particular, this allows us to apply our partial regularity theorem to a pseudodifferential equation admitting a solution blowing up in finite time. Thus, while the conditions placed on an amenable operator suffice to prove a partial regularity result for its associated pseudodifferential

equation, they are insufficient to prove regularity for all time. We note that while previous results prove partial regularity of the solution in space-time, our result only concludes that the solution is “partially regular” in space at the blowup time. Thus, although our technique does not require the local energy inequality (and this is one reason why we can generalize to equations arising from amenable bilinear operators) our theorem has a weaker conclusion and does not prove a space-time partial regularity result.

This chapter is organized as follows. In the first section we prove Theorem 1.2.3. In particular, given the vector field solution u , we construct a set E of Hausdorff dimension $n + 2 - 4\alpha$ and prove that it contains the singular set S_T . The proof necessitates the use of the “barrier” construction also found in [57] and [73], which is the main obstacle to applying our method to the case of classical dissipation. The energy estimates used in the first section are proven in the subsequent section, where we test the pseudodifferential equation against “wavelet” localizations of the solution (localizations in both frequency and physical space). Then, we show that Tao’s method is flexible enough to construct a blowup solution to the pseudodifferential equation associated to some amenable bilinear operator, which, together with Theorem 1.2.3, proves Theorem 1.2.1. In the last section, we prove some technical lemmas that are used throughout the chapter.

1.3 Uniqueness

This chapter presents our work in [30], which is joint work with Jincheng Yang.

The Navier-Stokes equations are a model for the motion of viscous incompressible fluid. The viscosity of the fluid causes an energy dissipation that is typical of parabolic partial differential equations (like the heat equation). In this sense, we may consider the Navier–Stokes equations as dissipating the energy of a fluid until it approaches a limiting end state, which we expect should be a stationary solution of the equations. Our quotidian observation of fluids further suggests that the end-state should be a fluid at rest, i.e. with Eulerian velocity $u \equiv 0$. Rigorous mathematical proof of this expectation has so far been elusive, but work on the stationary Navier–Stokes equations remains an important part of the field of mathematical fluid mechanics. The present chapter thus

concerns the stationary Navier–Stokes equations in $\mathbb{R}^3$:

$$-\Delta u + u \cdot \nabla u + \nabla p = 0, \quad (1.7)$$

$$\operatorname{div} u = 0, \quad (1.8)$$

where u denotes the velocity field of the fluid and p denotes the scalar pressure field of the fluid. Weak solutions with $u \in L^3_{loc}(\mathbb{R}^3)$ of Equation (1.7) are smooth, see Theorem X.1.1 in Galdi's monograph [44]. For this reason, we may always assume that u and p are smooth functions on $\mathbb{R}^3$. We shall also assume that u satisfies the natural vanishing condition at infinity:

$$\lim_{|x| \rightarrow +\infty} u(x) = 0. \quad (1.9)$$

The question remains open, however, whether there exist any nontrivial solutions (i.e. solutions besides $u \equiv 0$). In particular, do there exist nontrivial solutions of Equation (1.7) with finite Dirichlet energy, i.e. with

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx < \infty? \quad (1.10)$$

Solutions with finite Dirichlet energy and satisfying the vanishing condition Equation (1.9) are often called D -solutions, and the question whether nontrivial D -solutions exist is called the Liouville problem for the three-dimensional Navier–Stokes equations. We note here that for the stationary Navier–Stokes equations in $\mathbb{R}^n$ with $n \neq 3$, there are no nontrivial D -solutions.

We now survey some of the progress done towards resolving this question. Observe that the vanishing condition in Equation (1.9), the assumption of finite Dirichlet energy as in Equation (1.10), and the Sobolev embedding in $\mathbb{R}^3$ together imply that a D -solution u lies in $L^6(\mathbb{R}^3)$. Galdi proves in [44] that if one assumes more decay at infinity, in particular if one assumes that $u \in L^{9/2}(\mathbb{R}^3)$, then u is trivial. Chae and Wolf prove a logarithmically improved version of the $L^{9/2}(\mathbb{R}^3)$ criterion in [18].

Another direction for resolving the Liouville problem seems to have begun with the work of

Seregin, Silvestre, Šverák, and Zlatos in [82], where the authors consider linear elliptic and parabolic systems with a divergence-free drift term. In particular, they prove a Liouville theorem for the Laplace equation with drift when the drift term is divergence-free and in the scale-invariant space BMO^{-1} . We recall that a divergence-free drift vector field b_i can be written as the divergence of some tensor T_{ij} , and the assumption that $T \in BMO$ implies $b \in BMO^{-1}$. The authors of [82] suggest that proving deep properties, like a Liouville theorem for an elliptic system, requires scale-invariant information: one should not be able to break the scaling. The assumption that $u \in BMO^{-1}$ lies exactly in this category of hypotheses for the stationary Navier–Stokes system as well. Indeed, Seregin proves in [80] that if a solution u of Equation (1.7) is in $L^6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)$, then u is trivial (we note that the finite Dirichlet energy assumption is not directly used in Seregin’s proof).

Seregin proves the first Liouville-type result with a quantitative version of the BMO^{-1} condition in [81]. We denote balls of radius R centered at the origin as $B_R(0)$ or also as B_R when there is no confusion. Suppose $u = \operatorname{curl} \psi$ is a smooth solution of Equation (1.7). Seregin proves in [81] that with the assumption

$$\left(\int_{B_R} |\psi - (\psi)_{B_R}|^s dx \right)^{\frac{1}{s}} \leq CR^{\alpha(s)},$$

for some $s > 3$ and constant $C > 0$ with $\alpha(s) = \frac{s-3}{6(s-1)}$, the smooth solution u of Equation (1.7) is trivial. In their article [19], Chae and Wolf prove that if $u = \operatorname{div} T$ for some tensor T with

$$\left(\int_{B_R} |T - (T)_{B_R}|^s dx \right)^{\frac{1}{s}} \leq CR^{\alpha(s)}$$

for some $s > 3$ and constant $C > 0$ with $\alpha(s) = \min\{\frac{1}{3} - \frac{1}{s}, \frac{1}{6}\}$, then u is trivial. The recent work [6] of Bang and Yang provides a logarithmic improvement of the Chae–Wolf result. Lastly, we recommend Section 6 in the survey article [83] as another short summary of what has been done in this direction for the Liouville problem. The references in [83] are exhaustive up to the year of its publication.

The goal of the present work is to present a novel approach to the Liouville problem of the

stationary Navier–Stokes equations. The main difference to our approach, in comparison with previous methods of attack, is that we perform local estimates on domains called *capsules* that we allow to vary in size from point to point, depending on the local Dirichlet energy and local average velocity. The main point is that if the velocity does not *stretch* too much, we can prove a Liouville theorem. The best elucidation of our new approach will be in proving the following theorems:

Theorem 1.3.1. *Suppose $u \in C^\infty(\mathbb{R}^3)$ is a D -solution of the stationary Navier–Stokes equations. Let $s \geq 1$ be arbitrary. If $u = \operatorname{curl} \psi$ and satisfies*

$$\left(\int_{B_R(x_0)} |\psi - (\psi)_{B_R(x_0)}|^s dx \right)^{\frac{1}{s}} \leq CR^\alpha, \quad \forall R > 1, \forall x_0 \in \mathbb{R}^3 \quad (1.11)$$

for some constants $C > 0$ and $0 < \alpha < 1$, then $u \in L^{p,\infty}$ for any $p > \frac{4}{1-\alpha}$. Moreover, if $\alpha < \frac{1}{9}$, then $u \equiv 0$.

Theorem 1.3.2. *Suppose $u \in C^\infty(\mathbb{R}^3)$ is a D -solution of the stationary Navier–Stokes equations. If u satisfies:*

$$\int_{x_0}^x u \cdot d\ell \leq C|x - x_0|^\beta, \quad \forall x_0, x \in \mathbb{R}^3, \quad (1.12)$$

for some constants $C > 0$ and $0 < \beta < 1$, then $u \in L^{p,\infty}$ for any $p > \frac{4-34\beta/29}{1-\beta}$. Moreover, if $\beta < \frac{29}{193}$, then $u \equiv 0$.

Our Theorem 1.3.1 improves the result of Seregin by allowing L^s control on the mean oscillation for $1 \leq s \leq 3$ as well. In addition, our hypothesis is weaker than Seregin's for $3 < s < 7$. Likewise, our Theorem 1.3.2 improves the Chae–Wolf result: our hypothesis is weaker than their hypothesis when $3 < s < 9/2$. We also remark that our threshold $\alpha = 1/9$ is uniform across the exponents.

Our Theorem 1.3.2 shows that $u \equiv 0$ when we control the line integral of u . The integral $\int_{x_0}^x$ means integration on a straight line segment connecting x_0 and x . For $\beta < 1$, the assumption in Equation (1.12) holds automatically if $|x - x_0|$ is small since u is bounded. Therefore, Equation (1.12) is also a type of growth condition.

We remark that the hypotheses of our results encode quantitative information about the antiderivative of solutions of the stationary Navier–Stokes equations, in line with the previous works outlined above. Indeed, if u is a potential flow, then the left-hand side of (1.12) defines its potential function $\phi(x)$ with $u = \nabla\phi$. However, in this scenario the triviality of u follows directly from the Liouville theorem for harmonic functions: there are no non-constant entire harmonic functions with sublinear growth. The stream function ψ can be defined by a similar line integral of (1.12) with $-u^\perp$ replacing u for 2D incompressible flows. In this way, the stream function can also be understood as a type of antiderivative.

The rest of this chapter lies in setting up our *capsule* approach and applying it to prove our theorems. We also prove a more general conditional Liouville theorem with a hypothesis on the capsules alone, quantifying the *stretching* condition mentioned before.

In the first section of this chapter, we prove local estimates for the Navier–Stokes system. In the following section, we use the *capsules* to get global control on the shearing of the solution of Equation (1.7). Then we prove some conditional Liouville theorems as consequences of the capsule method. We perform some technical estimates on the kernel of the Poisson equation with drift in the final section. A short remark on notation: an inequality denoted with $\lesssim$ will always mean an inequality up to an absolute computable constant. For scalars or vectors f and g , $f \approx g$ means $|f - g| \lesssim \min\{|f|, |g|\}$. Lastly, we denote the positive part of a real number α by $\alpha_+ := \max(0, \alpha)$.

1.4 Non-Uniqueness

This chapter presents our work in [29], which is joint work with Stan Palasek.

Recall the initial value problem for the unforced incompressible Navier–Stokes equations on $\mathbb{T}^3$ or $\mathbb{R}^3$:

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \\ u(0, x) &= U^0(x) \end{aligned} \tag{1.13}$$

where the given initial data $U^0(x)$ is divergence-free, and $u(x, t)$ and $p(x, t)$ are the unknown

velocity and pressure fields. Initiated in 1934 by the pioneering work of Leray [64], there is now a well-developed theory of local well-posedness of (1.13) from initial data in many reasonable classes of functions, as well as global well-posedness from data that is sufficiently small in certain scale-invariant norms; see, for instance, [61] for an extensive survey. The strongest result in this direction is the following well-known theorem of Koch and Tataru [59]:

Theorem 1.4.1. *There exists $\epsilon > 0$ such that if U^0 is divergence-free with $\|u\|_{BMO^{-1}} < \epsilon$, then there exists a unique global-in-time solution that is regular for $t > 0$.*

To this point, it was unknown whether any of the positive results for small data are sharp in the sense that global well-posedness can fail for large initial data. This question was asked, for instance, by Koch and Tataru in [59] in relation to Theorem 1.4.1. We answer their question in the affirmative:

Theorem 1.4.2. *There exists divergence-free initial data $U^0 \in BMO^{-1}$ such that the Cauchy problem (1.13) admits two distinct global solutions*

$$u^{(1)}, u^{(2)} \in C_{t,x}^\infty((0, \infty) \times \mathbb{T}^3) \cap L^\infty([0, \infty); BMO^{-1}(\mathbb{T}^3)) \cap C^0([0, \infty); \dot{W}^{-1,p}(\mathbb{T}^3))$$

for all $p < \infty$.

Remark 1.4.1. *The solutions constructed in the proof of Theorem 1.4.2 belong precisely to the same class as those constructed by Koch and Tataru; the only difference is that they are not perturbative around zero. Indeed, one can verify that our solutions obey the following bounds:*

$$\|u\|_{X_{KT}} := \sup_{t>0} t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{T}^3)} + \sup_{x_0 \in \mathbb{T}^3} \sup_{R>0} \left(R^{-3} \int_0^{R^2} \int_{B(R)} |u(x,t)|^2 dx dt \right)^{1/2} < \infty.$$

In other words, they lie in the natural path space corresponding to solutions of (1.13) (or even the heat equation) with initial values in BMO^{-1} . It was shown in [59] that the bilinear operator appearing in the mild form of the Navier–Stokes equations is bounded on X_{KT} , which allows one to use Kato’s method to construct global solutions from small data.

Remark 1.4.2. *The BMO^{-1} norm is critical with respect the Navier–Stokes scaling symmetry. In other words, if $u(t, x)$ is a global solution of (1.13) with initial data U^0 , then*

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad U_\lambda^0(x) = \lambda U^0(\lambda x) \quad (1.14)$$

is also a solution for any $\lambda > 0$, and furthermore

$$\|u_\lambda\|_{X_{KT}} = \|u\|_{X_{KT}}, \quad \|U_\lambda^0\|_{BMO^{-1}} = \|U^0\|_{BMO^{-1}}.$$

Critical spaces are a natural setting for the questions we consider because they are at the threshold of the well-posedness theory; there are many theorems to the effect that certain classes of critical solutions are well-behaved [41, 45, 7, 8, 20, 3, 69, 68, 67, 62, 24, 50]. To our knowledge, Theorem 1.4.2 is the first ill-posedness result in a critical space strong enough to support well-posedness for small data. Moreover, it appears to be the first example of non-uniqueness for the Navier–Stokes equations from data in any critical space that does not rely on the presence of an external force. See §1.4.2 for a comparison of our solutions to those constructed previously in the literature.

Remark 1.4.3. *Let us emphasize that the solutions constructed in Theorem 1.4.2 are not in the Leray–Hopf class due to the fact that the initial velocity is not in $L^2(\mathbb{T}^3)$. The construction here uses a simplified version of the mechanism proposed by Palasek in [74] and this simplification requires the data to have infinite energy. If the full mechanism from [74] could be implemented, we expect it would yield solutions of (1.13) in a similar class as those of Jia and Šverák [54, 55] (but without exact self-similarity), namely Leray–Hopf solutions belonging to improved critical spaces at least as good as $B_{p,\infty}^{-1+3/p}$ for $p > 3$. There does not appear to be any obstruction in principle to realizing the mechanism proposed in [74], but there exists a substantial technical challenge in finding the correct building blocks that embed the behavior of the Obukhov dyadic model (cf. the discussion in Section 1.3 of [89]).*

Remark 1.4.4. *The initial data U^0 is smooth outside of a measure zero set $\Sigma \subset \mathbb{T}^3$. Indeed, it can be written as $U^0 = \sum_{k=1}^{\infty} V_k^0$ where each component V_k^0 is smooth on $\mathbb{T}^3$ and the supports are*

monotone, $\text{supp } V_k^0 \subset \text{supp } V_{k-1}^0$. Thus $\Sigma \subset \bigcap_{k \geq 0} \tilde{\Omega}_k$, with $\tilde{\Omega}_k$ as in Definition 4.2.1. The claim then follows because $|\tilde{\Omega}_k| \leq 2^{-k}$ from Lemma 4.2.3.

Remark 1.4.5. For convenience, we carry out the construction on the torus $\mathbb{T}^3 = \mathbb{R}^3 / (2\pi\mathbb{Z})^3$. The argument should extend with straightforward modifications to the Navier–Stokes equations posed on $\mathbb{R}^3$ with decay at spatial infinity. In principle, our methods extend to the two-dimensional case as well (which is likewise covered by the small data well-posedness results of [59], see also [46]), but there are technical difficulties similar to those faced in the approach to Onsager’s conjecture due to the geometry of Mikado flows. We leave open the problem of non-unique solutions with critical data on $\mathbb{T}^2$ and $\mathbb{R}^2$, but we point out the preprint of Albritton et al. in [2] as an example of recent work relevant to the non-uniqueness question in the two-dimensional case.

We also note that Cheskidov, Zeng, and Zhang posted a preprint [27] in which they use convex integration to construct wild weak solutions of the Navier–Stokes and MHD from *any* $H^{\frac{1}{2}}$ initial data. This is an interesting contribution that implies the lack of weak-strong uniqueness of dissipative solutions. We wish to emphasize that the solutions produced in [27] do not overlap with Theorem 1.4.2 at all; while they have the advantage that the data is in $L^2(\mathbb{R}^3)$, they do not appear to possess any further regularity for $t > 0$, nor does the energy inequality hold. The present chapter is concerned with the question of well-posedness of mild solutions, which is orthogonal to the setting treated in [27]. This can be seen from the fact that they construct distinct solutions even for *small* data in $H^{\frac{1}{2}}(\mathbb{R}^3)$.

1.4.1 Well-posedness and Ill-posedness at the Critical Regularity

We recall the scale of Besov spaces $B_{p,q}^s$ equipped with the norms $\|u\|_{B_{p,q}^s} = \|N^s \|P_N u\|_{L^p}\|_{\ell_N^q}$ where P_N are the Littlewood–Paley projections defined in §4.1.2. The spaces $B_{p,q}^{-1+3/p}$ are critical for the three-dimensional Navier–Stokes equations for all $p, q \in [1, \infty]$, although they primarily play a role in the negative regularity regime $p > 3$. They are related to the other well-known critical spaces as

follows:

$$H^{\frac{1}{2}} \subset L^3 \subset B_{p,\infty}^{-1+3/p} \subset BMO^{-1} \subset B_{\infty,\infty}^{-1}$$

where $3 < p < \infty$. Global well-posedness from small data in the first three spaces was proved by Fujita–Kato [43], Kato [56], Cannone [17], and Planchon [75], at which point the belief was that the natural endpoint of the hierarchy is the space $B_{\infty,\infty}^{-1}$ which is so large it contains *all* translation-invariant spaces of tempered distributions at the critical regularity. The insight of Koch and Tataru [59] was that BMO^{-1} , defined as the distributions with

$$\|u\|_{BMO^{-1}} := \sup_{R>0} \sup_{x_0 \in \mathbb{R}^3} \left(\int_0^{R^2} \int_{B(x_0,R)} |e^{t\Delta} u|^2 dx dt \right)^{\frac{1}{2}} < \infty,$$

falls in between $B_{p,\infty}^{-1+3/p}$ and $B_{\infty,\infty}^{-1}$ and is such that the Navier–Stokes bilinear operator

$$u \mapsto - \int_0^t e^{(t-t')\Delta} \mathbb{P} \operatorname{div} u(t') \otimes u(t') dt'$$

is bounded, allowing one to implement Kato’s method for constructing mild solutions from small data.

One can argue that BMO^{-1} is the largest reasonable critical space with this property. Indeed, to make sense of the nonlinearity, even in the sense of distributions, u must be L_{loc}^2 in spacetime. This must be true in particular for the zeroth Picard iterate $e^{t\Delta} u_0$, so we need $\|e^{t\Delta} u_0\|_{L_{t,x}^2([0,1] \times B(1))} = C < \infty$. Due to the symmetries of (1.13), this bound must be invariant under spatial translation as well as the scaling defined in (1.14). Thus, one is led precisely to the above definition of BMO^{-1} , corresponding to the Carleson formulation of BMO .

In critical spaces weaker than BMO^{-1} in which Theorem 1.4.1 does not apply, there has been progress in showing ill-posedness in the form of norm inflation. The first such result was due to Bourgain and Pavlović [11], who showed that arbitrarily small data in $B_{\infty,\infty}^{-1}$ can become arbitrarily large in a short time. In other words, the data-to-solution map is not continuous at 0 in the $B_{\infty,\infty}^{-1}$

topology. Germain [47] ruled out the fixed point argument for constructing mild solutions from data in $B_{\infty,q}^{-1}$ for $q > 2$. Explicit ill-posedness in this class was proven by Yoneda [99] following the ideas of [11]. Later, Wang [95] proved the surprising result that norm inflation from small data is possible in the spaces $\dot{B}_{\infty,q}^{-1}$ even for $q \in [1, 2]$, which are continuously embedded in BMO^{-1} . Thus, although these solutions are globally regular and stay small in the BMO^{-1} norm, they nonetheless become large in the stronger norm $B_{\infty,q}^{-1}$. The fact that the presence of norm inflation is not “monotone” in the class of data is interesting and suggests that non-uniqueness (or the lack thereof) may be a better benchmark for the well-posedness of (1.13). Let us also mention a slight strengthening of these results in a logarithmically subcritical Besov space due to Cui [36].

When one allows a force, recent works by Córdoba, Martínez-Zoroa, and Zheng demonstrate that finite-time blow-up is possible for Euler, hypodissipative Navier–Stokes, and other related models. For instance, see [34] and [35]. These constructions have a similar spirit to the one in the present work in the sense that they build an explicit multi-scale initial datum whose evolution forward in time can be reasonably well-understood.

1.4.2 Known Approaches towards Non-Uniqueness

Up to this point, there have been two strategies for producing non-unique solutions to (1.13) and other equations of fluid dynamics. Here we briefly summarize the state of the art.

Convex Integration

Inspired by Nash’s C^1 isometric embedding theorem [71], De Lellis and Székelyhidi introduced convex integration to the field of mathematical fluid dynamics in their seminal papers [39, 38]. Whereas for the Euler equations these methods produced essentially sharp results (culminating, for instance, in the resolution of Onsager’s conjecture [53]), for the Navier–Stokes equations there remain some persistent gaps between the regularity reached by convex integration schemes and the known well-posedness results. This area has been highly active so we refer the reader to [15] for a more thorough survey.

The first example of non-unique weak solutions of (1.13) was constructed in the space $C_t H_x^\epsilon$

for $\epsilon > 0$ small in the breakthrough work of Buckmaster and Vicol [14]. For comparison, the critical space in this scale in dimension three is $C_t H_x^{1/2}$, and Leray–Hopf solutions additionally lie in $L_t^2 H_x^1$. Tao [90] observed that one can reach regularity $C_t H_x^{\frac{1}{2}-\epsilon}$ by taking the dimension n large depending on ϵ , but then the critical Sobolev space is $C_t H_x^{\frac{n}{2}-1}$, which is far out of reach. Among the subsequent convex integration constructions following [14], for instance [12, 66, 25, 26], the sharpest in terms of the scaling of the space are those of Cheskidov and Luo: first in [25], they construct solutions in $L_t^{2-\epsilon} L_x^\infty(\mathbb{R}^n)$ for $n \geq 2$ and all $\epsilon > 0$. Later in [26], they exhibit solutions in the space $C_t L_x^{2-\epsilon}(\mathbb{R}^2)$, where $\epsilon > 0$ can be taken arbitrarily small. For comparison, $L_t^2 L_x^\infty$ and $L_t^\infty L_x^n$ are critical in dimension n ; thus, their solutions can get arbitrarily close to certain critical spaces on the Prodi–Serrin–Ladyzhenskaya scale and possess some additional favorable properties outside of a small set of times.

Unfortunately, convex integration at or above critical regularity appears out of reach. Indeed, the nonlinearity must be at least as large as the Laplacian in order for the error correction to function, so there is no room to lose anything during the iteration. Moreover, convex integration is only known to be applicable in settings where the solution space has flexibility, which is certainly not the case for solutions that immediately regularize like the ones in Theorem 1.4.2, nor (one might expect) for Leray–Hopf solutions, which exhibit weak-strong uniqueness. There are additional difficulties related to finding sufficiently intermittent building blocks which present an obstacle to, among other things, proving the sharpness of the full Prodi–Serrin–Ladyzhenskaya scale.

The Jia–Šverák Program

The other known strategy for demonstrating non-uniqueness was set forth by Jia and Šverák in [54, 55]. They propose to consider (1.13) in self-similar coordinates $\xi = x/t^{1/2}$, $\tau = \log t$ with the ansatz $u(t, x) = t^{-1/2} U(\xi, \tau)$. Thus the question of non-uniqueness for u at $t = 0$ is transferred to U at $\tau = -\infty$, which would be settled if one can find a steady solution $U(\xi, \tau) = \bar{U}(\xi)$ that is nonlinearly unstable in the self-similar dynamics. This scenario seems plausible in light of compelling numerical evidence by Guillod and Šverák [49]. Recently, Albritton, Brué, and Colombo [1] used this framework to construct non-unique Leray–Hopf solutions to the forced Navier–Stokes

equations by building a profile $\bar{U}$ out of the unstable vortex constructed by Vishik in [93, 94]. The force is necessary for their construction because $\bar{U}$ does not satisfy the Navier–Stokes in self-similar coordinates. Removing the force appears extremely difficult because the only suspected unstable self-similar profile is the one numerically discovered in [49]. It follows that one would have to verify spectral properties of the equation linearized around a numerical profile. Hou, Wang, and Yang [52] have announced a computer-assisted proof that would complete the Jia–Šverák Program.

1.4.3 Strategy of the Proof

The non-uniqueness mechanism that leads to Theorem 1.4.2 is based on the one proposed by Palasek in the context of the Obukhov dyadic model in [74], with certain simplifications to make the construction feasible in the full PDE setting. Let us now sketch the strategy of the proof, neglecting some details for the purpose of the exposition.

Non-Uniqueness Mechanism

The initial data is constructed as a lacunary series $U^0 = \sum_{k \geq 0} V_k^0$ where each V_k^0 is (approximately) localized in Fourier space to a band around $|\xi| = N_k$, with N_k a rapidly growing sequence of frequency scales. The essence of the construction is the following claim: there exists a choice of the $(V_k^0)_{k \in \mathbb{N}}$ such that at each frequency level k , there exist two evolutions forward in time that are consistent with (1.13), up to a small error:

- *The Heat-Dominated Flow:* let $v_k(x, t)$ be the flow of the data under the heat equation, namely

$$\begin{aligned} \partial_t v_k - \Delta v_k &= l.o.t. \\ v_k|_{t=0} &= V_k^0 \end{aligned} \tag{1.15}$$

where we add some lower order terms as a matter of convenience. Clearly $\|v_k(t)\|_{L^\infty}$ decays exponentially on the time scale N_k^{-2} .

- *The Inverse Cascade-Dominated Flow*: let $\bar{v}_k(x, t)$ be the flow of the data under the equation

$$\begin{aligned} \partial_t \bar{v}_k + P_{\sim N_k} \mathbb{P} \operatorname{div} v_{k+1} \otimes v_{k+1} &= 0 \\ \bar{v}_k|_{t=0} &= V_k^0 \end{aligned} \tag{1.16}$$

where v_{k+1} is the “heat-dominated flow” emanating from the data V_{k+1}^0 as described above, and $P_{\sim N_k}$ is a Littlewood–Paley-type projection to a frequency shell around N_k . Crucially, by a particular choice of V_{k+1}^0 , we can arrange that $\|\bar{v}_k(t)\|_{L^\infty}$ decays exponentially to zero on the time scale N_{k+1}^{-2} .

Taking for granted the assertions made so far (which, indeed, most of the chapter will be occupied with justifying), we can write down two distinct *approximate* solutions of (1.13) with data $U^0 = \sum_{k \geq 0} V_k^0$:

$$v^{(1)} := v_0 + \bar{v}_1 + v_2 + \bar{v}_3 + \cdots \quad \text{and} \quad v^{(2)} := \bar{v}_0 + v_1 + \bar{v}_2 + v_3 + \cdots. \tag{1.17}$$

The distinctness is immediate from the fact that v_k and $\bar{v}_k$ have different decay rates.

Let us give a more heuristic description of how the non-uniqueness arises. Fixing $k \geq 1$, one should interpret (1.16) as asserting that V_{k+1}^0 is capable of using its nonlinear self-interaction to “annihilate” the mode below that has initial data V_k^0 , and this annihilation occurs on time scale N_{k+1}^{-2} . Likewise, V_k^0 is capable of annihilating the mode at V_{k-1}^0 on time scale N_k^{-2} . However, these two events are incompatible because the k th mode would vanish on a time scale (i.e., N_{k+1}^{-2}) much shorter than the time it needs to act on v_{k-1} (i.e., N_k^{-2}). Thus, if V_k^0 is in fact annihilated by V_{k+1}^0 (i.e., its continuation $\bar{v}_k(t)$ satisfies (1.16)), then there are no nonlinear dynamics to act on V_{k-1}^0 , which then evolves as the heat-dominated flow $v_{k-1}(t)$ according to (1.15).

The conclusion of the discussion above is that these dynamics are only self-consistent if (1.15) and (1.16) occur at alternating sequences of k ’s, which justifies the choice (1.17). The non-uniqueness then is a consequence of the fact that (1.15) can occur at *either* every odd k *or* every even k .

Structure of the Principal Part

An obvious difficulty is that even if we can arrange (1.15)–(1.16), there remains a plethora of nonlinear interactions between the v_k ’s and $\bar{v}_k$ ’s that need to be small for us to claim these as approximate solutions. It turns out that if the data is chosen appropriately, the terms appearing in (1.15)–(1.16) are the only ones that are non-perturbative. Toward this end, we borrow certain elements from convex integration constructions where a central difficulty is also minimizing nonlinear interactions between solution components at different frequencies.

Now we give the rough ideas for how to select data obeying (1.15)–(1.16) while keeping all other interactions minimized. The initial data $V_k^0(x)$ is (very roughly, and locally) of the form $a_k(x)\theta \sin(N_k x \cdot \eta)$ for some fixed $\eta, \theta \in \mathbb{Z}^3$ with $\eta \cdot \theta = 0$, where $a_k(x)$ is a scalar coefficient that is principally supported at frequencies $|\xi| \ll N_k$. Then, according to (1.15), we should take $v_k(x, t) = a_k(x)\theta \sin(N_k x \cdot \eta) \exp(-N_k^2 |\eta|^2 t)$ up to various small errors. Having defined the $(v_k)_{k \in \mathbb{N}}$ in this way, the requirement (1.16) becomes

$$\begin{aligned} \partial_t \bar{v}_k &= -P_{\sim N_k} \mathbb{P} \operatorname{div} \left(a_{k+1}^2 \theta \otimes \theta \sin^2(N_{k+1} x \cdot \eta) \right) \exp(-2N_{k+1}^2 |\eta|^2 t) \\ \bar{v}_k|_{t=0} &= V_k^0. \end{aligned}$$

One recalls that $a_{k+1}^2(x)$ is mainly localized to frequencies $\ll N_{k+1}$, while $\sin^2(N_{k+1} x \cdot \eta)$ is supported at frequencies $|\xi| = 0$ and $|\xi| = 2|\eta|N_{k+1}$. Thus the interaction of a_{k+1}^2 with the high-frequency part of $\sin^2$ cannot concentrate significantly in the shell $|\xi| \sim N_k$, so only the zeroth mode of $\sin^2$ contributes. This lets us reduce (1.16) by integrating in time to

$$\bar{v}_k(t) = V_k^0 - CN_{k+1}^{-2} P_{\sim N_k} \mathbb{P} \operatorname{div}(a_{k+1}^2 \theta \otimes \theta) (1 - \exp(-2N_{k+1}^2 |\eta|^2 t))$$

up to errors, where C is a computable constant. Thus, our claim that $\|\bar{v}_k(t)\|_{L^\infty}$ decays on time scale N_{k+1} is achievable if a_{k+1} is chosen so that

$$V_k^0 = CN_{k+1}^{-2} P_{\sim N_k} \mathbb{P} \operatorname{div}(a_{k+1}^2 \theta \otimes \theta). \quad (1.18)$$

As stated, this is not quite possible, but the reader familiar with convex integration will recognize the issue is analogous to a problem first encountered by Nash in [71] that can be solved, for instance, by replacing the simple ansatz $V_k^0 = a_k(x)\theta \sin(N_k x \cdot \eta)$ with one based on a Mikado flow, i.e., a superposition of spatially disjoint shear flows. These have the advantage that the rank-1 stress $a_{k+1}^2 \theta \otimes \theta$ now becomes a linear combination $\sum_j a_{j,k+1}^2 \theta_j \otimes \theta_j$. We observe that V_k^0 can be constructed to be in the form ψ_k^0 which makes it not only divergence free, but also in the form $\operatorname{div} R_k$ for some symmetric tensor R_k . Thus a_{k+1} can be chosen to satisfy (1.18) exactly.

Imposing (1.18) at every $k \geq 0$ creates a recursive dependence between the modes of the initial data: having chosen V_0^0 , the requirement (1.18) determines a_1 , which in turn determines V_1^0 , which then determines a_2 , and so on. Based on this, we can estimate the amplitudes of the V_k^0 . Roughly speaking, $\|V_k^0\|_{L^\infty} \sim \|a_k\|_{L^\infty}$. Then (1.18) leads to

$$\|V_{k+1}^0\|_{L^\infty} \approx C_1 N_{k+1} \left(\frac{\|V_k^0\|_{L^\infty}}{N_k} \right)^{\frac{1}{2}}. \quad (1.19)$$

As an immediate consequence, we should expect the k th component of the data V_k^0 to have amplitude on the order N_k , in accordance with the critical scaling of (1.13). Two important observations are in order. First, (1.19) shows why there is no danger from losing constants during the iteration, which is a fatal issue for convex integration schemes at the critical regularity. Indeed, any large constants (independent of k) that appear in constructing V_{k+1}^0 from V_k^0 are removed by successively taking the $1/2$ power. Second, (1.19) shows why this method cannot produce solutions with small data (which would contradict Theorem 1.4.1). We are free to initiate the process with an arbitrarily small choice of V_0^0 , but as $k \rightarrow \infty$ the amplitude invariably approaches the fixed point of (1.19), which is presumably large.

One may notice at this point that the solutions described above belong to $B_{\infty,\infty}^{-1}$ but not BMO^{-1} . The required improvement is now a consequence of improving the spatial support of each V_k^0 . Recall that Mikado flows are supported on a periodic collection of pipe regions, and there is no difficulty in making the volume of the support smaller than, say, $|\mathbb{T}^3|/100$. If one is sufficiently careful, it is further possible to select a_{k+1} satisfying (1.18) such that $\operatorname{supp} a_{k+1} \subset \operatorname{supp} V_k^0$. It

follows inductively that the support of V_k^0 is contained in the intersection of the supports of the Mikado flows that were used to define $v_0^0, v_1^0, \dots, V_k^0$. Because these flows are at different scales, the volume of this intersection is multiplicative and one can bound $|\text{supp } V_k^0| \leq 2^{-k} |\mathbb{T}^3|$, which suffices to improve the regularity to BMO^{-1} .

Construction of the Perturbation

Finally, let us address the last challenge of the proof which is constructing perturbations $w^{(1)}$ and $w^{(2)}$ that correct the errors, i.e., so that $u^{(i)} = v^{(i)} + w^{(i)}$ are exact solutions of (1.13) with data U^0 . The idea is in a similar spirit to the constructions by Chemin and Gallagher [21, 22, 23] of global solutions of (1.13) from initial data that is large in BMO^{-1} (or even $B_{\infty,\infty}^{-1}$) while satisfying a nonlinear smallness condition. Suppose $v^{(i)}$ solves (1.13) with a small error in divergence form,

$$\begin{aligned} \partial_t v^{(i)} - \Delta v^{(i)} + \mathbb{P} \operatorname{div} v^{(i)} \otimes v^{(i)} &= \operatorname{div} F^{(i)} \\ v^{(i)}|_{t=0} &= U^0. \end{aligned}$$

Then $w^{(i)}$ must satisfy the perturbed Navier–Stokes equations

$$\begin{aligned} \partial_t w^{(i)} - \Delta w^{(i)} + \mathbb{P} \operatorname{div}(2v^{(i)} \odot w^{(i)} + w^{(i)} \otimes w^{(i)}) &= -\operatorname{div} F^{(i)} \\ w^{(i)}|_{t=0} &= 0. \end{aligned}$$

This equation is of the same type faced in [21, 22, 23]. The obvious strategy is to construct solutions via a fixed point argument, but this is complicated by the additional linear terms which are not contractive when $v^{(i)}$ is large. In the papers cited, this difficulty is handled by taking the initial data U^0 to be in $H^{1/2}$ (or at least $B_{\infty,2}^{-1}$), leading to $e^{t\Delta}U^0 \in L_t^2 L_x^\infty$. Unfortunately, the $v^{(i)}$ constructed above are not contained in $L_t^2 L_x^\infty$ (nor is that the case for, say, the self-similar solutions posited by Jia and Šverák) so those methods break down.

Instead, we proceed as follows. For $i = 1, 2$, let $S^{(i)}(t, t')$ be the semigroup associated to the

linearized equation for $w^{(i)}$, defined for all $0 < t' \leq t$. Then $w^{(i)}$ should obey the mild formulation

$$w^{(i)}(t) = \int_0^t S^{(i)}(t, t')(w^{(i)} \otimes w^{(i)} - F^{(i)}) dt'. \quad (1.20)$$

Note that $w^{(i)}$ has zero initial data, so to make sense of this mild formulation, we need only define the semigroup for $t' > 0$. Of course, it is also important that the estimates on $S^{(i)}(t, t')$ do not deteriorate too severely as $t'/t \rightarrow 0$. We show that $S^{(i)}(t, t')$ has estimates slightly worse than $e^{(t-t')\Delta} \mathbb{P} \operatorname{div}$. This loss is tolerable because by taking the sequence N_k to increase faster than exponentially, the errors $F^{(i)}$ become not just small, but small when measured in a subcritical norm; for instance $\|F^{(i)}(t)\|_{L^\infty} \lesssim t^{-1+\alpha}$ for an $\alpha > 0$ (cf. the critical power $\alpha = 0$). Combining all these elements, we construct the $w^{(1)}$ and $w^{(2)}$ as fixed points of the operator in the mild form (1.20) in a small ball around zero measured in a slightly subcritical norm. We conclude by showing that $w^{(i)} \rightarrow 0$ weakly near the initial time and that $w^{(i)}$ are small enough to not disturb the distinctness of $v^{(1)}$ and $v^{(2)}$.

1.4.4 Plan of the Chapter

In §4.1, we state some of the basic Littlewood–Paley theory and related estimates (§4.1.2–4.1.3) and introduce some of the key parameters that go into the construction (§4.1.4). In §4.2 we build the data U^0 and the leading parts $v^{(i)}$ of the solutions: first we construct the Mikado building blocks for the initial data and prove estimates on the supports (§4.2.1); then we run the iterative procedure to construct the initial data and prove estimates (§4.2.2); followed by construction of the leading parts of the two distinct solutions forward in time (§4.2.3). §4.3 is concerned with constructing the perturbation $w^{(i)}$ that will correct the principal parts $v^{(i)}$: first we estimate the residuals $F^{(i)}$ that are to be corrected (§4.3.1); then we construct the semigroup $S^{(i)}$ of the Navier–Stokes linearized around the principal parts $v^{(i)}$ (§4.3.2); and finally we construct the perturbations with a fixed point argument (§4.3.3). In §4.4, we complete the proof of Theorem 1.4.2, proving that the two solutions constructed attain the initial data and are distinct. Sections 4.5–4.7 give some technical tools used in the proofs.

Chapter 2

Partial Regularity and Blowup

2.1 Partial Regularity

2.1.1 Covering the Singular Set

Let $\mathbb{P}$ denote the Leray projection. Let $\mathcal{F}$ denote the Fourier transform. We shall employ a Littlewood-Paley partition of frequency space. In particular, for $j \in \mathbb{Z}$ we have pseudodifferential operators P_j with positive and smooth symbols $p_j(\xi)$ that are supported in $\frac{2}{3}2^j < |\xi| < 3 \cdot 2^j$ and that also satisfy $p_j(\xi) = p_0(2^{-j}\xi)$ and

$$\sum_{j \in \mathbb{Z}} p_j(\xi) = 1. \quad (2.1)$$

We also define $\tilde{P}_j := \sum_{k=-2}^2 P_{j+k}$ to be the sum of all Littlewood-Paley projections whose symbols' supports intersect the support of $p_j(\xi)$. Likewise we may define $\tilde{p}_j(\xi)$ as $\sum_{k=-2}^2 p_{j+k}(\xi)$. We can consider $P_j f$ as a combination of wavelets supported on cubes of sidelength 2^{-j} . Keeping this heuristic in mind, we localize on cubes of sidelength almost equal to 2^{-j} . That is, we fix an $\epsilon > 0$ and localize within slightly larger cubes of sidelength $2^{-j(1-\epsilon)}$. Such a localization will allow us to commute cutoffs in frequency space with cutoffs in physical space, with only the addition of a negligible error. Since the Hausdorff dimension estimate is a closed condition, an approximate localization like this will be acceptable. Our localization is achieved with the use of particularly

chosen bump functions that we describe below. See also [57] and [73].

Let Q be a cube of sidelength greater than $2^{-j(1-\epsilon)}$. We denote λQ to be the cube with the same center as Q and with $\text{sidelength}(\lambda Q) = \lambda \cdot \text{sidelength}(Q)$. We define a bump function $1 \geq \phi_{Q,j} \geq 0$ such that $\phi_{Q,j} = 1$ on Q and is zero outside of $(1+2^{-\epsilon j})Q$. We also require that for every multi-index α , there is a constant C_α depending on α but independent of Q and j such that

$$|D^\alpha \phi_{Q,j}| \leq C_\alpha 2^{|\alpha|j(1-\epsilon)} \quad (2.2)$$

Following [57], we call these bump functions of type j . In line with our previous discussion, we consider $\phi_{Q,j}P_j$ as a projection onto a localized wavelet, and, if Q is a cube with side length exactly equal to $2^{-j(1-\epsilon)}$, we denote the “wavelet coefficient” by

$$\|\phi_{Q,j}P_j f\|_2 := f_Q$$

and say that Q is at level j . For convenience, we shall use the following notation for a particular dilation of a cube at level j :

$$Q_j := (1 + 2^{-\epsilon j})Q.$$

Also, if Q is a cube at level j , we denote:

$$u_{\mathcal{N}^1(Q)} := \|\phi_{Q_j,j}P_{j-2 \leq k \leq j+2}u\|_{L^2},$$

which is a shorthand for the kinetic energy supported on nearby frequencies. We shall also use the notation $N(A)$ to denote the cardinality of a set A .

Throughout this chapter we use various properties satisfied by the bump functions $\phi_{Q,j}$ and the Littlewood-Paley projections P_j . Most of these properties can be classified as “commutator estimates” that allow us to move bump functions across Littlewood-Paley projections and vice versa with only the addition of a negligible error. For the benefit of exposition, we defer the technical proofs of these lemmas to the last section of this chapter; however, we shall use these

technical results freely here.

We shall now prove a general partial regularity theorem for systems of pseudodifferential equations. These systems will generalize the Navier-Stokes equations in the sense that they arise from the Stokes equations equipped with a nonlinearity of the form $B(u, u)$, where B is an amenable operator in the sense of Definition 1.2.2. Without loss of generality, we may assume in this section that the scaling term λ in the fifth part of Definition 1.2.2 is equal to 2.

We denote the Leray Projection of B by $\hat{B} := \mathbb{P}B$ for the rest of this section. By the incompressibility condition on u in the statement of the theorem, we can instead consider the following system of pseudodifferential equations:

$$\begin{aligned}\partial_t u + (-\Delta)^\alpha u + \hat{B}(u, u) &= 0 \\ u(t=0) &= u_0.\end{aligned}\tag{2.3}$$

Then we prove Theorem 1.2.3, which we reformulate as

Theorem 2.1.1. *Let $n \geq 3$ and $(n+1)/4 < \alpha < (n+2)/4$, let B be an amenable bilinear operator, and let u_0 be a Schwartz vector field. Suppose that $u(t)$ is a smooth vector field on $\mathbb{R}^n \times [0, T) \rightarrow \mathbb{R}^n$ that solves Equation (2.3). If T is the time of first blowup, then the closed set S_T has Hausdorff dimension at most $n + 2 - 4\alpha$.*

The next subsection is devoted to proving Theorem 2.1.1. We remark that if we take

$$B(u, v) = \frac{1}{2} \left((u \cdot \nabla) v + (v \cdot \nabla) u \right)$$

then we have the usual α -dissipative Navier-Stokes equations in Equation (2.3). Subsequently, we get Corollary 1.2.4.

We recall that our solution u is smooth up to time T and that the initial data u_0 is Schwartz. These are facts that we shall use frequently and usually without reference. Pairing Equation (2.3) with u and using the cancellation identity in Equation (1.5), we get:

$$\langle \partial_t u, u \rangle + \langle (-\Delta)^\alpha u, u \rangle = 0.\tag{2.4}$$

Integrating in time then gives us what we shall refer to as the **energy dissipation law**:

$$\|u(t)\|_2^2 = \|u_0\|_2^2 - 2 \int_0^t \|(-\Delta)^{\alpha/2} u\|_2^2 dt \leq \|u_0\|_2^2 \quad \forall t \in [0, T]. \quad (2.5)$$

We shall examine the dynamics of the “wavelet coefficients” $u_Q := \|\phi_{Q,j} P_j f\|_2$, where $\phi_{Q,j}$ is a bump function of type j and Q is a cube at level j . To this end, we pair Equation (2.3) with $P_j \phi_{Q,j}^2 P_j u$, and get, after simplification:

$$\partial_t \left(\frac{1}{2} u_Q^2 \right) = \langle -\hat{B}(u, u), P_j \phi_{Q,j}^2 P_j u \rangle - \langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle. \quad (2.6)$$

We note that the “wavelet coefficients” u_Q are continuous in time, see Lemma 2.4.24 and Corollary 2.4.25. We prove the two main estimates for the terms on the right-hand-side of Equation (2.6) in a later section. We estimate the “dissipation” term by

$$\langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle \geq -K 2^{-200j} + K 2^{2\alpha j} u_Q^2 - K 2^{(2\alpha-\epsilon)j} u_{\mathcal{N}^1(Q)}^2. \quad (2.7)$$

We estimate the “nonlinear” term by

$$\begin{aligned} \left| \langle -\hat{B}(u, u), P_j \phi_{Q,j}^2 P_j u \rangle \right| &\leq K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q + \\ &+ K \sum_{k=\delta j}^{j-10} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 + \\ &K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 + K 2^{-150j}. \end{aligned} \quad (2.8)$$

We have used the following notation: for a cube Q at level j we denoted

$$k < j \Rightarrow Q_k := 2^{(j-k)(1-\epsilon)} Q, \quad k \geq j \Rightarrow Q_k = (1 + 2^{-\epsilon k}) Q.$$

The estimates hold for all cubes Q at level $j \geq 0$, and we shall now clarify the role of the symbol K .

Throughout the rest of this chapter, the symbol K will denote a constant appropriately chosen so that every inequality where it appears is satisfied. The choice of K will be clear at each instance of its use, and it will only depend on the L^2 norm of the initial data, the blowup time T , the dimension $n \geq 3$, and the parameters ϵ, α , and γ . Also, we introduce another parameter δ that depends on α and the dimension $n \geq 3$. The parameter δ will be used later when we estimate the nonlinear term in Equation (2.6). Given $n \geq 3$ and $\alpha > (n+1)/4$, we shall carefully choose the parameters δ, γ , and ϵ so that our argument closes.

We now describe the construction of a collection of cubes, away from which we have “critical regularity”, as in [57] and [73]. We call a cube Q at level j **bad** if and only if:

$$\int_0^T \int \sum_{k \geq j} 2^{2\alpha k} |\phi_{Q_j, j} P_k u|^2 dx dt \geq 2^{-(n+2-4\alpha)j - \epsilon j}.$$

We shall consider the reverse inequality as the hypothesis of an ϵ -regularity statement compatible with the scaling of the equations. In other words,

$$(2^m)^{2\alpha-1} u((2^m)x, (2^m)^{2\alpha} t)$$

is a solution of Equation (2.3) for any $m \in \mathbb{Z}$, if $u(x, t)$ is a solution of the same equation. We define the set M_j to be the union of cubes Q for all bad cubes Q at level j . We first obtain a convenient cover of M_j :

Proposition 2.1.2. *There exists a constant K depending only on ϵ so that for all $j \geq K$, there is a covering $\mathcal{C}_j$ of M_j by cubes at level j such that for some other constant K independent of j :*

$$N(\mathcal{C}_j) \leq K 2^{(n+2-4\alpha)j + \epsilon j}.$$

Proof. Let $\mathcal{A}$ be the collection of cubes $2Q$ where Q is bad and at level j . By the Vitali Covering Lemma (Lemma 2.4.22), there are disjoint cubes $2Q'$ in $\mathcal{A}$ such that the collection of cubes $10Q'$ covers M_j . However, any cube of sidelength $10 \cdot 2^{-j(1-\epsilon)}$ can be covered by K cubes at level j (K depending on ϵ and $n \geq 3$ only), so we are led to define $\mathcal{C}_j$ to be the collection of the cubes at level j

that cover the cubes $10Q'$. Now since the cubes Q' are bad with cubes $2Q'$ disjoint, the cubes Q'_j are also disjoint. Indeed, by always choosing $j \geq 100/\epsilon$, each such cube $Q'_j = (1 + 2^{-\epsilon j})Q' \subset (3/2)Q'$ is strictly contained in the cube $2Q'$. By the energy dissipation law and Lemma 2.4.13 we may conclude:

$$\begin{aligned}
& N(\mathcal{C}_j) 2^{-(n+2-4\alpha)j-\epsilon j} \leq \\
& \leq K \sum_{Q'} \sum_{k \geq j} 2^{2\alpha k} \int_0^T \int \left| \phi_{Q'_j, j} P_k u \right|^2 dx dt \leq K \sum_{k \geq j} 2^{2\alpha k} \int_0^T \int \sum_{Q'} \left| \phi_{Q'_j, j} P_k u \right|^2 dx dt \leq \\
& \leq K \sum_{k \geq j} 2^{2\alpha k} \int_0^T \int_{\mathbb{R}^n} |P_k u|^2 dx dt \leq K,
\end{aligned}$$

which shows that there are at most $K 2^{(n+2-4\alpha)j+\epsilon j}$ cubes in the collection. $\square$

2.1.2 Critical Regularity Away from Bad Cubes

Note that the cube Q in the following proposition may actually be bad. The only condition that we place on Q is that some large dilations of Q are not bad, and this will be sufficient for the following proposition.

Proposition 2.1.3. *There exists j_0 sufficiently large so that for any cube Q at level j such that the cubes Q_k are not bad for any $k \in [\delta j, j - 10]$, we have for all $t \in [0, T]$:*

$$u_Q(t) < 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)}.$$

Proof. The claim is true for sufficiently small $t \geq 0$ because our initial data is Schwartz. Let $t_0 > 0$ be the first time when the claim is not true. More explicitly, we assume that for any j_0 there exists Q at some level j such that the cubes Q_k are not bad for all $k \in [\delta j, j - 10]$ but $u_Q(t_0) \geq 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)}$. Let $t_1 \in (0, t_0)$ be the largest time so that $u_Q(t_1) \leq \frac{1}{2} 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)}$, so that

$$\frac{1}{2} 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)} \leq u_Q(t) \leq 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)} \quad (2.9)$$

for all $t \in (t_1, t_0)$. Recall that our estimate in Equation (2.8) states that for any $\delta > 0$:

$$\begin{aligned} \left| \langle -\hat{B}(u, u), P_j \phi_{Q,j}^2 P_j u \rangle \right| &\leq K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q + \\ + K \sum_{k=\delta j}^{j-10} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_{Q'} + K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_{Q'} \|\phi_{Q_j,j} P_k u\|_2^2 + \\ + K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 + K 2^{-150j}. \end{aligned}$$

We thus conclude from Equation (2.6) and Equation (2.7) that

$$\begin{aligned} \frac{3}{4} 2^{-j(n+2-4\alpha+\epsilon)} &\leq u_Q^2(t_0) - u_Q^2(t_1) \leq \int_{t_1}^{t_0} \left(K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q + \right. \\ + K \sum_{k=\delta j}^{j-10} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 + \\ + K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 \Big) dt - \int_{t_1}^{t_0} K 2^{2\alpha j} u_Q^2 dt + \\ \int_{t_1}^{t_0} K 2^{(2\alpha-\epsilon)j} u_{\mathcal{N}^1(Q)}^2 + K 2^{-150j}. \end{aligned} \quad (2.10)$$

We observe that since Q_{j-10} is not a bad cube we have, for some constant K depending on α, ϵ, n :

$$\int_0^T \int \sum_{k \geq j-10} 2^{2\alpha k} |\phi_{Q_{j-10},j-10} P_k u|^2 dx dt \leq K 2^{-(n+2-4\alpha+\epsilon)j} \quad (2.11)$$

and in particular

$$\int_0^T u_Q^2 dt \leq \int_0^T u_{\mathcal{N}^1(Q)}^2 dt \leq \int_0^T \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 dt \leq K 2^{-(n+2-2\alpha+\epsilon)j}. \quad (2.12)$$

In addition, since Q_k is not bad for any $k \in [\delta j, j-10]$, we have

$$\int_0^T u_{Q_k}^2 dt \leq K 2^{-(n+2-2\alpha+\epsilon)k} \quad (2.13)$$

for any $k \in [\delta j, j - 10]$. We shall reach a contradiction by appropriately bounding the terms in the estimate of the nonlinearity. First we observe by the Cauchy-Schwarz inequality:

$$2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} \int_{t_1}^{t_0} u_{\mathcal{N}^1(Q)} u_Q \leq 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} \left(\int_{t_1}^{t_0} u_{\mathcal{N}^1(Q)}^2 dt \right)^{1/2} \left(\int_{t_1}^{t_0} u_Q^2 dt \right)^{1/2}.$$

Using Equation (2.12), we have that

$$K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} \int_{t_1}^{t_0} u_{\mathcal{N}^1(Q)} u_Q \leq K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3}-(n+2-2\alpha+\epsilon))j} \leq K 2^{-(n+2-4\alpha+\epsilon)j} 2^{(1+\frac{n\delta}{2}+\frac{n\gamma}{3}-2\alpha)j}.$$

Since $\alpha > (n+1)/4$, we may choose $\delta < (n-1)/n$ and γ sufficiently small so that

$$1 + \frac{n\delta}{2} + \frac{n\gamma}{3} - 2\alpha < \frac{n\gamma}{3} + 2\left(\frac{n+1}{4} - \alpha\right) < 0.$$

Thus, choosing j_0 sufficiently large, we have for all $j \geq j_0$:

$$K 2^{(1+\frac{n\delta}{2}+\frac{n\gamma}{3}-2\alpha)j} \leq \frac{1}{1000}.$$

Therefore

$$K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} \int_{t_1}^{t_0} u_{\mathcal{N}^1(Q)} u_Q \leq \frac{1}{1000} 2^{-(n+2-4\alpha+\epsilon)j}. \quad (2.14)$$

We continue our analysis with the low-high terms. We observe by the Cauchy-Schwarz inequality and Equation (2.9) that

$$2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} \int_{t_1}^{t_0} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q \leq 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)} \left(\int_{t_1}^{t_0} u_{Q_k}^2 dt \right)^{1/2} \left(\int_{t_1}^{t_0} u_{\mathcal{N}^1(Q)}^2 dt \right)^{1/2}.$$

Thus, by Equation (2.12) and Equation (2.13), we have

$$K 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} \int_{t_1}^{t_0} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q \leq K 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)} 2^{-(n+2-2\alpha+\epsilon)\frac{k}{2}} 2^{-(n+2-2\alpha+\epsilon)\frac{j}{2}}.$$

Simplifying, we get

$$K 2^{\frac{nk}{2} + j + \frac{n\gamma j}{3}} \int_{t_1}^{t_0} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q \leq K 2^{-j(n+2-4\alpha+\epsilon)} 2^{-(2-2\alpha+\epsilon)\frac{j}{2}} 2^{j + \frac{n\gamma j}{3} - \alpha j}. \quad (2.15)$$

Recall that $n \geq 3$ and $\alpha > (n+1)/4 \geq 1$, so we get (also assuming $0 < \epsilon < 2\alpha - 2$) from Equation (2.15):

$$\sum_{k=\delta j}^{j-10} K 2^{\frac{nk}{2} + j + \frac{n\gamma j}{3}} \int_{t_1}^{t_0} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q \leq K 2^{-j(n+2-4\alpha+\epsilon)} 2^{-(2-2\alpha+\epsilon)\frac{j}{2}} 2^{j + \frac{n\gamma j}{3} - \alpha j}.$$

Assuming that $\gamma < 3\epsilon/(2n)$ gets us

$$-(1 - \alpha + \epsilon/2) + 1 + \frac{n\gamma}{3} - \alpha = n\gamma/3 - \epsilon/2 < 0.$$

Then choosing j_0 large enough, we may conclude that

$$K 2^{-(2-2\alpha+\epsilon)\frac{j}{2}} 2^{j + \frac{n\gamma j}{3} - \alpha j} \leq \frac{1}{1000}$$

for all $j \geq j_0$. This allows us to conclude that

$$\sum_{k=\delta j}^{j-10} K 2^{\frac{nk}{2} + j + \frac{n\gamma j}{3}} \int_{t_1}^{t_0} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q \leq \frac{1}{1000} 2^{-j(n+2-4\alpha+\epsilon)}. \quad (2.16)$$

We proceed to examine the high-high terms. By Equation (2.9) and Equation (2.11) we have

$$\begin{aligned} & \int_{t_1}^{t_0} K \sum_{k>j+10} 2^{nj(\frac{1}{2} - \frac{2\gamma}{3}) + k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 dt \leq \\ & \leq 2^{-(n+2-4\alpha+\epsilon)\frac{j}{2}} \int_{t_1}^{t_0} K \sum_{k>j+10} 2^{nj(\frac{1}{2} - \frac{2\gamma}{3}) + k(n\gamma+1)} \|\phi_{Q_j,j} P_k u\|_2^2 dt \leq \\ & \leq K 2^{-(n+2-4\alpha+\epsilon)\frac{j}{2}} 2^{\frac{nj}{2} - \frac{2\gamma nj}{3}} 2^{-j(n+2-4\alpha+\epsilon)} \sum_{k>j+10} 2^{k(n\gamma+1-2\alpha)}. \end{aligned}$$

Since $\alpha > 1$, and since we can always assume that $\gamma < 1/n$, we have that the sum converges, yielding

$$\begin{aligned} & \int_{t_1}^{t_0} K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 dt \leq \\ & \leq K 2^{-(n+2-4\alpha+\epsilon)j} 2^{j(\frac{n}{2}-\frac{2\gamma n}{3}-\frac{(n+2-4\alpha+\epsilon)}{2}+n\gamma+1-2\alpha)}. \end{aligned}$$

However, choosing $\gamma < \frac{3\epsilon}{2n}$, we can guarantee that

$$\frac{n}{2} - \frac{2\gamma n}{3} - \frac{(n+2-4\alpha+\epsilon)}{2} + n\gamma + 1 - 2\alpha = \frac{n\gamma}{3} - \frac{\epsilon}{2} < 0,$$

so that, for some j_0 large enough, we in fact have

$$\int_{t_1}^{t_0} K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 dt \leq \frac{1}{1000} 2^{-(n+2-4\alpha+\epsilon)j}. \quad (2.17)$$

The local terms are dealt with by using Equation (2.12) and Equation (2.9). First, we see that

$$\begin{aligned} & \int_{t_1}^{t_0} K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10}\|_{L^2}^2 dt \leq \\ & \leq K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}-2\alpha j} 2^{-\frac{j}{2}(n+2-4\alpha+\epsilon)} 2^{-(n+2-4\alpha+\epsilon)j}. \end{aligned}$$

Then, choosing $\gamma < \frac{3\epsilon}{2n}$, we can guarantee that

$$\frac{n+2}{2} + \frac{n\gamma}{3} - 2\alpha - \frac{(n+2-4\alpha+\epsilon)}{2} = \frac{n\gamma}{3} - \frac{\epsilon}{2} < 0,$$

so that, for some j_0 large enough, we in fact have

$$\int_{t_1}^{t_0} K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10}\|_{L^2}^2 dt \leq \frac{1}{1000} 2^{-(n+2-4\alpha+\epsilon)j}. \quad (2.18)$$

Finally, by Equation (2.12), there exists j_0 large enough so that

$$\int_{t_1}^{t_0} K 2^{(2\alpha-\epsilon)j} u_{N^1(Q)}^2 dt \leq 2^{-\epsilon j} K 2^{-(n+2-4\alpha+\epsilon)j} \leq \frac{1}{1000} 2^{-(n+2-4\alpha+\epsilon)j}. \quad (2.19)$$

By Equation (2.10), Equation (2.14), Equation (2.16), Equation (2.17), Equation (2.18), and Equation (2.19), we conclude that

$$\frac{3}{4} 2^{-j(n+2-4\alpha+\epsilon)} \leq \frac{1}{100} 2^{-j(n+2-4\alpha+\epsilon)} - K \int_{t_1}^{t_0} 2^{2\alpha j} u_Q^2 dt \leq \frac{1}{100} 2^{-j(n+2-4\alpha+\epsilon)},$$

which implies that $3/4 \leq 1/100$, a contradiction. $\square$

2.1.3 The Barrier

We follow the construction of the barrier as presented by Ożański in [73], and we attempt to use similar notation, wherever we can. Recall that we denoted the union of all bad cubes at level j by M_j and we found a collection of cubes at level j , denoted $\mathcal{C}_j$, that covers M_j and satisfies

$$N(\mathcal{C}_j) \leq K 2^{(n+2-4\alpha+\epsilon)j}.$$

As in [73], we wish to find another (possibly larger) cover of M_j , $\mathcal{B}_j$, of cubes at level j satisfying the **barrier property**: for all $x \notin \cup_{Q \in \mathcal{B}_j} Q$, there exists $0 < r < 1$ such that $\partial(rQ_j(x)) \cap \cup_{Q \in \mathcal{B}_k} Q = \emptyset$ for all $k \geq j$. We denoted the cube at level j centered at x by $Q_j(x)$, and we shall call the set $\partial(rQ_j(x))$ the **barrier at x** .

The following geometric lemma is proven in [73] for cubes in $\mathbb{R}^3$, but it in fact carries over to cubes in any spatial dimension $\mathbb{R}^n$, whenever $n \geq 3$, since the proof only involves the use of dilations. See [73] for details of the proof.

Lemma 2.1.4 ([73]). *Let $n \geq 3$. Let x, y be two different points in $\mathbb{R}^n$ and let $Q(y)$ and $Q'(x)$ be cubes of sidelength $2a$ and $2b$, respectively. If $\partial(rQ(y)) \cap Q'(x) \neq \emptyset$, then $r \in [r_{Q'} - b/a, r_{Q'} + b/a]$, where $r_{Q'} > 0$ is such that $x \in \partial(r_{Q'}Q)$.*

We now proceed to construct a collection of cubes at level $j \geq 0$ that satisfies the barrier property and covers the set M_j .

Lemma 2.1.5. *There exists a collection $\mathcal{B}_j$ of cubes at level $j \geq 0$ that satisfies the barrier property and covers the set M_j . In addition, the cardinality of $\mathcal{B}_j$ is at most*

$$N(\mathcal{B}_j) \leq K 2^{j(n+2-4\alpha+\epsilon)}. \quad (2.20)$$

Proof. First we find a cover $\mathcal{B}'_j$ of M_j by cubes at level j so that for any cube Q disjoint from every element of $\mathcal{B}'_j$, there exists a number $r > 0$ such that $\partial(rQ)$ is disjoint from every bad cube at level k for all $k \geq j$. In addition the cover $\mathcal{B}'_j$ will satisfy

$$N(\mathcal{B}'_j) \leq K 2^{j(n+2-4\alpha+\epsilon)}.$$

Then, the cover $\mathcal{B}_j$ will be obtained by replacing each cube Q in the cover $\mathcal{B}'_j$ by its dilation $3Q$ and then covering each dilated cube by at most 10^n cubes at level j . This process will result in a cover $\mathcal{B}_j$ having the desired cardinality bound, as in Equation (2.20), but with a slightly larger constant K . In addition, for any point $x \notin \mathcal{B}_j$, $Q_j(x)$ is a cube disjoint from all the elements $\mathcal{B}'_j$, so the barrier property also gets transferred to the cover $\mathcal{B}_j$. We thus proceed to construct the cover $\mathcal{B}'_j$.

Following [73], we call a cube Q at level j **k -naughty** if and only if Q intersects more than $\eta 2^{(k-j)(n+2-4\alpha+2\epsilon)}$ elements of $\mathcal{C}_k$, where η is a parameter to be chosen later in this proof. We remark here that, for sufficiently large k , there are in fact no k -naughty cubes because there are only at most $K 2^{k(n+2-4\alpha+\epsilon)}$ cubes in the entire collection $\mathcal{C}_k$. We call a cube Q at level j **naughty** if and only if it is k -naughty for all $k \geq j$. As before, we remark that this imposes only finitely many conditions on the cube Q .

We construct a cover $\mathcal{B}_{j,k}$ of all the k -naughty cubes at level j such that

$$N(\mathcal{B}_{j,k}) \leq K \eta^{-1} 2^{j(n+2-4\alpha+\epsilon)} 2^{\epsilon(j-k)}.$$

Let Q^1 be a k -naughty cube at level j and given the k -naughty cubes $Q^1, \dots, Q^l$, let Q^{l+1} be a k -naughty cube at level j disjoint from $3Q^i$ for all $i \in \{1, \dots, l\}$. Then Q^{l+1} must intersect at least $\eta 2^{(k-j)(n+2-4\alpha+2\epsilon)}$ elements of $\mathcal{C}_k$ that are disjoint from the elements of $\mathcal{C}_k$ intersected by the cubes Q^i for $i \in \{1, \dots, l\}$. Since we have a definite bound on the number of cubes in $\mathcal{C}_k$, we see that this iterative process terminates after

$$L \leq N(\mathcal{C}_k) / (\eta 2^{(k-j)(n+2-4\alpha+2\epsilon)}) \leq K \eta^{-1} 2^{(n+2-4\alpha)j} 2^{\epsilon(j-k)}$$

steps. In particular the family $\{3Q^1, \dots, 3Q^L\}$ covers the k -naughty cubes at level j . Covering each dilated cube $3Q^i$ by at most 4^n cubes at level j gives us our desired cover.

Next we let

$$\mathcal{B}'_j := \cup_{k \geq j} \mathcal{B}_{j,k},$$

which covers the set M_j , because $\mathcal{B}_{j,j}$ covers all the j -naughty cubes at level j . Let Q be a cube disjoint from all elements of $\mathcal{B}'_j$. Recall that we want to find some $0 < r < 1$ such that $\partial(rQ)$ does not intersect any bad cube at level k for any $k \geq j$. Let $C^k(Q)$ be the collection of cubes Q' from $\mathcal{C}_k$ intersecting Q . Since Q is not naughty, we have

$$N(C^k(Q)) \leq \eta 2^{(k-j)(n+2-4\alpha+2\epsilon)}.$$

Now let Q' be an arbitrary element of $\mathcal{C}_k$ that intersects $\partial(rQ)$ with $2^{-(1-\epsilon)} < r < 1$. Since $r < 1$, we necessarily have that $Q' \in C^k(Q)$, and since $r > 2^{-(1-\epsilon)}$, we necessarily have that the center of Q' is different than the center of Q (this is a requirement on r that is present but unexplained in [57] while omitted in [73]). Now we may let $r_{Q'}$ be such that $\partial(r_{Q'}Q)$ contains the center of Q' , so by the geometric Lemma 2.1.4 (with $2a = 2^{-j(1-\epsilon)}$ and $2b = 2^{-k(1-\epsilon)}$), we have

$$\partial(rQ) \cap Q' \neq \emptyset \implies r \in [r_{Q'} - 2^{(1-\epsilon)(j-k)}, r_{Q'} + 2^{(1-\epsilon)(j-k)}].$$

Thus if $f_k(r)$ is the number of bad cubes at level k that intersect $\partial(rQ)$, we may estimate

$$f_k(r) \leq \sum_{Q' \in C^k(Q)} \chi_{[r_{Q'} - 2^{(1-\epsilon)(j-k)}, r_{Q'} + 2^{(1-\epsilon)(j-k)}]}(r).$$

Integrating we have

$$\|f_k(r)\|_{L^1(2^{-(1-\epsilon)}, 1)} \leq 2N(C^k(Q))2^{(1-\epsilon)(j-k)} \leq 2\eta 2^{(4\alpha-n-1-3\epsilon)(j-k)}.$$

Denoting $\sum_{k \geq j} f_k := f$, we conclude, by using that $\alpha > (n+1)/4$ (choosing $0 < \epsilon < (4\alpha - n - 1)/3$) and summing, that

$$\|f\|_{L^1((2^{-(1-\epsilon)}, 1))} \leq c\eta.$$

By choosing η small enough we can use the fact that f takes only integer values to guarantee the existence of $r \in (2^{-(1-\epsilon)}, 1)$ such that $f(r) = 0$. This number r is the number desired so that $\mathcal{B}'_j$ has the barrier property. Indeed, by Chebyshev's inequality and assuming η sufficiently small (here $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$)

$$m(\{r \in (2^{-(1-\epsilon)}, 1) : f(r) \geq 1\}) \leq \int_{2^{-(1-\epsilon)}}^1 |f| dr < \frac{1}{1000}(1 - 2^{-(1-\epsilon)}),$$

so there certainly exists some $r \in (2^{-(1-\epsilon)}, 1)$, with the desired property. $\square$

We let

$$E := \limsup_{j \rightarrow \infty} \mathcal{B}_j$$

and our goal is to show

$$S_T \subset E. \tag{2.21}$$

By the bound in Equation (2.20) and Lemma 2.4.21, we know that $\mathcal{H}(E) \leq n + 2 - 4\alpha + \epsilon$, which combined with Equation (2.21) and the arbitrary smallness of ϵ , will finish the proof of Theorem 1.2.3.

2.1.4 Regularity Away from Naughty Cubes

We prove Equation (2.21). Our partial regularity result requires the use of the barrier in an essential way. The barrier construction allows the critical regularity proven in Proposition 2.1.3 to propagate throughout an open neighborhood of every point $x \notin E$, which is essential for showing regularity within an open set (in the sense of $u(x, t)$ having two continuous spatial derivatives).

Theorem 2.1.6. *Let $x \notin E$. There exists $j_1(x) \in \mathbb{Z}$ such that*

$$u_Q(t) < 2^{-j\rho(Q)/2}$$

for all times $t \in [0, T]$ and for every cube $Q \subset rQ_{j_1}(x)$ at any level j . Here $r := r(x, j_1) \in (2^{-(1-\epsilon)}, 1)$ is the number found in the proof of Lemma 2.1.5,

$$\rho(Q) := n + 2 - 4\alpha + \min(2n + 7, \epsilon\delta(Q)/10),$$

and $\delta(Q)$ denotes the smallest integer k such that Q_{j-k} intersects $\partial(rQ_{j_1}(x))$.

Proof. First we observe that $x \notin E$ implies that there exists some integer $j_2 \geq 0$ such that $x \notin Q$ for any $Q \in \mathcal{B}_j$ and for any $j \geq j_2$. Recall that j_0 is the integer found in the proof of Proposition 2.1.3. Suppose the claim of the Theorem is not true, then for any $j_1 \geq (\max(j_0, j_2) + 10)/\delta^2$ (we assume that j_1 is much larger than j_0 in order to apply Proposition 2.1.3 on dilations of cubes) we can let $t_0 > 0$ be the first time such that

$$u_{Q'}(t) \leq 2^{-k\rho(Q')/2} \tag{2.22}$$

for all $t \in [0, t_0]$ and all cubes $Q' \subset rQ_{j_1}(x)$ at any level k while there exists some j and some cube $Q \subset rQ_{j_1}$ at level j such that

$$u_Q(t_0) > 2^{-j\rho(Q)/2}. \tag{2.23}$$

That Equation (2.23) holds follows from continuity of the energy integral proven in Lemma 2.4.24, also compare with Equation (3.32) in [73]. Once again, let t_1 be the largest time such that $u_Q(t_1) =$

$\frac{1}{2}2^{-j\rho(Q)/2}$, so that

$$\frac{1}{2}2^{-j\rho(Q)/2} \leq u_Q(t) \leq 2^{-j\rho(Q)/2} \quad (2.24)$$

for all $t \in [t_1, t_0]$. Recall that our estimate in Equation (2.8) states that for any $\delta > 0$:

$$\begin{aligned} & \left| \langle -\hat{B}(u, u), P_j \phi_{Q,j}^2 P_j u \rangle \right| \leq K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q + \\ & + K \sum_{k=\delta j}^{j-10} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_{Q'} + K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_{Q'} \|\phi_{Q_j,j} P_k u\|_2^2 + \\ & + K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 + K 2^{-150j}. \end{aligned}$$

We thus conclude from Equation (2.6) and Equation (2.7) that

$$\begin{aligned} \frac{3}{4}2^{-j\rho(Q)} &= u_Q^2(t_0) - u_Q^2(t_1) \leq \int_{t_1}^{t_0} \left(K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q + \right. \\ & + K \sum_{k=\delta j}^{j-10} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 + \\ & \left. + K 2^{\frac{(n+2)j}{2}+\frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 \right) dt - \int_{t_1}^{t_0} K 2^{2\alpha j} u_Q^2 dt + \\ & + \int_{t_1}^{t_0} K 2^{(2\alpha-\epsilon)j} u_{\mathcal{N}^1(Q)}^2 + K 2^{-150j}. \end{aligned} \quad (2.25)$$

We first claim that $\delta(Q) \geq 11$. Otherwise if $\delta(Q) \leq 10$ (which implies $\rho(Q) \leq n+2-4\alpha+\epsilon$), then there exists some $k_0 \leq 10$ such that the cube Q_{j-k_0} intersects the barrier $\partial(rQ_{j_1}(x))$. It follows then that the larger cubes Q_k also intersect the barrier for all $k \in [\delta j, j-10]$. By Proposition 2.1.3, and since $\delta j > j_0$, we conclude that

$$u_Q(t_0) < 2^{-j(n+2-4\alpha+\epsilon)/2} \leq 2^{-j\rho(Q)/2},$$

which contradicts Equation (2.23). We take note of one consequence of the fact that $\delta(Q) > 10$

separately in

$$\rho(Q) \geq n + 2 - 4\alpha + \epsilon. \quad (2.26)$$

Having shown that the cube Q is necessarily “far” from the barrier, we proceed to show a contradiction to our contradiction hypothesis. First we show that large dilations of Q have critical regularity, or that:

$$u_{Q_k}(t) < K2^{-(n+2-4\alpha+\epsilon)k/2} \quad (2.27)$$

for all times $t < t_0$ and for all $k \in [\delta j, j - 5]$. If $\delta(Q_k) \geq 11$, then $Q_k \subset rQ_{j_1}(x)$ and $\rho(Q_k) \geq n + 2 - 4\alpha + \epsilon$. Indeed, in this case, the smallest dilation of Q_k that still intersects the barrier is at worst Q_{k-11} , so the original cube Q_k remains within the cube rQ_{j_1} . But then by Equation (2.22) we conclude, when $\delta(Q_k) \geq 11$:

$$u_{Q_k}(t) < 2^{-k\rho(Q_k)/2} < K2^{-(n+2-4\alpha+\epsilon)k/2}.$$

In the case when $\delta(Q_k) \leq 10$, we argue as before and get that the larger cubes Q_{l-10} intersect the barrier for all $l \in [\delta k, k]$, so by Proposition 2.1.3, and since $k \geq \delta j > j_0$, we obtain the desired critical regularity.

Next we show that cut-offs at high frequency have better than critical regularity. Let $k \geq j + 10(2n + 7)/\epsilon$ be arbitrary, we aim to show that

$$\|\phi_{Q_j,j}P_k u\|_2 < K2^{-nj}2^{-(n+9-4\alpha)k/2} \quad (2.28)$$

for all times $t < t_0$. Recall that for $k \geq j$, $Q_k := (1+2^{-\epsilon k})Q$, so, by also choosing j_1 sufficiently large (i.e. $j_1 > 1/\epsilon$), we have that $Q_j \subset (3/2)Q$. We may thus cover $(3/2)Q$ by at most $K2^{n(k-j)(1-\epsilon)}$ cubes at level k (we denote the cover by $S_k(3Q/2)$). Therefore we estimate:

$$\|\phi_{Q_j,j}P_k u\|_2 \leq K \sum_{Q' \in S_k(3Q/2)} u_{Q'}.$$

Each $Q' \in S_k(3Q/2)$ is a cube at level $k \geq j + 10(2n + 7)/\epsilon$, so $Q'_j = 2^{-(j-k)(1-\epsilon)}Q' \subset Q_{j-2}$, and

in particular $\delta(Q') = \delta(Q'_j) + k - j \geq \delta(Q_{j-2}) + 10(2n+7)/\epsilon > 10(2n+7)/\epsilon$. It follows from the definition that $\rho(Q') \geq 3n+9-4\alpha$. Then we conclude:

$$\|\phi_{Q_j,j}P_k u\|_2 \leq N(S_k(3Q/2))2^{-(3n+9-4\alpha)k/2} \leq K2^{-nj}2^{\epsilon n(j-k)}2^{-(n+9-4\alpha)k/2} \leq 2^{-nj}2^{-(n+9-4\alpha)k/2}.$$

For the local frequencies, we would like to show for all $t < t_0$ that for all $k \in (j-5, j+10(2n+7)/\epsilon]$, we have

$$\|\phi_{Q_j,j}P_k u\|_{L^2} \leq K2^{-j(\rho(Q)-\frac{2\epsilon}{5})/2}. \quad (2.29)$$

Let j_1 be sufficiently large so that $Q_j \subset (3/2)Q$. Let $k \in (j-5, j+10(2n+7)/\epsilon]$, and let $S_k((3/2)Q)$ be a cover of $(3/2)Q$ by cubes Q' at level k . There are at most K such cubes Q' , where K depends only on ϵ and n . Moreover, by the fact that $\delta(Q) \geq 11$, we have

$$\delta(Q_k) \geq \delta(Q) + k - j \geq \delta(Q) - 4 \geq 7,$$

so, since $r > 2^{-(1-\epsilon)}$, we have $Q_k \subset rQ_{j_1}(x)$, $\rho(Q_k) \geq \rho(Q) - \frac{2\epsilon}{5}$, and Equation (2.22) applies. Thus, we can estimate, if $k < j$,

$$\|\phi_{Q_j,j}P_k u\|_{L^2} \leq u_{Q_k} \leq 2^{-\rho(Q_k)k/2} \leq 2^{-(\rho(Q)-\frac{2\epsilon}{5})k/2} \leq K2^{-(\rho(Q)-\frac{2\epsilon}{5})j/2}.$$

If $k \geq j$, then $Q'_j \subset Q_{j-2}$, so $\delta(Q') = \delta(Q'_j) + k - j \geq \delta(Q_{j-2}) = \delta(Q) - 2$, from which $\rho(Q') \geq \rho(Q) - \epsilon/5$. By Equation (2.22) we conclude

$$u_{Q'} \leq 2^{-\rho(Q')k/2} \leq 2^{-j(\rho(Q)-\epsilon/5)/2}.$$

From which we conclude

$$\|\phi_{Q_j,j}P_k u\|_{L^2} \leq K2^{-j(\rho(Q)-\epsilon/5)/2},$$

which completes the proof of Equation (2.29). Now we begin estimating the terms on the right-hand-side of Equation (2.25). Since $\alpha > (n+1)/4 \geq 1$, we can assume that δ, ϵ, γ are sufficiently

small so that

$$1 + \frac{n\delta}{2} + \frac{n\gamma}{3} + \epsilon < 2 < 2\alpha.$$

Then, by Equation (2.24) and our estimate on the local frequencies in Equation (2.29), we have

$$\int_{t_1}^{t_0} K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q dt \leq K(t_0 - t_1) 2^{(2\alpha-\rho(Q)-4\epsilon/5)j}. \quad (2.30)$$

Likewise, by our estimate on the local frequencies in Equation (2.29), we have

$$K \int_{t_1}^{t_0} 2^{(2\alpha-\epsilon)j} u_{\mathcal{N}^1(Q)}^2 dt \leq K(t_0 - t_1) 2^{j(2\alpha-\rho(Q)-3\epsilon/5)}. \quad (2.31)$$

Similarly, by Equation (2.24), Equation (2.27), and Equation (2.29), we can estimate the low-high terms:

$$\begin{aligned} K \sum_{k=\delta j}^{j-10} \int_{t_1}^{t_0} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q dt &\leq \sum_{k=\delta j}^{j-10} \int_{t_1}^{t_0} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} 2^{-k(n+2-4\alpha+\epsilon)/2} 2^{j(-\rho(Q)+\frac{\epsilon}{5})} dt \leq \\ &\leq \sum_{k=\delta j}^{j-10} \int_{t_1}^{t_0} 2^{j+\frac{n\gamma j}{3}-\rho(Q)j+\frac{\epsilon j}{5}} 2^{-k(2-4\alpha+\epsilon)/2} dt \leq K(t_0 - t_1) 2^{j+\frac{n\gamma j}{3}-\rho(Q)j+\frac{\epsilon j}{5}} 2^{-j(2-4\alpha+\epsilon)/2}. \end{aligned}$$

Where we have summed over k after using that $\alpha > 1$. Simplifying we have

$$K(t_0 - t_1) 2^{j+\frac{n\gamma j}{3}-\rho(Q)j+\frac{\epsilon j}{5}} 2^{2\alpha j} 2^{-j(2+\epsilon)/2} \leq K(t_0 - t_1) 2^{(2\alpha-\rho(Q)-\frac{3\epsilon}{10}+\frac{n\gamma}{3})j}.$$

We can choose $\gamma < 3\epsilon/(100n)$ so that

$$K \sum_{k=\delta j}^{j-10} \int_{t_1}^{t_0} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q dt \leq K(t_0 - t_1) 2^{j(2\alpha-\rho(Q)-\frac{\epsilon}{10})j}. \quad (2.32)$$

We estimate the high-high terms as follows:

$$\int_{t_1}^{t_0} K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 \leq$$

$$\begin{aligned}
& \int_{t_1}^{t_0} K \sum_{k > j+10(2n+7)/\epsilon} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_{j,j}} P_k u\|_2^2 + \\
& + \int_{t_1}^{t_0} K \sum_{k > j+10}^{j+10(2n+7)/\epsilon} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_{j,j}} P_k u\|_2^2 \leq \\
& \leq K \int_{t_1}^{t_0} 2^{-\rho(Q)j/2} 2^{nj(1/2-2\gamma/3)} 2^{-2nj} \sum_{k > j+10} 2^{k(n\gamma+1-(n+9-4\alpha))} + \\
& K \int_{t_1}^{t_0} 2^{-\rho(Q)j/2} 2^{nj(1/2-2\gamma/3)} 2^{j(n\gamma+1-\rho(Q)+\frac{2\epsilon}{5})}.
\end{aligned}$$

Since $\alpha < (n+2)/4$, we can choose γ small enough (e.g. $\gamma < 6/n$) so that the infinite sum is convergent, yielding

$$\begin{aligned}
& \int_{t_1}^{t_0} K \sum_{k > j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_{j,j}} P_k u\|_2^2 \leq \\
& \leq K(t_0 - t_1) 2^{-\rho(Q)j/2} 2^{nj(1/2-2\gamma/3)} 2^{-2nj} 2^{j(n\gamma+1-(n+9-4\alpha))} + \\
& K(t_0 - t_1) 2^{-\rho(Q)j/2} 2^{nj(1/2-2\gamma/3)} 2^{j(n\gamma+1-\rho(Q)+\frac{2\epsilon}{5})}.
\end{aligned}$$

Recall that by Equation (2.26), we know $-\rho(Q) \leq -(n+2-4\alpha+\epsilon)$. It follows that (after choosing $\gamma < 3\epsilon/(100n)$):

$$K 2^{-\rho(Q)j/2} 2^{nj(1/2-2\gamma/3)} 2^{j(n\gamma+1-\rho(Q)+\frac{2\epsilon}{5})} \leq K 2^{(2\alpha-\rho(Q)-\epsilon/20)j}.$$

Recall that (by definition) $\rho(Q) \leq 3n+9-4\alpha$, so that (choosing $\gamma < 3/n$ and $\epsilon < n+5/2$)

$$\begin{aligned}
& K 2^{-\rho(Q)j/2} 2^{nj(1/2-2\gamma/3)} 2^{-2nj} 2^{j(n\gamma+1-(n+9-4\alpha))} \leq 2^{j(-\rho(Q)+\rho(Q)/2+n/2+n\gamma/3-2n+1-n-(9-4\alpha))j} \leq \\
& \leq K 2^{j(-\rho(Q)+3n/2+(9-4\alpha)/2-5n/2+n\gamma/3-(8-4\alpha))j} \leq K 2^{j(-\rho(Q)-n+2\alpha-7/2+n\gamma/3)} \leq K 2^{j(2\alpha-\rho(Q)-\epsilon)}.
\end{aligned}$$

We can now conclude that:

$$\int_{t_1}^{t_0} K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} u_Q \|\phi_{Q_j,j} P_k u\|_2^2 \leq K(t_0 - t_1) 2^{j(2\alpha-\epsilon/20-\rho(Q))}. \quad (2.33)$$

The estimate for the local terms proceeds as:

$$\begin{aligned} \int_{t_1}^{t_0} K 2^{\frac{(n+2)j}{2} + \frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 dt &\leq K \int_{t_1}^{t_0} 2^{j(-\rho(Q)-\rho(Q)/2 + \frac{n+2}{2} + \frac{n\gamma}{3} + \frac{2\epsilon}{5})} dt \leq \\ &\leq K \int_{t_1}^{t_0} 2^{j(-\rho(Q)-\frac{n}{2}-1+2\alpha-\frac{\epsilon}{2} + \frac{n+2}{2} + \frac{n\gamma}{3} + \frac{2\epsilon}{5})} dt \leq K(t_0 - t_1) 2^{j(2\alpha-\rho(Q)-\frac{\epsilon}{10} + \frac{n\gamma}{3})}, \end{aligned}$$

where we have used Equation (2.26). Choosing $\gamma < 3\epsilon/(100n)$, we conclude

$$\int_{t_1}^{t_0} K 2^{\frac{(n+2)j}{2} + \frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 dt \leq K(t_0 - t_1) 2^{(2\alpha-\rho(Q)-\epsilon/20)j}. \quad (2.34)$$

From Equation (2.30), Equation (2.31), Equation (2.32), Equation (2.33), and Equation (2.34), we know, choosing j_1 large enough and ϵ small enough:

$$\frac{3}{4} 2^{-j\rho(Q)} \leq K^{-150j} - K \int_{t_1}^{t_0} 2^{2\alpha j} u_Q^2 dt + K 2^{(2\alpha-\epsilon/20-\rho(Q))j}.$$

Since $u_Q \geq 2^{-\rho(Q)j/2}/2$ on the domain of time-integration, we can choose j_1 large enough so that

$$\frac{3}{4} 2^{-j\rho(Q)} \leq -K(t_0 - t_1) 2^{(2\alpha-\rho(Q))j} (1 - K 2^{-\epsilon j/20}) < 0$$

which implies $\frac{3}{4} < 0$, a contradiction. $\square$

Corollary 2.1.7. *Let $x \notin E$. There exists a neighborhood U of x such that $\|u(t)\|_{C^\infty(U)}$ is bounded for all $t \in [0, T]$.*

Proof. By changing the definition of $\rho(Q)$, we can get arbitrarily stronger control on the energy away, as long as we are far from the barrier, which would allow us to conclude arbitrarily high regularity in the proof that now follows. It thus suffices to show here that, for example, there exists

a neighborhood U of x such that $\|u(t)\|_{C^2(U)}$ is bounded for all $t \in [0, T]$. We have proven that there exists j_1 and r such that $u_Q(t) \leq 2^{-j\rho(Q)/2}$ for all $t \in [0, T]$ and for all cubes $Q \subset rQ_{j_1}(x)$ at level j . Let j_2 be the smallest integer such that $\delta(Q) \geq 10(2n+7)/\epsilon$ for all cubes $Q \subset Q_{j_2}$ at level j , so that $\rho(Q) \geq 3n+9-4\alpha$ for any such cube Q at level j . Let $U := Q_{j_2+2}(x)$. Let $S_j(U)$ be a cover of U by cubes at level j . Then for $j \geq j_2$:

$$\begin{aligned} \|P_j u\|_{C^2(U)} &\leq \|P_j u\|_{W^{3+n/2,2}(U)} \leq 2^{j(3+n/2)} \|P_j u\|_{L^2} \leq 2^{j(3+n/2)} \sum_{Q \in S_j(U)} \|\phi_Q P_j u\|_{L^2} \leq \\ &\leq K 2^{nj(1-\epsilon)} 2^{nj/2+3j} u_Q \leq 2^{(n(1-\epsilon)+n/2+3-\rho(Q))j/2} \leq K 2^{-j/2}. \end{aligned}$$

Thus, summing over j , we have

$$\|u\|_{C^2(U)} \leq K(j_2).$$

□

Corollary 2.1.7, the fact that $\mathcal{H}(E) \leq n+2-4\alpha+\epsilon$, and the arbitrary smallness of ϵ finishes the proof of Theorem 1.2.3.

2.2 Estimates for the Local Energy

2.2.1 Estimate for the Dissipation Term

We first consider the dissipation term in Equation (2.6).

Proposition 2.2.1. *Let Q be a cube at level $j \geq 0$. We have:*

$$\langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle \geq -K 2^{-200j} + K 2^{2\alpha j} u_Q^2 - K 2^{(2\alpha-\epsilon)j} u_{N^1(Q)}^2$$

Proof. We have:

$$\langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle = \langle (-\Delta)^\alpha \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle + \langle [(-\Delta)^\alpha, \phi_{Q,j} P_j] u, \phi_{Q,j} P_j u \rangle. \quad (2.35)$$

It remains to estimate the two terms on the right-hand-side of Equation (2.35). We begin by using Lemma 2.4.15 to state:

$$\left| \langle (-\Delta)^\alpha \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle - \langle (-\Delta)^\alpha \tilde{P}_j \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle \right| \leq K 2^{-400j} \quad (2.36)$$

and

$$\left| \langle \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle - \langle \tilde{P}_j \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle \right| \leq K 2^{-400j}. \quad (2.37)$$

Then, because of the localization in frequency space, we have

$$\begin{aligned} & \langle (-\Delta)^\alpha \tilde{P}_j \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle = \\ & = \int |\xi|^{2\alpha} \tilde{p}_j(\xi) |\mathcal{F}(\phi_{Q,j} P_j u)(\xi)|^2 d\xi \geq K 2^{2\alpha j} \langle \tilde{P}_j \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle. \end{aligned} \quad (2.38)$$

Using Equation (2.36), Equation (2.37), and Equation (2.38) we conclude that

$$\langle (-\Delta)^\alpha \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle \geq -K 2^{-200j} + 2^{2\alpha j} u_Q^2. \quad (2.39)$$

Now we turn to the second term on the right-hand-side of Equation (2.35).

First, by Lemma 2.4.4, we have $[(-\Delta)^\alpha, \phi_j P_j] \in OPS_{1,1-\epsilon}^{2\alpha-\epsilon}$. Thus, the symbol of this operator, which we denote by $q(x, \xi)$, satisfies

$$|q(x, \xi)| \leq K((1 + |\xi|^2)^{1/2})^{2\alpha-\epsilon}.$$

Because of this bound and the localization of frequency, we get, proceeding as above:

$$|\langle [(-\Delta)^\alpha, \phi_j P_j] u, \phi_{Q,j} P_j u \rangle| \leq K(2^{(2\alpha-\epsilon)j} + 1) |\langle u, \phi_{Q,j} P_j u \rangle| \leq K 2^{(2\alpha-\epsilon)j} |\langle u, \phi_{Q,j} P_j u \rangle|.$$

Letting $\tilde{Q} = (1 + 2^{-\epsilon j})Q$, we conclude using Lemma 2.4.15 (to commute Littlewood-Paley projections), Lemma 2.4.19 (to commute bump functions), the energy dissipation law, and the Cauchy-

Schwartz inequality that:

$$\begin{aligned}
|\langle [(-\Delta)^\alpha, \phi_j P_j] u, \phi_{Q,j} P_j u \rangle| &\leq K 2^{(2\alpha-\epsilon)j} \left| \langle u, \phi_{\tilde{Q},j} \tilde{P}_j \phi_{Q,j} P_j u \rangle \right| + K 2^{-150j} \leq \\
&\leq K 2^{(2\alpha-\epsilon)j} \|\phi_{\tilde{Q},j} \tilde{P}_j u\|_2 \cdot u_Q + K 2^{-150j} \leq K 2^{(2\alpha-\epsilon)j} \|\phi_{\tilde{Q},j} \tilde{P}_j u\|_2^2 + K 2^{-150j}.
\end{aligned}$$

Finally using Lemma 2.4.20, we have

$$|\langle [(-\Delta)^\alpha, \phi_j P_j] u, \phi_{Q,j} P_j u \rangle| \leq K 2^{(2\alpha-\epsilon)j} u_{\mathcal{N}^1(Q)}^2 + K 2^{-150j}. \quad (2.40)$$

Combining Equations (2.35), (2.39), and (2.40) yields the desired inequality. $\square$

2.2.2 Estimate for the Nonlinear Term

The next term we consider from Equation (2.6) is

$$\langle -\hat{B}(u, u), P_j \phi_{Q,j}^2 P_j u \rangle = \langle -\phi_{Q,j} P_j \hat{B}(u, u), \phi_{Q,j} P_j u \rangle.$$

We shall use a scheme for estimating multilinear functions that is inspired by the Bony-Coifman-Meyer paradifferential calculus (for the details of which, see [10], [31], and [91]). It involves the following partition of frequency space, which is also used by the authors of [57] and [73]. We may write

$$P_j \hat{B}(u, u) = H_{j,lh} + H_{j,hl} + H_{j,hh} + H_{j,loc}$$

where

$$\text{low-high frequencies: } H_{j,lh} = \sum_{k < j-10} P_j \hat{B}(P_k u, \tilde{P}_j u)$$

$$\text{high-low: } H_{j,hl} = \sum_{k < j-10} P_j \hat{B}(\tilde{P}_j u, P_k u)$$

$$\text{high-high: } H_{j,hh} = \sum_{k > j+10} P_j \hat{B}(\tilde{P}_k u, P_k u) + \sum_{k > j+10} P_j \hat{B}(P_k u, \tilde{P}_k u)$$

$$\text{local: } H_{j,loc} = \sum_{j-10 < k < j+10} P_j \hat{B}(\tilde{P}_k u, P_k u) + \sum_{j-10 < k < j+10} P_j \hat{B}(P_k u, \tilde{P}_k u).$$

For the rest of this subsection we fix a cube Q at level j and introduce the following notation:

$$k < j \Rightarrow Q_k := 2^{(j-k)(1-\epsilon)} Q, \quad k \geq j \Rightarrow Q_k = (1 + 2^{-\epsilon k}) Q.$$

We first estimate the low-high frequency terms:

Lemma 2.2.2. *There exists a constant $K > 0$ such that for any arbitrary $\delta > 0$ and $j \in \mathbb{Z}^+$ we have:*

$$|\langle -\phi_{Q,j} H_{j,lh}, \phi_{Q,j} P_j u \rangle| \leq K 2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q + K \sum_{k=\delta j}^{j-10} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + K 2^{-150j}.$$

Proof. We first deal with the case when $\delta j \leq k < j - 10$ and consider a term in the sum $H_{j,lh}$:

$$\langle -\phi_{Q,j} P_j \hat{B}(P_k u, \tilde{P}_j u), \phi_{Q,j} P_j u \rangle.$$

Now by an application of Lemma 2.4.15, we know that for any $f \in L^2$:

$$\|\phi_{Q,j} P_j f - \tilde{P}_j \phi_{Q,j} P_j f\|_2 \leq K 2^{-200j}. \quad (2.41)$$

Since B is an amenable bilinear operator, we have by assumption that $B_v^1(u) = \hat{B}(u, v)$ and $B_v^2(u) = \hat{B}(v, u)$ are pseudodifferential operators with symbols in some class $S_{1,1}^m$. Thus by an application of Lemma 2.4.11, also using Equation (2.41), we see:

$$\|\phi_{Q,j} P_j \hat{B}(P_k u, \tilde{P}_j u) - \phi_{Q,j} P_j \hat{B}(\phi_{Q_k,k} P_k u, \tilde{P}_j u)\|_2 \leq K 2^{-200j}. \quad (2.42)$$

We may therefore write

$$\langle -\phi_{Q,j} P_j \hat{B}(P_k u, \tilde{P}_j u), \phi_{Q,j} P_j u \rangle = \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q_k,k} P_k u, \tilde{P}_j u), \phi_{Q,j} P_j u \rangle + R_{j,k}$$

where $R_{j,k}$ is an error term with absolute value less than $K2^{-195j}$ for all k . In a similar fashion, we can commute bump functions to get:

$$\begin{aligned} & \left| \langle -\phi_{Q,j} P_j \hat{B}(P_k u, \tilde{P}_j u), \phi_{Q,j} P_j u \rangle \right| \leq \\ & \left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q_k,k} P_k u, \phi_{Q_j,j} \tilde{P}_j u), \phi_{Q,j} P_j u \rangle \right| + K2^{-180j}. \end{aligned} \quad (2.43)$$

We therefore have the estimate:

$$\left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q_k,k} P_k u, \phi_{Q_j,j} \tilde{P}_j u), \phi_{Q,j} P_j u \rangle \right| \leq K \|\hat{B}(\phi_{Q_k,k} P_k u, \phi_{Q_j,j} \tilde{P}_j u)\|_2 \cdot u_Q.$$

Since B is an amenable bilinear operator, we use the property in Equation (1.6) and get

$$\begin{aligned} & \left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q_k,k} P_k u, \phi_{Q_j,j} \tilde{P}_j u), \phi_{Q,j} P_j u \rangle \right| \leq \\ & u_Q \cdot \|\phi_{Q_k,k} P_k u\|_\infty (\|\nabla \phi_{Q_j,j} \tilde{P}_j u\|_{q(\gamma)} + \|\phi_{Q_j,j} \tilde{P}_j u\|_{q(\gamma)}), \end{aligned} \quad (2.44)$$

where $q(\gamma) := 6/(3-2\gamma)$. For the rest of this proof we refer to $q(\gamma)$ by q only. Now, we use Lemma 2.4.18 to get:

$$\|\phi_{Q_k,k} P_k u\|_\infty \leq K2^{nk/2} u_{Q_k} + K2^{-(250/\delta)k} \leq K2^{nk/2} u_{Q_k} + K2^{-250j} \quad (2.45)$$

$$\|\phi_{Q_j,j} \tilde{P}_j u\|_q \leq K2^{nj(1/2-1/q)} u_{\mathcal{N}^1(Q)} + K2^{-250j} = K2^{\frac{n\gamma j}{3}} u_{\mathcal{N}^1(Q)} + K2^{-250j}. \quad (2.46)$$

We recall that derivatives of a Sobolev-smoothing operator remain Sobolev smoothing, so we may use the “commutator” lemmas of the final section of this chapter “within derivatives” as well. Now, to deal with the gradient term, we use $\tilde{P}_j P_j = P_j$ and Lemma 2.4.15 to commute Littlewood-Paley projections and get:

$$\nabla(\phi_{Q_j,j} \tilde{P}_j u) = \nabla(P_j \phi_{Q_j,j} \tilde{P}_j u) + R_j = P_j \nabla(\phi_{Q_j,j} \tilde{P}_j u) + R_j,$$

where R_j is an error term of absolute value at most $K2^{-300j}$. By Lemma 2.4.13, we thus have

$$\|\nabla(\phi_{Q,j}\tilde{P}_j u)\|_q \leq 2^{j+\frac{n\gamma j}{3}} u_{\mathcal{N}^1(Q)} + K2^{-200j}. \quad (2.47)$$

Combining our estimates in Equations (2.43), (2.44), (2.45), (2.46), (2.47), we get:

$$\left| \langle -\phi_{Q,j} P_j \hat{B}(P_k u, \tilde{P}_j u), \phi_{Q,j} P_j u \rangle \right| \leq 2^{nk/2+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + K2^{-150j}.$$

Summing over $\delta j \leq k < j - 10$ and recognizing that there are only around j terms gets us the first term with the sum. For the case when $k < \delta j$, we directly use the inequality satisfied by $\hat{B}$, our commutator estimates from the final section, and the energy dissipation law to get:

$$\begin{aligned} & \left| \langle -\phi_{Q,j} P_j \hat{B}(P_k u, \tilde{P}_j u), \phi_{Q,j} P_j u \rangle \right| \leq \\ & \leq 2^j u_Q \|P_k u\|_\infty \|\phi_{Q,j} \tilde{P}_j u\|_q + K2^{-150j} \leq K2^{j+n\delta j/2+\frac{n\gamma j}{3}} u_Q u_{\mathcal{N}^1(Q)} + K2^{-150j}, \end{aligned}$$

which finishes the proof. $\square$

By symmetry of $\hat{B}$, the same result holds for the high-low terms:

Lemma 2.2.3. *There exists a constant $K > 0$ such that for any arbitrary $\delta > 0$ and $j \in \mathbb{Z}^+$ we have:*

$$\begin{aligned} & |\langle -\phi_{Q,j} H_{j,h\ell}, \phi_{Q,j} P_j u \rangle| \leq \\ & K2^{j(1+\frac{n\delta}{2}+\frac{n\gamma}{3})} u_{\mathcal{N}^1(Q)} u_Q + K \sum_{k=\delta j}^{j-10} 2^{\frac{nk}{2}+j+\frac{n\gamma j}{3}} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + K2^{-150j}. \end{aligned}$$

Our next effort is towards estimating the high-high frequency terms.

Lemma 2.2.4. *For some constant K and for all $j \in \mathbb{Z}^+$ we have*

$$|\langle \phi_{Q,j} H_{j,h\ell}, \phi_{Q,j} P_j u \rangle| \leq K \sum_{k>j+10} 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+\gamma j+k(n\gamma+1)} u_Q \|\phi_{Q,j} P_k u\|_2^2 + K2^{-150j}.$$

Proof. As usual, we examine and wish to estimate a single term in the high-high sum where $k > j + 10$, which by symmetry we may write:

$$\langle \phi_{Q,j} P_j \hat{B}(\tilde{P}_k u, P_k u), \phi_{Q,j} P_j u \rangle.$$

As in the proof of Lemma 2.2.2, we can commute bump functions across our operators to get

$$\begin{aligned} & \left| \langle \phi_{Q,j} P_j \hat{B}(\tilde{P}_k u, P_k u), \phi_{Q,j} P_j u \rangle \right| \leq \\ & K \left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q,j} \tilde{P}_k u, \phi_{Q,j} P_k u), \phi_{Q,j} P_j u \rangle \right| + K 2^{-200j}. \end{aligned} \quad (2.48)$$

Since B is amenable we can bound the trilinear expression:

$$\begin{aligned} & \left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q,j} \tilde{P}_k u, \phi_{Q,j} P_k u), \phi_{Q,j} P_j u \rangle \right| \leq \\ & (\|\phi_{Q,j} \tilde{P}_k u\|_{p_1} + \|\nabla \phi_{Q,j} \tilde{P}_k u\|_{p_1}) \|\phi_{Q,j} P_k u\|_{p_2} \|\phi_{Q,j} P_j u\|_r \end{aligned}$$

where

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{\gamma}{3} + \frac{1}{r} = 1.$$

Henceforth, we can always assume that $0 < \gamma < 1/2$. With this assumption, we can pick p_1, p_2, r so that

$$\frac{1}{p_1} = \frac{1}{2}, \quad \frac{1}{p_2} = \frac{1}{2} - \gamma, \quad \frac{1}{r} = \frac{2\gamma}{3}$$

By Lemma 2.4.18 we know that

$$\|\phi_{Q,j} P_j u\|_r \leq 2^{nj(\frac{1}{2} - \frac{2\gamma}{3})} u_Q + K 2^{-200j} \quad (2.49)$$

and

$$\|\phi_{Q,j} P_k u\|_{p_2} \leq 2^{nk\gamma} \|\phi_{Q,j} P_k\|_2 + K 2^{-200j}. \quad (2.50)$$

By the same reasoning as in the proof of Lemma 2.2.2, we get

$$\|\nabla \phi_{Q_j,j} P_k u\|_2 \leq K 2^k \sum_{l=-2}^{l=2} \|\phi_{Q_j,j} P_{k+l} u\|_2 \quad (2.51)$$

Combining our estimates from Equations (2.48), (2.49), (2.50), (2.51) yields:

$$|\langle \phi_{Q,j} H_{j,hh}, \phi_{Q,j} P_j u \rangle| \leq K \sum_{k>j+10} u_Q 2^{nj(\frac{1}{2}-\frac{2\gamma}{3})+k(n\gamma+1)} \|\phi_{Q_j,j} P_k u\|_2^2 + K 2^{-150j}.$$

□

Next we deal with the “local” frequencies around level j :

Lemma 2.2.5. *For some constant K we have for all $j \in \mathbb{Z}^+$:*

$$|\langle \phi_{Q,j} H_{j,loc}, \phi_{Q,j} P_j u \rangle| \leq K 2^{\frac{(n+2)j}{2} + \frac{n\gamma j}{3}} u_Q \|\phi_{Q_j,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 + K 2^{-150j}.$$

Proof. We can explicitly write out the term we wish to bound by

$$\langle \phi_{Q,j} H_{j,loc}, \phi_{Q,j} P_j u \rangle = 2 \sum_{l=-2}^{l=2} \sum_{j-10 < k}^{k < j+10} \langle \phi_{Q,j} P_j \hat{B}(P_{k+l} u, P_k u), \phi_{Q,j} P_j u \rangle.$$

As in the proof of Lemma 2.2.2, we can use the triangle inequality and commute bump functions to get:

$$|\langle \phi_{Q,j} H_{j,loc}, \phi_{Q,j} P_j u \rangle| \leq K 2^{-200j} + K \sum_{l=-2}^{l=2} \sum_{j-10 < k}^{k < j+10} \left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q_j,j} P_{k+l} u, \phi_{Q_j,j} P_k u), \phi_{Q,j} P_j u \rangle \right| \quad (2.52)$$

For appropriate choices of $k = k_0, l = l_0$ that maximize the expression in the sum in Equation (2.52), and since our constants K are allowed to depend on ϵ , we have:

$$K \sum_{l=-2}^{l=2} \sum_{j-10 < k < j+10} \left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q_j,j} P_{k+l} u, \phi_{Q_j,j} P_k u), \phi_{Q,j} P_j u \rangle \right| \leq$$

$$\leq K \left| \langle \phi_{Q,j} P_j \hat{B}(\phi_{Q,j} P_{k_0+l_0} u, \phi_{Q,j} P_{k_0} u), \phi_{Q,j} P_j u \rangle \right| \quad (2.53)$$

We remark that by Lemma 2.4.20:

$$\|\phi_{Q,j} P_k u\|_2 \leq \|\phi_{Q,j} P_{j-10 \leq k \leq j+10}\|_{L^2} \quad (2.54)$$

and by Lemma 2.4.18:

$$\|\phi_{Q,j} P_j u\|_\infty \leq 2^{nj/2} u_Q + K 2^{-200j}. \quad (2.55)$$

As in the proof of Lemma 2.2.2, we have, where $q = 6/(3 - 2\gamma)$:

$$\|\nabla(\phi_{Q,j} P_{k_0+l_0} u)\|_q \leq 2^{j+\frac{n\gamma j}{3}} \|\phi_{Q,j} P_{j-10 \leq k \leq j+10}\|_{L^2} + K 2^{-200j}. \quad (2.56)$$

Combining our estimates from Equations (2.52), (2.53), (2.54), (2.55), (2.56), we get:

$$|\langle \phi_{Q,j} H_{j,loc}, \phi_{Q,j} P_j u \rangle| \leq K 2^{\frac{(n+2)j}{2} + \frac{n\gamma j}{3}} u_Q \|\phi_{Q,j} P_{j-10 \leq k \leq j+10} u\|_{L^2}^2 + K 2^{-150j}.$$

□

2.3 Partial Regularity and Blowup

In this section we deal only with functions and vector fields defined on $\mathbb{R}^3$. For this reason, the parameter n does not denote the dimension of space but rather an index of summation.

2.3.1 Flexibility of Blowup

We shall assume that the reader has a familiarity with the paper [89] of Tao. We first give an overview of the construction in [89]. Afterwards, we show the flexibility of the blowup result therein. Indeed, Tao notes himself (in footnote 12 of [89]), that his methods show blowup even when hyperdissipation approaches the critical value of $\alpha = 5/4$. In this subsection we give some details showing why this is the case while also modifying the technique to prove the existence of a

blowup solution to the pseudodifferential equation associated to some amenable operator.

As before, we let $\langle \cdot, \cdot \rangle$ be the L^2 pairing for vector fields. Let $1 < \lambda < 2$. We denote by $H_{df}^{10}(\mathbb{R}^3)$ the space of vector fields on $\mathbb{R}^3$ that are divergence free in the distributional sense and whose first ten weak derivatives are square integrable.

A basic local cascade operator is a bilinear operator $C : H_{df}^{10}(\mathbb{R}^3) \times H_{df}^{10}(\mathbb{R}^3) \rightarrow H_{df}^{10}(\mathbb{R}^3)^*$ defined via duality by:

$$\langle C(u, v), w \rangle = \sum_{n \in \mathbb{Z}} \lambda^{5n/2} \langle u, \psi_{1,n} \rangle \langle v, \psi_{2,n} \rangle \langle w, \psi_{3,n} \rangle$$

where for $i \in \{1, 2, 3\}$,

$$\psi_{i,n}(x) = \lambda^{3n/2} \psi_i(\lambda^n x)$$

and ψ_i is a Schwartz vector field whose Fourier transform is compactly supported within a small spherical shell around the unit sphere in $\mathbb{R}^3$. We also define a local cascade operator to be a finite linear combination of basic local cascade operators. Lastly, we call a basic local cascade operator a zero-momentum basic local cascade operator if its constituent Schwartz vector fields satisfy

$$\int_{\mathbb{R}^3} \psi_i dx = 0.$$

Likewise, we may define a zero-momentum local cascade operator.

In [89], Tao proves the existence of a local cascade equation that admits a solution blowing up in finite time. In particular, Tao proves the following:

Theorem 2.3.1 ([89]). *Let $1 < \lambda < 2$ be arbitrary. Then there exists a symmetric local cascade operator C and a Schwartz divergence-free vector field u_0 such that the cancellation identity holds*

$$\langle C(u, u), u \rangle = 0 \text{ for all } u \in H_{df}^{10}(\mathbb{R}^3) \quad (2.57)$$

and there does not exist any global mild solution $u : [0, \infty) \rightarrow H_{df}^{10}(\mathbb{R}^3)$ to the initial value problem

$$\begin{aligned}\partial_t u - \Delta u + C(u, u) &= 0 \\ u(\cdot, t=0) &= u_0\end{aligned}\tag{2.58}$$

In this subsection we show that Tao's result is flexible in the sense that we find a local cascade operator with more restraints on the constituent Schwartz vector fields ψ_i whose corresponding local cascade equation admits a blowup solution with fractional dissipation. In particular we prove:

Theorem 2.3.2. *Let $1 < \lambda < 2$ and $0 < \alpha < 5/4$ be arbitrary. Then there exists a zero-momentum symmetric local cascade operator C and a Schwartz divergence-free vector field u_0 such that the cancellation identity holds*

$$\langle C(u, u), u \rangle = 0 \text{ for all } u \in H_{df}^{10}(\mathbb{R}^3)\tag{2.59}$$

and there does not exist any global mild solution $u : [0, \infty) \rightarrow H_{df}^{10}(\mathbb{R}^3)$ to the initial value problem

$$\begin{aligned}\partial_t u + (-\Delta)^\alpha u + C(u, u) &= 0 \\ u(\cdot, t=0) &= u_0\end{aligned}\tag{2.60}$$

First, we determine the structure our local cascade operator must have in order to fulfill some basic requirements like the cancellation property. In particular, following Tao, we consider four balls B_1, B_2, B_3, B_4 in the spherical shell $\{\xi : 1 < |\xi| \leq \frac{\lambda+1}{2}\}$, and we choose the balls so that the unions $B_i \cup (-B_i)$ are mutually disjoint for $i \in \{1, 2, 3, 4\}$. We choose four zero-momentum divergence-free Schwartz vector fields ψ_i such that $\mathcal{F}(\psi_i)$ has support in $B_i \cup -B_i$, and we normalize them so that $\|\psi_i\|_2 = 1$. We have the L^2 -rescaled functions $\psi_{i,n} := \lambda^{3n/2} \psi_i(\lambda^n x)$ that also satisfy $\|\psi_{i,n}\|_2 = 1$. Let $S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. We define the following local cascade operator (in other words, a linear combination of basic local cascade operators):

$$C(u, v) := \sum_{n \in \mathbb{Z}} \sum_{\substack{i_1, i_2, i_3 \in \{1, 2, 3, 4\} \\ (\mu_1, \mu_2, \mu_3) \in S}} a_{i_1, i_2, i_3, \mu_1, \mu_2, \mu_3} \lambda^{5n/2} \langle u, \psi_{i_1, n+\mu_1} \rangle \langle v, \psi_{i_2, n+\mu_2} \rangle \psi_{i_3, n+\mu_3}$$

To ensure that C is a symmetric bilinear operator we require that

$$a_{i_1, i_2, i_3, \mu_1, \mu_2, \mu_3} = a_{i_2, i_1, i_3, \mu_2, \mu_1, \mu_3}.$$

The cancellation property is satisfied if we require

$$\sum_{\{a, b, c\}=\{1, 2, 3\}} a_{i_a, i_b, i_c, \mu_a, \mu_b, \mu_c} = 0.$$

With these first conditions imposed on C , we are ready to study the basic dynamics of the pseudodifferential equation associated to C . Whenever applicable, we define the following Fourier projections that will act as “wavelets”:

$$u_{i,n}(t)(x) := \mathcal{F}^{-1}(\chi_{\xi \in \lambda^n(B_i \cup -B_i)}(\xi) \cdot \mathcal{F}(u(t))(\xi))$$

and the following functions of time that behave like “wavelet coefficients”:

$$X_{i,n}(t) := \langle u(t), \psi_{i,n} \rangle = \langle u_{i,n}(t), \psi_{i,n} \rangle.$$

Lastly, we define the following “local energies”:

$$E_{i,n}(t) := \frac{1}{2} \|u_{i,n}(t)\|_2^2.$$

The behavior of the pseudodifferential equation is determined by the dynamics of the functions $X_{i,n}(t)$. This is described in Lemma 4.1 of [89]. Below, we only note the modifications necessary in the case of fractional dissipation (assuming that ψ is a zero-momentum vector field has no effect on the lemma’s conclusions).

Lemma 2.3.3. *Suppose u is a global mild solution to the pseudodifferential equation associated to*

C with initial data ψ_{1,n_0} for some n_0 integer. For any i, n we have

$$\partial_t X_{i,n} = \sum_{i_1, i_2 \in \{1,2,3,4\}} \sum_{(\mu_1, \mu_2, \mu_3) \in S} a_{i_1, i_2, i, \mu_1, \mu_2, \mu_3} \lambda^{5(n-\mu_3)/2} X_{i_1, n-\mu_3+\mu_1} X_{i_2, n-\mu_3+\mu_2} + O(\lambda^{2\alpha n} E_{i,n}^{1/2})$$

and

$$\frac{1}{2} X_{i,n}^2(t) \leq E_{i,n}(t) \leq \frac{1}{2} X_{i,n}^2(t) + O\left(\lambda^{2\alpha n} \int_0^t E_{i,n}(s) ds\right).$$

Proof. The vector field $u(t)$ is a mild solution of the pseudodifferential equation associated to C .

Thus we write

$$u(t) = e^{-t(-\Delta)^\alpha} u_0 + \int_0^t e^{(s-t)(-\Delta)^\alpha} C(u(s), u(s)) ds. \quad (2.61)$$

As noted before, we have the “wavelet” decomposition of u :

$$u(t) = \sum_{n,i} u_{i,n}(t)$$

with

$$X_{i,n}(t) = \langle u(t), \psi_{i,n} \rangle = \langle u_{i,n}(t), \psi_{i,n} \rangle.$$

We pair Equation (2.61) with $\psi_{i,n}$ and get

$$u_{i,n}(t) = e^{-t(-\Delta)^\alpha} X_{i,n}(0) \psi_{i,n} + \psi_{i,n} \left(\sum_{i_1, i_2 \in \{1,2,3,4\}, (\mu_1, \mu_2, \mu_3) \in S} a_{i_1, i_2, i, \mu_1, \mu_2, \mu_3} \lambda^{5(n-\mu_3)/2} \int_0^t \langle u, \psi_{i_1, n+\mu_1-\mu_3} \rangle \langle u, \psi_{i_2, n+\mu_2-\mu_3} \rangle ds \right).$$

If we differentiate with respect to the time variable t and simplify we obtain

$$\begin{aligned} \partial_t u_{i,n} &= -(-\Delta)^\alpha u_{i,n} + \\ &\sum_{i_1, i_2 \in \{1,2,3,4\}, (\mu_1, \mu_2, \mu_3) \in S} a_{i_1, i_2, i, \mu_1, \mu_2, \mu_3} \lambda^{5(n-\mu_3)/2} X_{i_1, n-\mu_3+\mu_1} X_{i_2, n-\mu_3+\mu_2} \psi_{i,n} \end{aligned} \quad (2.62)$$

Now, by the localization in frequency space, we observe that (where K depends only on ψ_i):

$$|\langle -(-\Delta)^\alpha u_{i,n}, \psi_{i,n} \rangle| = |\langle u_{i,n}, -(-\Delta)^\alpha \psi_{i,n} \rangle| \leq K E_{i,n}^{1/2}(t) \lambda^{2\alpha n}. \quad (2.63)$$

Pairing Equation (2.62) with $\psi_{i,n}$ and using Equation (2.63) gets us the first statement in the lemma. The first inequality in the second statement follows from Cauchy-Schwarz. It remains to prove the second inequality. We recall the “local energy inequality” found by Tao, which is independent of the dissipation parameter α . Equation (4.11) in [89] is

$$\partial_t E_{i,n} \leq \sum_{i_1, i_2} \sum_{(\mu_1, \mu_2, \mu_3) \in S} a_{i_1, i_2, i, \mu_1, \mu_2, \mu_3} \lambda^{5(n-\mu_3)/2} X_{i_1, n-\mu_3+\mu_1} X_{i_2, n-\mu_3+\mu_1} X_{i,n}. \quad (2.64)$$

Now, if we multiply the evolution inequality for $X_{i,n}$ by $X_{i,n}$, subtract this from Equation (2.64), and integrate in time, we have the second desired inequality

$$E_{i,n}(t) \leq \frac{1}{2} X_{i,n}^2(t) + K \lambda^{2\alpha n} \int_0^t E_{i,n}(s) ds.$$

□

The only significant change from the work in [89] is the presence of α in the exponent of λ in the dissipation term.

Tao’s construction of a blowup solution to the pseudodifferential equation associated to C continues with an intricate analysis of an infinite dimensional ODE system whose dynamics are contained within the dynamics of C . Tao proves that there exist a sequence of times t_m converging to some finite T so that for some $\epsilon_0 > 0$, at least $\lambda^{-\epsilon_0 n}$ of the energy is concentrated in the ball of radius λ^{-n} around the spatial origin.

An essential part of the blowup construction is that the aforementioned concentration of energy occurs on timescales of order $t_{n+1} - t_m = \lambda^{(-\frac{5}{2} + O(\epsilon_0))n}$ which are significantly smaller than the dissipation time scale which (for fractional dissipation of order α) is $\lambda^{2\alpha n}$ by Lemma 2.3.3. This can occur so long as $\alpha < 5/4$. Since Tao never uses any assumption about the integral of the functions

$\psi_{i,n}$, one can follow through Tao's construction and get a blowup solution to the pseudodifferential equation (with hyperdissipation of order $\alpha < 5/4$) associated to some zero-momentum cascade operator. This is nothing less than Theorem 2.3.2 above.

2.3.2 Additional Properties of Local Cascade Operators

We have established Theorem 2.3.2, where we constructed a zero-momentum, symmetric, local cascade operator $C(u, v)$ satisfying the cancellation identity for divergence-free vector fields whose associated α -dissipative pseudodifferential equation has a solution blowing up in finite time from Schwartz initial data. Recall that we proved a partial regularity result for α -dissipative pseudodifferential equations with nonlinearities arising from amenable bilinear operators. It follows that if we desire to use Theorem 2.1.1 and get a partial regularity result for the pseudodifferential equation from Theorem 2.3.2, we have to verify that $C(u, v)$ is an amenable bilinear operator.

That $C(u, v)$ is defined for Schwartz vector fields is clear. That $C(u, v)$ satisfies the cancellation identity

$$\langle \mathbb{P}C(u, u), u \rangle = 0 \quad \text{for divergence-free } u$$

is equally clear. Indeed, since $C(u, u)$ is always divergence-free, we have $\mathbb{P}C(u, u) = C(u, u)$. It follows that

$$\langle \mathbb{P}C(u, u), u \rangle = \langle C(u, u), u \rangle = 0 \text{ for all divergence-free } u.$$

The scaling property, with the choice of scaling parameter λ , is clearly satisfied by C . Tao also makes this observation. Thus, it remains to show that $C_v^1(u) = \mathbb{P}C(u, v)$ is a pseudodifferential operator in some class $OPS_{1,1}^m$ and to show the bilinear operator bound. First we need to prove an auxiliary lemma.

Lemma 2.3.4. *Let ψ be a Schwartz divergence-free vector field that has zero momentum, i.e. $\int \psi = 0$. Let Ψ_i be the vector field given by Lemma 2.4.23 so that $\operatorname{div} \Psi_i = \psi_i$, the i^{th} component of ψ . Let $\psi_n := \lambda^{3n/2} \psi(\lambda^n x)$ be the L^2 rescaling of ψ and let $\Psi_{i,n}$ be the L^2 rescaling of Ψ_i . Then*

for any vector field $v \in H_{df}^{10}$, we have:

$$\lambda^n \langle \psi_n, v \rangle = - \sum_i \int \Psi_{i,n} \cdot \nabla v_i$$

Proof. Since ψ has zero momentum, we can use Lemma 2.4.23 and write

$$\lambda^n \psi_{i,n}(x) = \lambda^{5n/2} \psi_i(\lambda^n x) = \lambda^{5n/2} (\operatorname{div} \Psi_i)(\lambda^n x) = \lambda^{3n/2} \operatorname{div} (\Psi_i(\lambda^n x)) = \operatorname{div} \Psi_{i,n}(x).$$

Then we integrate by parts:

$$\lambda^n \langle \psi_n, v \rangle = \sum_i \int \lambda^{5n/2} \psi_i(\lambda^n x) v_i dx = \sum_i \int v_i \operatorname{div} \Psi_{i,n} dx = - \sum_i \int \Psi_{i,n} \cdot \nabla v_i dx$$

□

Proposition 2.3.5. *Let $C(u, v)$ be a zero-momentum, symmetric local cascade operator. Then, for arbitrary $0 < \gamma < 1$ and for any $1 \leq p_1, p_2, r \leq \infty$ satisfying*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{\gamma}{3} = \frac{1}{r},$$

we have that the following inequality holds

$$\|C(u, v)\|_{L^r} \leq K \|u\|_{L^{p_1}} (\|v\|_{L^{p_2}} + \|\nabla v\|_{L^{p_2}})$$

for all $u \in L^{p_1}$ and $v \in W^{1,p_2}$. The constant K depends on γ, λ , and the vector fields ψ_i used in the definition of $C(u, v)$.

Proof. It suffices to prove the proposition for any (zero-momentum, symmetric) basic local cascade operator:

$$C(u, v) = \sum_{n \in \mathbb{Z}} \lambda^{5n/2} \langle u, \psi_{1,n} \rangle \langle v, \psi_{2,n} \rangle \psi_{3,n}(x).$$

Since the sum above is absolutely convergent, we can divide the operator C into two parts:

$$C^+(u, v) := \sum_{n \geq 0} \lambda^{5n/2} \langle u, \psi_{1,n} \rangle \langle v, \psi_{2,n} \rangle \psi_{3,n}(x)$$

and $C^-(u, v) := C(u, v) - C^+(u, v)$. By a change of variables we observe that

$$\|\psi_{3,n}\|_{L^r} = \lambda^{3n/2} \|\psi_3(\lambda^n x)\|_{L^r} = \lambda^{3n/2-3n/r} \|\psi_3(u)\|_{L^r} = \lambda^{3n/2-3n/r} \|\psi_3\|_{L^r}. \quad (2.65)$$

By the absolute convergence of the sum, we may bound each component in turn. We begin with C^- , which is the more straightforward term. We observe that

$$\begin{aligned} \|C^-(u, v)\|_{L^r} &= \left\| \sum_{n < 0} \lambda^{5n/2} \langle u, \psi_{1,n} \rangle \langle v, \psi_{2,n} \rangle \psi_{3,n}(x) \right\|_{L^r} \leq \\ &\leq \sum_{n < 0} \left| \lambda^{5n/2} \langle u, \psi_{1,n} \rangle \langle v, \psi_{2,n} \rangle \right| \cdot \|\psi_{3,n}(x)\|_{L^r} \leq \\ &\leq \|\psi_3\|_{L^r} \sum_{n < 0} \lambda^{4n-3n/r} |\langle u, \psi_{1,n} \rangle| |\langle v, \psi_{2,n} \rangle| \leq \\ &\leq \|\psi_3\|_{L^r} \sum_{n < 0} \lambda^{4n-3n/r} \|u\|_{L^{p_1}} \|\psi_{1,n}\|_{L^{p'_1}} \|v\|_{L^{p_2}} \|\psi_{2,n}\|_{L^{p'_2}} \end{aligned}$$

where the first inequality is the triangle inequality, the second is due to Equation (2.65), and the third inequality is Holder's inequality, where p' denotes the Holder conjugate exponent of a number p . Then, by the same reasoning that justifies Equation (2.65), we can perform a change of variables in the integration and get

$$\|C^-(u, v)\|_{L^r} \leq \|\psi_3\|_{L^r} \|\psi_1\|_{L^{p'_1}} \|\psi_2\|_{L^{p'_2}} \sum_{n < 0} \lambda^{7n-3n(1/r+1/p'_1+1/p'_2)} \|u\|_{L^{p_1}} \|v\|_{L^{p_2}}.$$

Now by the hypothesis on our exponents, it follows that

$$\frac{1}{r} + \frac{1}{p'_1} + \frac{1}{p'_2} = \frac{1}{r} + 2 - \frac{1}{p_1} - \frac{1}{p_2} = 2 + \frac{\gamma}{3}.$$

Thus, we conclude, since $0 < \gamma < 1$ and $\lambda > 1$:

$$\|C^-(u, v)\|_{L^r} \leq \|\psi_3\|_{L^r} \|\psi_1\|_{L^{p'_1}} \|\psi_2\|_{L^{p'_2}} \sum_{n < 0} \lambda^{n(1-\gamma)} \|u\|_{L^{p_1}} \|v\|_{L^{p_2}} \leq K \|u\|_{L^{p_1}} \|v\|_{L^{p_2}}$$

where K depends only on λ, γ, ψ_i . Now we proceed to estimate C^+ . As with C^- we can use Holder's inequality, a change of variables in the integration, and Lemma 2.3.4 (to exchange λ^n for a derivative) to get:

$$\|C^+(u, v)\|_{L^r} \leq \|\psi_3\|_{L^r} \|\psi_1\|_{L^{p'_1}} \sup_i \|\Psi_{2,i}\|_{L^{p'_2}} \sum_{n \geq 0} \lambda^{-\gamma n} \|u\|_{L^{p_1}} \|\nabla v\|_{L^{p_2}}$$

where $\operatorname{div} \Psi_{2,i} = \psi_{2,i}$. This is where we use the zero-momentum condition. Because $\gamma > 0$ we conclude that

$$\|C^+(u, v)\|_{L^r} \leq K \|u\|_{L^{p_1}} \|\nabla v\|_{L^{p_2}}$$

where K depends only on λ, γ, ψ_i . Combining our estimates with one final application of the triangle inequality gets us

$$\|C(u, v)\|_{L^r} \leq K \|u\|_{L^{p_1}} (\|v\|_{L^{p_2}} + \|\nabla v\|_{L^{p_2}})$$

where K depends only on λ, γ, ψ_i . □

We note that the conclusion of Proposition 2.3.5 is in fact stronger than what is necessary for C to be amenable. To finish showing that $C(u, v)$ is an amenable bilinear operator, it remains to prove that $C_v^1(u) = \mathbb{P}C(u, v)$ is a pseudodifferential operator in some class $S_{1,1}^m$. That $C_v^2(u)$ is also in the same symbol class would follow immediately from the symmetry of C .

Proposition 2.3.6. *Let $C(u, v)$ be a symmetric local cascade operator, then $C_v^1(u)$ is a pseudodifferential operator with symbol in the class $S_{1,1}^{5/2}$.*

Proof. It suffices to prove the proposition for basic local cascade operators of the form:

$$C(u, v) = \sum_{n \in \mathbb{Z}} \lambda^{5n/2} \langle u, \psi_{1,n} \rangle \langle v, \psi_{2,n} \rangle \psi_{3,n}(x)$$

where we only deal with scalar functions. Using the fact that the Fourier transform is a unitary operator on L^2 and the absolute convergence of the sum, we write

$$C_v^1(u) = \iint \sum_{n \in \mathbb{Z}} \lambda^{5n/2} \psi_{3,n}(x) \langle v, \psi_{2,n} \rangle \mathcal{F}(\psi_{2,n})(\xi) \mathcal{F}(u)(\xi) d\xi.$$

Expanding this and putting it into the form of a pseudodifferential operator we write (up to some dimensional constant):

$$C_v^1(u) = \iint \left(\sum_{n \in \mathbb{Z}} \lambda^{5n/2} \psi_3(\lambda^n x) \langle v, \psi_{2,n} \rangle \mathcal{F}(\psi_2)(\lambda^{-n} \xi) e^{-ix \cdot \xi} \right) \mathcal{F}(u)(\xi) e^{ix \cdot \xi} d\xi.$$

The infinite sum, however, has only one nonzero term for any given choice of ξ , because $\mathcal{F}(\psi_2)$ has nicely chosen compact support. In particular, the only nonzero term for any given ξ is the n^{th} term where $|\xi| \sim \lambda^n$. It follows readily then that the symbol of the pseudodifferential operator C_v^1 , the “infinite” sum in the expression above, is in fact in the symbol class $S_{1,1}^{5/2}$. $\square$

As a consequence of the two previous propositions as well as Theorems 2.3.2 and 2.1.1, we have:

Theorem 2.3.7. *Let $C(u, v)$ be the bilinear operator from Theorem 2.3.2. Let T be the blowup time to the associated pseudodifferential equation with $1 < \alpha < 5/4$. Then the closed set S_T has Hausdorff dimension at most $5 - 4\alpha$.*

We recall here the following general theorem about local cascade operators proven by Tao in [89].

Theorem 2.3.8 (Theorem 3.2 in [89]). *Let $\lambda_0 > 1$ be an absolute constant sufficiently close to 1. Then every local cascade operator (not necessarily zero-momentum) is an averaged (in the sense of Equation (1.4)) Euler bilinear operator.*

Due to the theorem above, we have the following corollary of Theorem 2.3.7 (taking λ near enough to 1):

Corollary 2.3.9. *Let $1 < \alpha < 5/4$ be arbitrary. There exists a symmetric averaged Euler bilinear operator obeying the cancellation identity for $u \in H_{df}^{10}(\mathbb{R}^3)$ and a Schwartz divergence-free vector*

initial vector field so that there is no global-in-time solution to the associated α -dissipative pseudodifferential equation. Moreover if T is the time of first blowup, then the closed set S_T has Hausdorff dimension at most $5 - 4\alpha$.

2.4 Technical Lemmas

Throughout this subsection we shall work in the general Euclidean space $\mathbb{R}^n$. All our spaces will be of functions with domain $\mathbb{R}^n$, unless otherwise stated. Our shorthand for the L^p norm will be $\|\cdot\|_p$, and we denote the Fourier transform by $\mathcal{F}$. We denote the fractional (inhomogeneous) Sobolev space of order s and integrability p by:

$$W^{s,p}(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(f)(\xi)) \in L^p(\mathbb{R}^n) \right\}.$$

Lastly, we denote $W^{s,2}(\mathbb{R}^n)$ by $H^s(\mathbb{R}^n)$ and use the shorthand $\|\cdot\|_s$ for the norm of $H^s(\mathbb{R}^n)$. Since we almost always deal with the spaces $H^s(\mathbb{R}^n)$, the use of this shorthand will always be clear in context. The following embedding theorem is standard. A proof may be found in [5].

Lemma 2.4.1 (Theorem 1.66 in [5]). *The space $H^s(\mathbb{R}^n)$ embeds continuously in the Hölder Space $C^{k,\alpha}(\mathbb{R}^n)$ provided that $s \geq n/2 + k + \alpha$.*

We shall also require the use of the Schwartz-Paley-Wiener theorem, which we state as the following lemma.

Lemma 2.4.2. *The vector space $C_c^\infty(\mathbb{R}^n)$ of smooth and compactly supported functions is algebraically and topologically isomorphic, via the Fourier transform, to the space of entire functions F on $\mathbb{C}^n$ which satisfy the following growth bounds: there exists $B > 0$ such that for every $M \in \mathbb{Z}^+$ there exists real constant K_M such that for all $\xi \in \mathbb{C}^n$,*

$$|F(\xi)| \leq K_M (1 + |\xi|)^{-M} e^{B|\operatorname{Im}(\xi)|}.$$

2.4.1 Pseudodifferential Operators

A pseudodifferential operator is an operator P on Schwartz functions $u \in \mathcal{S}$ with the property that:

$$P(x, D)u := (2\pi)^{-n/2} \int e^{ix \cdot \xi} p(x, \xi) \mathcal{F}(u)(\xi) d\xi.$$

where $p(x, \xi)$ is a smooth function on $\mathbb{R}^{2n}$. It is clear that such operators map the Schwartz space into itself. We can write $Pu = \mathcal{F}^{-1}(p \cdot \mathcal{F}(u))$ if the function $p(x, \xi)$ is a function of ξ only. These operators are often called Fourier multipliers. Operators that are Fourier multipliers commute. In particular, if P_1, P_2 are two Fourier multipliers, then $P_1 P_2 = P_2 P_1$.

We call $p(x, \xi)$ the symbol of P , and we say that p is in the symbol class $S_{\rho, \delta}^m$ where $m \in \mathbb{R}, 0 \leq \rho, \delta \leq 1$ if and only if

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha, \beta} ((1 + |\xi|^2)^{1/2})^{m - \rho|\alpha| + \delta|\beta|}$$

for some constants $C_{\alpha, \beta}$. The associated operator P is said to be in the class $OPS_{\rho, \delta}^m$. If $\delta < 1$, pseudodifferential operators in the class $S_{\rho, \delta}^m$ can be defined instead on the dual of the Schwartz space, which we denote $\mathcal{S}'$.

Let Λ^s be the pseudodifferential operator with symbol $(1 + |\xi|^2)^{s/2}$, which is in the class $S_{1,0}^s$. We may recognize the Sobolev space $H^s(\mathbb{R}^n)$ as $\Lambda^{-s} L^2(\mathbb{R}^n)$. We call a pseudodifferential operator, P , Sobolev-smoothing if and only if $P(H^s) \subset H^t$ for any real numbers s and t . In other words, a Sobolev-smoothing operator P continuously maps any Sobolev space into any other Sobolev space. We also define the symbol class $S_{\rho, \delta}^{-\infty}$ to be

$$S_{\rho, \delta}^{-\infty} := \bigcap_{m \in \mathbb{R}} S_{\rho, \delta}^m$$

However, there is really only one class of symbols of infinitely negative order, a fact which we note below.

Lemma 2.4.3. *Let $0 \leq \rho, \delta \leq 1$ and let $p(x, \xi) \in S_{\rho, \delta}^{-\infty}$. Then $p(x, \xi) \in S_{1,0}^{-\infty}$.*

Proof. Suppose $p(x, \xi) \in S_{\rho, \delta}^{-\infty}$. Let α, β and real number N be arbitrary. By hypothesis we can

choose $m < N - (1 - \rho)|\alpha| - \delta|\beta|$ so that for all (x, ξ) :

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|^2)^{(m - \rho|\alpha| + \delta|\beta|)/2} \leq C_{\alpha, \beta} (1 + |\xi|^2)^{(N - |\alpha|)/2}.$$

The arbitrariness of N proves that $p(x, \xi) \in S_{1,0}^{-\infty}$. □

Henceforth we denote $S_{1,0}^{-\infty}$ by $S^{-\infty}$. We now recall three standard lemmas below. The proofs may be found in Chapter 0 of [91].

Lemma 2.4.4. *Let $P_j \in OPS_{\rho_j, \delta_j}^{m_j}$. Let $\rho = \min(\rho_1, \rho_2)$ and $\delta = \max(\delta_1, \delta_2)$. Suppose*

$$0 \leq \delta_2 < \rho_1 \leq 1.$$

Then $P_1 P_2 \in OPS_{\rho, \delta}^{m_1 + m_2}$, and we have $P_1 P_2 = T_N + R_N$ where

$$T_N(x, \xi) = \sum_{|\alpha| \leq N} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x, \xi) D_x^\alpha p_2(x, \xi)$$

and $R_N \in OPS_{\rho, \delta}^{m_1 + m_2 - N(\rho - \delta)}$.

Lemma 2.4.5. *If $P \in OPS_{\rho, \delta}^m$, $\rho > 0$, and $m < -n + \rho(n - 1)$, then*

$$P : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad \forall \quad 1 \leq p \leq \infty$$

is continuous.

Lemma 2.4.6. *If $P \in OPS_{\rho, \delta}^0$ and $0 \leq \delta < \rho \leq 1$, then*

$$P : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$$

is continuous.

The following lemma, which deals with the “exotic” symbol classes $S_{1,1}^m$, is proven in the appendix of [4]. It provides an asymptotic expansion for operators in the exotic class.

Lemma 2.4.7. *Let $0 \leq \delta < 1$. Suppose $P \in OPS_{1,1}^m$ and $Q \in OPS_{1,\delta}^\mu$. Then $PQ \in OPS_{1,1}^{m+\mu}$ and we have $PQ = T_N + R_N$ where*

$$T_N(x, \xi) = \sum_{|\alpha| \leq N} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi)$$

and $R_N \in OPS_{1,1}^{m+\mu-N(1-\delta)}$.

Our next lemma gives us an inclusion of the operators with symbols in $S^{-\infty}$ into the Sobolev-smoothing operators.

Lemma 2.4.8. *If P has its symbol in $S^{-\infty}$, then it is a Sobolev-smoothing operator.*

Proof. Let s, t be arbitrary real numbers, and suppose that $f \in H^t = \Lambda^{-t}L^2$, i.e. we may write $f = \Lambda^{-t}u$ for some function $u \in L^2$. We have to show that $Pf \in H^s = \Lambda^{-s}L^2$, but this amounts to showing that $\Lambda^s P \Lambda^{-t}u \in L^2$.

Since $P \in S^{-\infty}$, we have, for some m small enough, that $P \in S_{1,0}^m$ with $m + s - t < -1$. Then, by Lemma 2.4.4, $\Lambda^s P \Lambda^{-t} \in S_{1,0}^{m+s-t}$, which is continuous from $L^2 \rightarrow L^2$ by Lemma 2.4.5. This proves that P is Sobolev-smoothing. $\square$

The three lemmas that follow are also about pseudodifferential operators. The first two say that certain products of operators are guaranteed to be Sobolev-smoothing.

Lemma 2.4.9. *Let P be a pseudodifferential operator with symbol $p(\xi)$ in the class $S_{1,0}^m$. Let ϕ_1, ϕ_2 be two smooth functions with disjoint supports (and assume they are pseudodifferential operators in the class $S_{1,\delta}^0$ with $1 > \delta \geq 0$). Then, the composition $\phi_1 P \phi_2$ is a smoothing pseudodifferential operator.*

Proof. We first compute the symbol p_2 of $P_2 = P\phi_2$. Since $1 > \delta$, a standard result (see [91]) gets us the asymptotic expansion:

$$p_2(x, \xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(\xi) D_x^\alpha \phi_2(x)$$

where the sum and asymptotic expression is interpreted in the sense that the sum over $|\alpha| < N$ differs from $p_2(x, \xi)$ by an element of $S_{1,\delta}^{m-N(1-\delta)}$. Then, a similar asymptotic expansion can be found for the symbol of $Q = \phi_1 P_2 = \phi_1 P \phi_2$, which is given by:

$$q(x, \xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha \phi_1(x) D_x^\alpha p_2(x, \xi).$$

It follows that up to a smoothing operator we have:

$$q(x, \xi) \sim \phi_1(x) p_2(x, \xi).$$

Now, for any arbitrary N , we have that

$$q(x, \xi) \sim \phi_1(x) \sum_{N > \alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(\xi) D_x^\alpha \phi_2(x)$$

up to an operator in $S_{1,\delta}^{m-N(1-\delta)}$. Since ϕ_1, ϕ_2 have disjoint supports, the above sum is zero for all N . We conclude that $q(x, \xi) \in S_{1,\delta}^{-\infty}$, so $Q = \phi_1 P \phi_2$ is an operator in $OPS^{-\infty}$, which, by Lemma 2.4.8, means that Q is Sobolev-smoothing. $\square$

We separate the next lemma from the previous one since it deals with the exotic class. The proof, however, is the same as for Lemma 2.4.9.

Lemma 2.4.10. *Let ϕ_1, ϕ_2 be two smooth functions with disjoint supports in the class $S_{1,\delta}^0$, with $1 > \delta \geq 0$. Let P be a pseudodifferential operator in the class $OPS_{1,1}^m$. Then the composition $\phi_1 P \phi_2$ is a Sobolev-smoothing pseudodifferential operator.*

Proof. We first deal with the composition $P_2 = P \phi_2$, which by Lemma 2.4.7 lies in the class $OPS_{1,1}^m$. However, the same Lemma 2.4.7 also gives us the asymptotic expansion:

$$p_2(x, \xi) \sim \sum_{|\alpha| \leq N} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x, \xi) D_x^\alpha \phi_2(x)$$

up to a term in $OPS_{1,1}^{m-N(1-\delta)}$. But then the symbol of $Q = \phi_1 P \phi_2$ is given by:

$$q(x, \xi) \sim \phi_1(x) \sum_{|\alpha| \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x, \xi) D_x^\alpha \phi_2(x)$$

up to an operator in $S^{-\infty}$. Since ϕ_1, ϕ_2 have disjoint support, the above sum is zero, so it follows that $q \in S^{-\infty}$, as desired. $\square$

The next lemma gives us a quantitative version of the smoothing property:

Lemma 2.4.11. *Let $j \in \mathbb{Z}$ and let P_j be a Littlewood-Paley projection operator. Let β be an arbitrary real number, and assume that $\|f\|_{H^\beta} \leq 1$. Let Q be a pseudodifferential operator with symbol $q(x, \xi)$ in the class $S_{1,1}^m$. Let ϕ_1, ϕ_2 be two smooth functions with disjoint supports in the class $S_{1,\delta}^0$ where $1 > \delta \geq 0$. Then for any positive integer N , the composition $\phi_1 Q \phi_2$ satisfies the following quantitative bound*

$$\|P_j \phi_1 Q \phi_2 f\|_2 \leq K 2^{-jN}$$

with constant K depending only on N and β .

Proof. By Lemma 2.4.10, we know that $\tilde{Q} := \phi_1 Q \phi_2$ is a smoothing operator. In particular, we have for any real number s :

$$\|\tilde{Q}f\|_s \leq K \|f\|_\beta \leq K,$$

where the constant K depends only on s and β . By Lemma 2.4.13, we have

$$2^{sj} \|P_j \tilde{Q}f\|_2 \leq K \|\tilde{Q}f\|_s.$$

Together we get, for some other constant K depending only on s and β :

$$\|P_j \phi_1 Q \phi_2 f\|_2 \leq K 2^{-sj}.$$

Since s was arbitrary, we achieve what we desired. $\square$

2.4.2 Littlewood-Paley Decomposition and Localization

We use a Littlewood-Paley partition of frequency space. Namely, for $j \in \mathbb{Z}$ we have pseudodifferential operators P_j with smooth symbols $p_j(\xi)$ that are supported in $\frac{2}{3}2^j < |\xi| < 3 \cdot 2^j$, which also satisfy $p_j(\xi) = p_0(2^{-j}\xi)$ and

$$\sum_{j \in \mathbb{Z}} p_j(\xi) = 1. \quad (2.66)$$

We define $\tilde{P}_j := \sum_{k=-2}^2 P_{j+k}$ to be the sum of all Littlewood-Paley projections whose symbols' supports intersect the support of $p_j(\xi)$. Likewise we may define $\tilde{p}_j(\xi)$. We also define $\tilde{\tilde{P}}_j := \sum_{k=-4}^4 P_{j+k}$ to be the sum of all Littlewood-Paley projections whose symbols' supports intersect the support of $\tilde{p}_j$. Likewise we may define $\tilde{\tilde{p}}_j$.

We also note that

$$\tilde{P}_j P_j = P_j \quad (2.67)$$

These Littlewood-Paley projections satisfy the following inequalities

Lemma 2.4.12. *For any $2 \leq q \leq \infty$ there exists a constant K independent of j such that*

$$\|P_j f\|_q \leq K 2^{nj \left(\frac{1}{2} - \frac{1}{q}\right)} \|P_j f\|_2$$

Proof. Let $\phi_j = \mathcal{F}^{-1}(\tilde{p}_j)$. Recalling the scaling law satisfied by the symbols p_j , we see that

$$\phi_j(x) = 2^{nj} \phi_0(2^j x). \quad (2.68)$$

By definition, $\tilde{P}_j f = \phi_j * f$. By Young's convolution inequality (for which we need $q \geq 2$) and the projection property Equation (2.67) we have

$$\|P_j f\|_q = \|\tilde{P}_j P_j f\|_q \leq \|P_j f\|_2 \|\phi_j\|_{2q/(2+q)}$$

By the scaling property in Equation (2.68) and a change of variables we get:

$$\|\phi_j\|_{2q/(2+q)} = 2^{nj} \|\phi_0(2^j x)\|_{2q/(2+q)} = 2^{nj(1/2-1/q)} \|\phi_0\|_{2q/(2+q)}.$$

Combining the last two inequalities yields our result. $\square$

The next two lemmas may together be characterized as representative of the “finite band property” relating derivatives with Littlewood-Paley projections using the localization in frequency space. For more details on the finite band property, see [58] and Chapter 2 in [5]. The following result is just a restatement of a lemma from [5] in our notation.

Lemma 2.4.13 (Lemma 2.1 in [5]). *For any $s \geq 0$ there exists a constant K such that for any $j \in \mathbb{Z}$ and for all $1 \leq p \leq \infty$ we have*

$$\|P_j f\|_{W^{s,p}(\mathbb{R}^n)} \leq K 2^{js} \|P_j f\|_{L^p(\mathbb{R}^n)}$$

$$2^{js} \|P_j f\|_{L^p(\mathbb{R}^n)} \leq K \|P_j f\|_{W^{s,p}(\mathbb{R}^n)}.$$

We shall need the following L^∞ estimates on the high-frequency cutoff of each bump function ϕ_Q .

Lemma 2.4.14. *Let $\phi = \phi_{Q,j}$ be a bump function of type j (as before). Define*

$$\phi_2 := \mathcal{F}^{-1}(\chi_{|\xi| > \frac{1}{100} 2^j}(\xi) \mathcal{F}\phi(\xi)).$$

We call ϕ_2 the high frequency cutoff of ϕ . Then, given any N , there is a constant K depending only on N and ϵ such that:

$$\max(\|\phi_2(x)\|_\infty, \|\mathcal{F}(\phi_2)(\xi)\|_\infty) \leq K 2^{-jN}.$$

Proof. By the Schwartz-Paley-Wiener Theorem stated in Lemma 2.4.2, and since ϕ_2 is smooth and compactly supported with Fourier transform $\mathcal{F}(\phi_2)$ supported away from the origin of frequency

space, we may conclude that there is a constant K depending only on N such that

$$\|\mathcal{F}\phi_2(\xi)\|_\infty \leq K2^{-jN}.$$

We prove a similar inequality for $\|\phi_2(x)\|_\infty$ by a rescaling argument. Let N be arbitrary. Let ψ be a smooth function with compact support such that $\text{supp}(\mathcal{F}(\psi)) = \{\xi : |\xi| \geq \frac{1}{100}\}$. We claim that if $\|D^\alpha \psi(x)\|_\infty \leq C_\alpha 2^{(-\epsilon|\alpha|-n)j}$ for all α , then $\|\psi(x)\|_\infty \leq K(N, \epsilon) 2^{(-N-n)j}$, where $K(N, \epsilon)$ depends only on N and ϵ but not on j . Assuming the claim, let ψ be a function satisfying the hypotheses of the claim and consider the rescaled function

$$\tilde{\psi}(x) := 2^{nj} \psi(2^j x).$$

The Fourier transform of this function is

$$\mathcal{F}(\tilde{\psi})(\xi) = \mathcal{F}(\psi)(2^{-j}\xi).$$

The support of $\mathcal{F}(\tilde{\psi})$ is $\{\xi : |\xi| \geq \frac{1}{100} 2^j\}$, our claim yields

$$\|\tilde{\psi}(x)\|_\infty \leq 2^{-Nj},$$

and the following derivative bounds hold:

$$\left| D^\alpha \tilde{\psi}(x) \right| \leq C_\alpha 2^{(|\alpha|(1-\epsilon))j}.$$

Proving the L^∞ bound for $\tilde{\psi}$ assuming the above derivative bounds and high frequency support is equivalent to proving the L^∞ bound claimed before for ψ . We recognize ϕ_2 as one such function $\tilde{\psi}$, so we just have to prove the statement regarding ψ to finish the proof of the lemma.

Consider instead $f = 2^{nj} \psi$, and assume to the contrary that there exists an N such that for

any K , we have

$$\|f\|_\infty > K2^{-Nj} \quad \text{and} \quad \|D^\alpha f\|_\infty \leq C_\alpha 2^{-\epsilon|\alpha|j} \quad \forall \alpha > 0$$

However, the uniform derivative bounds preclude the arbitrary size of f , which is a contradiction. $\square$

In the statement of the following lemmas, the constant 100 appears. The particular value of this constant is unimportant and may be replaced by any large number of our choice. In addition, the hypothesis $k \geq j$ appears. This hypothesis can be weakened to $k \geq j - K$ for any given positive constant K without any significant modification to the arguments used. The next lemma is a commutator estimate similar to Proposition 5.2 in [57].

Lemma 2.4.15. *Let $2 < q < \infty$ be arbitrary. Suppose $\|f\|_2 \leq K$. Suppose ϕ is a bump function of type j and let $k \geq j - 2$ be arbitrary. Then, for a constant K depending only on q :*

$$\|\phi P_k f - \tilde{P}_k \phi P_k f\|_2 \leq K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)},$$

and

$$\|\phi \tilde{P}_k f - \tilde{P}_k \phi \tilde{P}_k f\|_2 \leq K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)}.$$

Proof. Given our bump function, we can decompose it into the sum $\phi = \phi_1 + \phi_2$ such that $\mathcal{F}\phi_2(\xi) = \chi_{|\xi| > \frac{1}{100} 2^j}(\xi) \mathcal{F}\phi(\xi)$ for some M . By Lemma 2.4.14, we know there is a constant K depending only on q such that:

$$\max(\|\phi_2(x)\|_\infty, \|\mathcal{F}\phi_2(\xi)\|_\infty) \leq K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)}.$$

On the other hand, we have:

$$\tilde{P}_k \phi_1 P_k f = \phi_1 P_k f \tag{2.69}$$

and

$$\tilde{P}_k \phi_1 \tilde{P}_k f = \phi_1 \tilde{P}_k f. \tag{2.70}$$

Indeed, up to a dimensional constant we can write

$$\tilde{P}_k \phi_1 P_k f = \iint e^{ix \cdot \xi} \tilde{p}_k(\xi) \mathcal{F}(\phi_1)(z) p_k(\xi - z) \mathcal{F}(f)(\xi - z) dz d\xi$$

and

$$\tilde{P}_k \phi_1 \tilde{P}_k f = \iint e^{ix \cdot \xi} \tilde{\tilde{p}}_k(\xi) \mathcal{F}(\phi_1)(z) \tilde{p}_k(\xi - z) \mathcal{F}(f)(\xi - z) dz d\xi.$$

Now if $\xi - z$ is in the support of p_k and $|z| < \frac{1}{100} 2^j$, then, since $k \geq j - 2$, we have that ξ remains well inside the support of $\tilde{p}_k$. It follows that the $\tilde{p}_k$ term in the integrand can be discounted, so Equation (2.69) is true. Likewise if $\xi - z$ is in the support of $\tilde{p}_k$ and $|z| < \frac{1}{100} 2^j$, then, since $k \geq j - 2$, we have that ξ remains well inside the support of $\tilde{\tilde{p}}_k$. It follows that the $\tilde{\tilde{p}}_k$ term in the integrand can be discounted, so Equation (2.70) is true. Now observing that

$$\phi P_k f - \tilde{P}_k \phi P_k f = \phi_1 P_k f - \tilde{P}_k \phi_1 P_k f + \phi_2 P_k f - \tilde{P}_k \phi_2 P_k f$$

and that $\|f\|_2 \leq K$, we conclude that

$$\begin{aligned} \|\phi P_k f - \tilde{P}_k \phi P_k f\|_2 &= \|\phi_2 P_k f - \tilde{P}_k \phi_2 P_k f\|_2 \leq K \max(\|\phi_2(x)\|_\infty, \|\mathcal{F}(\phi_2)(\xi)\|_\infty) \leq \\ &\leq K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)} \end{aligned}$$

Likewise we have

$$\phi \tilde{P}_k f - \tilde{\tilde{P}}_k \phi \tilde{P}_k f = \phi_1 \tilde{P}_k f - \tilde{\tilde{P}}_k \phi_1 \tilde{P}_k f + \phi_2 \tilde{P}_k f - \tilde{\tilde{P}}_k \phi_2 \tilde{P}_k f$$

and that $\|f\|_2 \leq K$, we conclude that

$$\begin{aligned} \|\phi \tilde{P}_k f - \tilde{\tilde{P}}_k \phi \tilde{P}_k f\|_2 &= \|\phi_2 \tilde{P}_k f - \tilde{\tilde{P}}_k \phi_2 \tilde{P}_k f\|_2 \leq K \max(\|\phi_2(x)\|_\infty, \|\mathcal{F}(\phi_2)(\xi)\|_\infty) \leq \\ &\leq K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)} \end{aligned}$$

□

One case of the next commutator estimate is used implicitly in [57], although neither a statement nor a proof is given there.

Lemma 2.4.16. *Let $2 < q < \infty$ be arbitrary. Suppose $\|f\|_2 \leq K$. Suppose ϕ is a bump function of type j and let $k \geq j - 2$ be arbitrary. Then, for a constant depending only on q :*

$$\|\phi P_k f - \tilde{P}_k \phi P_k f\|_\infty \leq K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)}.$$

Proof. As before, we decompose $\phi = \phi_1 + \phi_2$ where ϕ_2 is the high frequency cutoff. By Lemma 2.4.14, where we let $N = (100 \cdot \lfloor \frac{q}{q-2} \rfloor)$, there is a constant K depending only on q such that:

$$\max(\|\phi_2(x)\|_\infty, \|\mathcal{F}\phi_2(\xi)\|_\infty) \leq K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)}.$$

As in the proof of Lemma 2.4.15, we just have to estimate:

$$\|\phi_2 P_k f - \tilde{P}_k \phi_2 P_k f\|_\infty.$$

Now using the fact that $\|f\|_{L^2} \leq K$, our L^∞ bounds for ϕ_2 and $\mathcal{F}(\phi_2)$, and Lemma 2.4.12, we reach our desired conclusion. □

Lemma 2.4.17. *Let ϕ be a bump function of type j , let $k \geq j - 2$ and $2 < q < \infty$ be arbitrary. Suppose also that $\|f\|_2 \leq K$. Then:*

$$\|\phi P_k f\|_\infty \leq K 2^{nk/2} \|\phi P_k f\|_2 + K 2^{-j(100 \cdot \lfloor \frac{q}{q-2} \rfloor)}$$

Proof. This follows immediately from Lemma 2.4.16, Lemma 2.4.12, and the observation that

$$\phi P_k f = \phi P_k f - \tilde{P}_k \phi P_k f + \tilde{P}_k \phi P_k f.$$

□

The next lemma extends Lemma 5.3 in [57].

Lemma 2.4.18. *Let $2 \leq q \leq \infty$, $\|f\|_2 \leq K$, and let ϕ be a bump function of type j . Let $k \geq j - 2$ be arbitrary. Then:*

$$\|\phi P_k f\|_q \leq K 2^{nk(1/2-1/q)} \|\phi P_k f\|_2 + K 2^{-100j}$$

Proof. When $q = 2$ there is nothing to show. When $q = \infty$, the result is proven in Lemma 2.4.17, and the exponent in the error term can be made arbitrarily negative. We prove the remaining cases by interpolation. First, it is evident from Lemma 2.4.12 that the function $\phi P_k f \in L^q$ for all $2 \leq q \leq \infty$. In particular, since the L^p norms are log-convex:

$$\|\phi P_k f\|_q \leq \|\phi P_k f\|_2^{2/q} \|\phi P_k f\|_\infty^{1-2/q}$$

For the remainder of the proof, we denote $A := \|\phi P_k f\|_2$. Using Lemma 2.4.17, which is the case $q = \infty$ in the statement of our lemma, we have that:

$$\|\phi P_k f\|_q \leq A^{2/q} \left(2^{nk/2} A + K 2^{-100j \lfloor \frac{q}{q-2} \rfloor} \right)^{(q-2)/q}$$

Using the elementary inequality $(a + b)^p \leq 2^p(a^p + b^p)$ for $p \geq 0$ and the fact that $q > 2$ gets us:

$$\|\phi P_k f\|_q \leq 2^{(q-2)/q} A^{2/q} \left(2^{\frac{nk}{2} \cdot \frac{q-2}{q}} A^{(q-2)/q} + K 2^{-100j} \right)$$

Using the fact that $\|f\|_2 \leq K$ we conclude:

$$\|\phi P_k f\|_q \leq K 2^{nk(\frac{1}{2}-\frac{1}{q})} \|\phi P_k f\|_2 + K 2^{-100j}$$

as desired. □

We can commute bump functions with Littlewood-Paley projections, as long as we multiply with another bump function with slightly larger support.

Lemma 2.4.19. *Let $k \geq j - 2$, let $\phi_{Q,j}$ be a bump function at level j . Let $\beta < 100$ and consider f with $\|f\|_{H^\beta} \leq K$. Then*

$$\|(1 - \phi_{(1+2^{-\epsilon j})Q,j})P_k \phi_{Q,j} f\|_2 \leq K 2^{-100j}$$

and

$$\|(1 - \phi_{(1+2^{-\epsilon j})Q,j})\tilde{P}_k \phi_{Q,j} f\|_2 \leq K 2^{-100j}.$$

Proof. By Lemma 2.4.9, we see that

$$(1 - \phi_{(1+2^{-\epsilon j})Q,j})P_k \phi_{Q,j}$$

is a smoothing pseudodifferential operator. By Lemma 2.4.15 and since $\tilde{P}_k P_k = P_k$, we have

$$\begin{aligned} (1 - \phi_{(1+2^{-\epsilon j})Q,j})P_k \phi_{Q,j} &= P_k \phi_{Q,j} - \phi_{(1+2^{-\epsilon j})Q,j} P_k \phi_{Q,j} = \\ &= \tilde{P}_k P_k \phi_{Q,j} - \tilde{P}_k \phi_{(1+2^{-\epsilon j})Q,j} P_k \phi_{Q,j} + R_j = \tilde{P}_k ((1 - \phi_{(1+2^{-\epsilon j})Q,j})P_k \phi_{Q,j}) + R_j, \end{aligned}$$

where R_j is an error term satisfying $|R_j| \leq K 2^{-200j}$. By Lemma 2.4.11, we have a quantitative bound on the Sobolev smoothing operator that appears, thus proving our desired inequality. By Lemma 2.4.9, we see that

$$(1 - \phi_{(1+2^{-\epsilon j})Q,j})\tilde{P}_k \phi_{Q,j}$$

is a smoothing pseudodifferential operator. By Lemma 2.4.15 and since $\tilde{P}_k \tilde{P}_k = \tilde{P}_k$, we have

$$\begin{aligned} (1 - \phi_{(1+2^{-\epsilon j})Q,j})\tilde{P}_k \phi_{Q,j} &= \tilde{P}_k \phi_{Q,j} - \phi_{(1+2^{-\epsilon j})Q,j} \tilde{P}_k \phi_{Q,j} = \\ &= \tilde{P}_k \tilde{P}_k \phi_{Q,j} - \tilde{P}_k \phi_{(1+2^{-\epsilon j})Q,j} \tilde{P}_k \phi_{Q,j} + R_j = \tilde{P}_k ((1 - \phi_{(1+2^{-\epsilon j})Q,j})\tilde{P}_k \phi_{Q,j}) + R_j, \end{aligned}$$

where R_j is an error term satisfying $|R_j| \leq K 2^{-200j}$. By Lemma 2.4.11, we have a quantitative bound on the Sobolev smoothing operator that appears, thus proving our desired inequality. The

same proof works to get an arbitrarily negative exponent in the error term instead. $\square$

Note that in the previous lemma, the order of the bump functions in the composition was unimportant. The only essential fact was disjointness of support.

Lemma 2.4.20. *Let $\mathcal{A}$ be a collection of cubes at level j covering a measurable set E . For any $f \in L^2$, we have:*

$$\|\chi_E P_j f\|_2^2 \leq \sum_{Q \in \mathcal{A}} f_Q^2$$

Proof. We notice that

$$\int \chi_E |P_j f|^2 \leq \int \left(\sum_{Q \in \mathcal{A}} \phi_{Q,j}^2 \right) |P_j f|^2 = \sum_{Q \in \mathcal{A}} f_Q^2.$$

$\square$

2.4.3 Hausdorff Measure and Dimension

We denote a ball of radius r in $\mathbb{R}^n$ by B_r . Let $E \subset \mathbb{R}^n$. We recall that, up to a dimensional constant multiple, the d -dimensional Hausdorff measure of E is given by

$$H^d(E) := \sup_{\delta > 0} \inf_{\mathcal{C}_\delta(E)} \sum_{B_r \in \mathcal{C}_\delta(E)} r^d$$

where we have taken an infimum over all coverings $\mathcal{C}_\delta(E)$ of E by balls of radius less than or equal to δ . We define the Hausdorff dimension of E to be:

$$\mathcal{H}(E) = \inf\{d : H^d(E) = 0\}.$$

We need a way to compute the Hausdorff dimension that follows from a discretization by sets of scale 2^j . In particular we have the following lemma from [57]:

Lemma 2.4.21. *Let $\mathcal{A}_j$ be a sequence of collections of balls in $\mathbb{R}^n$ so that each element of $\mathcal{A}_j$ has radius 2^{-j} . Suppose that the number of balls in each $\mathcal{A}_j$, denoted $N(\mathcal{A}_j)$, is bounded by $N(\mathcal{A}_j) \leq$*

$C2^{jd}$ where C is independent of j . Let

$$E = \limsup_{j \rightarrow \infty} \mathcal{A}_j := \bigcap_{j \in \mathbb{N}} \bigcup_{k > j} \bigcup_{B \in \mathcal{A}_k} B$$

be the set of points in infinitely many of the unions $\bigcup_{B \in \mathcal{A}_j} B$. Then $\mathcal{H}(E) \leq d$.

Proof. By the definition of the Hausdorff dimension, it suffices to show that $H^s(E) = 0$ for all $s > d$. To that end, pick j large enough that $2^{-j} < \delta$. By construction of our set, E can be covered by $\bigcup_{k > j} \bigcup_{B \in \mathcal{A}_k} B$. Evidently, we have

$$H^s(E) \leq \sum_{k > j} N(\mathcal{A}_k) (2^{-k})^s \leq C \sum_{k > j} 2^{kd} (2^{-k})^s$$

and the right-hand side above converges to zero as $j \rightarrow \infty$ so long as $d < s$. $\square$

We also include in this subsection the standard Vitali covering lemma, which is often used in estimating the Hausdorff dimension of a set.

Lemma 2.4.22 (Vitali). *Let $\mathcal{A}$ be a collection of cubes. Then there is subcollection $\mathcal{A}'$ so that any two cubes in $\mathcal{A}'$ are pairwise disjoint and*

$$\bigcup_{Q \in \mathcal{A}} Q \subset \bigcup_{Q \in \mathcal{A}'} 5Q.$$

2.4.4 Zero-Momentum Schwartz Vector Fields

Lemma 2.4.23. *A Schwartz function $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ can be written as $\psi = \operatorname{div} \Psi$ for some Schwartz vector field Ψ if and only if $\int_{\mathbb{R}^3} \psi = 0$.*

Proof. One direction is relatively straightforward. Indeed suppose that $\psi = \operatorname{div} \Psi$ for some Schwartz vector field and let B_R be some ball centered at the origin of radius R . Then the divergence theorem gets us:

$$\int_{B_R} \psi = \int_{B_R} \operatorname{div} \Psi = \int_{\partial B_R} \Psi \cdot \mathbf{n}$$

Since Ψ is Schwartz and decays faster than any polynomial at infinity, the right hand side above tends to zero as $R \rightarrow \infty$ and the result is shown.

Now assume that $\int_{\mathbb{R}^3} \psi = 0$; we desire to construct Ψ . Let $f(z)$ be any real-valued function in $C_c^\infty(\mathbb{R})$. Let

$$\Gamma(x, y, z) = f(z) \cdot \int_{-\infty}^x \int_{-\infty}^y \int_{-\infty}^{\infty} \psi(r, s, t) dt ds dr$$

By our assumption on the integral of ψ , the function Γ is a Schwartz function. Now,

$$\partial_x \Gamma = f(z) \int_{-\infty}^y \int_{-\infty}^{\infty} \psi(x, s, t) dt ds$$

and

$$\partial_y \Gamma = f(z) \int_{-\infty}^x \int_{-\infty}^{\infty} \psi(r, y, t) dt dr.$$

So defining (a function that is also a Schwartz function by our assumption on ψ):

$$\Xi(x, y, z) = \int_{-\infty}^z (\psi - \partial_x \Gamma - \partial_y \Gamma)(x, y, t) dt$$

we observe that

$$\psi = \partial_x \Gamma + \partial_y \Gamma + \partial_z \Xi$$

so our desired vector field $\Psi = (\Gamma, \Gamma, \Xi)$. □

2.4.5 The Continuity of the Energy Integral

Lemma 2.4.24. *Let u be any smooth solution of Equation (2.3) with Schwartz initial value u_0 , let Q be any cube at level j , and let $t_0 < T$ be any time before blowup. Then the function of time $u_Q(t)$ is continuous on $[0, T)$ and the following equality holds:*

$$\lim_{\epsilon \rightarrow 0} \int_{t_0}^{T-\epsilon} \frac{d}{dt} u_Q^2 dt = \int_{t_0}^T \frac{d}{dt} u_Q^2 dt$$

Proof. The vector field $u(t)$ is a smooth (mild) solution of Equation (2.3),

$$\partial_t u + (-\Delta)^\alpha u + \hat{B}(u, u) = 0,$$

with Schwartz initial data u_0 . In particular, its L^2 norm is strongly continuous in time, so it remains to show that the localized norm u_Q remains continuous in time. For this we proceed as in the proof of Lemma 2.7 in [73].

We observe that

$$\|\phi_{Q,j} P_j u(x, t) - \phi_{Q,j} P_j u(x, s)\|_{L^2(\mathbb{R}^n)}^2 \leq \int_{2Q} \left| \int_{\mathbb{R}^n} p_j(x - \xi) (\mathcal{F}(u)(\xi, t) - \mathcal{F}(u)(\xi, s)) d\xi \right|^2 dx.$$

The integral inside the absolute value on the right-hand side above converges to zero because u is a smooth mild solution whose L^2 norm is strongly continuous in time and because the Fourier transform is a unitary operator on L^2 . Moreover, since $\|u\|_{L^2} \leq K$, the integral is dominated by:

$$\left| \int_{\mathbb{R}^n} p_j(x - \xi) (\mathcal{F}(u)(\xi, t) - \mathcal{F}(u)(\xi, s)) d\xi \right| \leq K \tilde{p}_j(x)$$

Thus, by the dominated convergence theorem, we conclude that the localization u_Q remains strongly continuous in time. The desired limit then follows in a straightforward manner. $\square$

As a corollary, we have:

Corollary 2.4.25. *Let $t_0 < T$ be any time. Then*

$$\int_{t_0}^T \frac{d}{dt} u_Q^2 dt = \lim_{t \rightarrow T} u_Q^2(t) - u_Q^2(t_0).$$

Proof. Indeed, we may write

$$\int_{t_0}^{T-\epsilon} \frac{d}{dt} u_Q^2 dt = u_Q^2(T - \epsilon) - u_Q^2(t_0)$$

and take the limit of the above expression as $\epsilon \rightarrow 0$, using Lemma 2.4.24. $\square$

2.4.6 The Critical Exponent in Higher Dimensions

We determine the critical exponent of hyperdissipation that guarantees global regularity for solutions of Equation (1.3). A proof along the same lines for the hyperdissipative three-dimensional Navier-Stokes equations may be found in [57].

Proposition 2.4.26. *Let $\alpha > (n + 2)/4$ and let $u(t) : \mathbb{R}^n \times [0, \epsilon) \rightarrow \mathbb{R}^n$ be a smooth solution of Equation (1.3) from Schwartz initial data. Then, $u(t)$ is smooth for all times $t \geq 0$.*

Proof. We let $\beta > 0$ and test Equation (1.3) against $(-\Delta)^\beta u$ to get:

$$\langle \partial_t u, (-\Delta)^\beta u \rangle + \langle (-\Delta)^\alpha u, (-\Delta)^\beta u \rangle + \langle (u \cdot \nabla) u, (-\Delta)^\beta u \rangle = 0.$$

Using that $\operatorname{div} u = 0$, we may rewrite the equation as:

$$\frac{d}{dt} \left(\frac{1}{2} \|u\|_\beta^2 \right) + \|u\|_{\alpha+\beta}^2 + \langle (-\Delta)^{\frac{\beta}{2}} \operatorname{div}(u \otimes u), (-\Delta)^{\frac{\beta}{2}} u \rangle = 0. \quad (2.71)$$

We estimate the trilinear term by:

$$\left| \langle (-\Delta)^{\frac{\beta}{2}} \operatorname{div}(u \otimes u), (-\Delta)^{\frac{\beta}{2}} u \rangle \right| \leq \|u\|_{W^{1,p}} \|u\|_{W^{\beta,q}} \|u\|_\beta \quad (2.72)$$

where $1/p + 1/q = 1/2$. Now, $\alpha = (n + 2)/4$ is the minimum value of α so that both inequalities

$$\frac{1}{2} - \frac{\alpha}{n} \leq \frac{1}{p} - \frac{1}{n}, \quad \text{and} \quad \frac{1}{2} - \frac{\alpha}{n} \leq \frac{1}{q}$$

hold for some choice (p, q) with $1/p + 1/q = 1/2$ depending on $n \geq 3$. Since these inequalities hold, we may use the Sobolev Embedding Theorem and then Young's inequality to conclude that:

$$\|u\|_{W^{1,p}} \|u\|_{W^{\beta,q}} \|u\|_\beta \leq \|u\|_\alpha \|u\|_{\alpha+\beta} \|u\|_\beta \leq \frac{1}{2} \|u\|_{\alpha+\beta}^2 + \frac{1}{2} \|u\|_\alpha^2 \|u\|_\beta^2. \quad (2.73)$$

Using Equations (2.71), (2.72), and (2.73), we get

$$\frac{d}{dt} \left(\frac{1}{2} \|u\|_{\beta}^2 \right) \leq \frac{1}{2} \|u\|_{\alpha}^2 \|u\|_{\beta}^2$$

It follows that

$$\|u(t)\|_{\beta}^2 \leq \|u(0)\|_{\beta}^2 \exp \left(\int_0^t \|u(s)\|_{\alpha}^2 ds \right).$$

However, by the energy dissipation law, we know

$$\int_0^t \|u(s)\|_{\alpha}^2 ds \leq \|u(0)\|_{L^2}$$

for all times t whenever $u(t)$ is smooth. Since β was arbitrary, u_0 was Schwartz, and $u(t)$ was smooth for small times $t \in [0, \epsilon)$, we get regularity for all times $t \geq 0$. $\square$

Chapter 3

Uniqueness

3.1 Local Estimates for the Navier–Stokes System

We define a *capsule* $\mathcal{C}$ of radius R , half-length $L \geq R$ in the direction $e \in \mathbb{S}^2$, and center x to be defined by

$$\mathcal{C}_{R,L,e}(x) := \bigcup_{t \in [-L+R, L-R]} (te + B_R(x)). \quad (3.1)$$

Here $B_R(x) \subset \mathbb{R}^3$ is a ball of radius R centered at x , and we shall abbreviate $B_R = B_R(0)$. Note that $\mathcal{C}_{R,R,e}(x) = B_R(x)$ for any e . Given a capsule $\mathcal{C}_{R,L,e}(x)$ and any $\lambda > 0$, we denote $\lambda \mathcal{C}_{R,L,e}(x) := \mathcal{C}_{\lambda R, \lambda L, e}(x)$. Where there is no confusion, we may omit dependencies on R, L, e , or x . Occasionally, we may denote the capsule by $\mathcal{C}_x$ to highlight the dependency on the center point x .

Take $\mathcal{C} = \mathcal{C}_{R,L,e_1}(0)$ to be a capsule centered at the origin, pointing in the direction of e_1 . In this section, suppose u is a smooth solution to the stationary Navier–Stokes equations in a dilated capsule $2\mathcal{C}$. We further assume there exist $U, \Xi > 0$, such that

$$\oint_{B_R} u \, dx = U e_1, \quad \oint_{2\mathcal{C}} |\nabla u|^2 \, dx = \Xi^2. \quad (3.2)$$

We shall now derive a local L^∞ bound in $\mathcal{C}$ on $u - Ue_1$. Let V denote the volume of the capsule $\mathcal{C}$, or $V = |\mathcal{C}|$. Note that $V \approx LR^2$.

Lemma 3.1.1. *If u is a smooth solution to the stationary Navier–Stokes equations satisfying the assumptions in Equation (3.2), then*

$$\|u - Ue_1\|_{L^\infty(\mathcal{C})} \lesssim \Xi R^{\frac{1}{2}} V^{\frac{1}{2}} \left(\frac{UR}{L} + \frac{1}{R} + L\Xi \right) (\Xi V^{\frac{2}{3}} + 1) + L\Xi. \quad (3.3)$$

The proof of this lemma will occupy the following three subsections.

3.1.1 Control of the Average Velocity

Denote

$$\bar{u}(t) := \oint_{B_R(te_1)} u \, dx.$$

Then $\bar{u}(0) = Ue_1$. In this subsection we show that $|\bar{u}(t) - Ue_1| \lesssim L\Xi$ provided that $|t| \leq 2(L - R)$.

Lemma 3.1.2. *For every $|t| \leq 2(L - R)$, we have*

$$|\bar{u}(t) - Ue_1| \lesssim L \left(\oint_{\mathcal{C}_{2R, 2L, e_1}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

To do this, we require the following short lemma.

Lemma 3.1.3. *Let $f(x)$ be a nonnegative function. Then for every $l > R$, we have*

$$\frac{\sqrt{3}-1}{2} R \int_{\mathcal{C}_{R/2, l+R/2, e_1}} f(x) \, dx \leq \int_{-l}^l \int_{B_R} f(x + te_1) \, dx \, dt \leq 2R \int_{\mathcal{C}_{R, l+R, e_1}} f(x) \, dx.$$

Proof. We first rewrite:

$$\int_{-l}^l \int_{B_R} f(x + te_1) \, dx \, dt = \int_{-l}^l \int_{B_R(te_1)} f(x) \, dx \, dt. \quad (3.4)$$

Let $\mathbf{1}_A(x)$ be the characteristic function of a set A . We rewrite

$$\int_{-l}^l \int_{B_R(te_1)} f(x) \, dx \, dt = \int_{-l}^l \int_{\mathbb{R}^3} f(x) \mathbf{1}_{B_R(te_1)}(x) \, dx \, dt.$$

Using the Fubini–Tonelli Theorem we have:

$$\int_{-l}^l \int_{B_R(te_1)} f(x) \, dx \, dt = \int_{\mathbb{R}^3} f(x) \left(\int_{-l}^l \mathbf{1}_{B_R(te_1)}(x) \, dt \right) dx. \quad (3.5)$$

Now observe that $x \in B_R(te_1)$ implies $|x - te_1| < R$, or equivalently

$$(t - x_1)^2 + x_2^2 + x_3^2 < R^2,$$

where $x = (x_1, x_2, x_3)$. We thus have that t necessarily lies in the interval

$$t \in \left(x_1 - \sqrt{R^2 - (x_2^2 + x_3^2)}, x_1 + \sqrt{R^2 - (x_2^2 + x_3^2)} \right).$$

The integral of the characteristic function is thus equal to

$$\begin{aligned} \int_{-l}^l \mathbf{1}_{B_R+te_1}(x) \, dt &= \max \left\{ \min \left\{ l, x_1 + \sqrt{R^2 - (x_2^2 + x_3^2)} \right\} \right. \\ &\quad \left. - \max \left\{ -l, x_1 - \sqrt{R^2 - (x_2^2 + x_3^2)} \right\}, 0 \right\} \\ &= \min \left\{ 2\sqrt{R^2 - (x_2^2 + x_3^2)}, \right. \\ &\quad \left. \left(\sqrt{R^2 - (x_2^2 + x_3^2)} + l - |x_1| \right)_+ \right\}. \end{aligned} \quad (3.6)$$

Let us compute (3.6). On the one hand, if $x \in \mathcal{C}_{R/2, l+R/2, e_1}$, which is equivalent to $(|x_1| - l)_+^2 + x_2^2 + x_3^2 < (R/2)^2$, then

$$\sqrt{R^2 - (x_2^2 + x_3^2)} > \sqrt{R^2 - (R/2)^2} = \sqrt{3}R/2.$$

Therefore

$$\int_{-l}^l \mathbf{1}_{B_{R+te_1}}(x) dt \geq \min \left\{ \sqrt{3}R, \sqrt{3}R/2 - R/2 \right\} = \frac{\sqrt{3}-1}{2}R.$$

This is true for any $x \in \mathcal{C}_{R/2, l+R/2}$, therefore we obtain a lower bound on the integral in Equation (3.6):

$$\int_{-l}^l \mathbf{1}_{B_{R+te_1}}(x) dt \geq \frac{\sqrt{3}-1}{2}R \cdot \mathbf{1}_{\mathcal{C}_{R/2, l+R/2}}(x).$$

On the other hand, for the integral (3.6) to be positive, one must have

$$\sqrt{R^2 - (x_2^2 + x_3^2)} + l - |x_1| > 0,$$

equivalent to $(|x_1| - l)_+^2 + x_2^2 + x_3^2 < R^2$, or $x \in \mathcal{C}_{R, l+R, e_1}$. Moreover, in this situation, the integral is always smaller than

$$\int_{-l}^l \mathbf{1}_{B_{R+te_1}}(x) dt \leq 2\sqrt{R^2 - (x_2^2 + x_3^2)} \leq 2R,$$

so we obtain an upper bound on the integral in Equation (3.6):

$$\int_{-l}^l \mathbf{1}_{B_{R+te_1}}(x) dt \leq 2R \cdot \mathbf{1}_{\mathcal{C}_{R, l+R, e_1}}(x).$$

Combine the lower and upper bound, the conclusion of the lemma is immediate from Equation (3.5). □

Using this auxiliary Lemma, we now prove Theorem 3.1.2.

Proof of Theorem 3.1.2. For any $t_1, t_2 \in [-2L + 2R, 2L - 2R]$, we have, using Equation (3.4) and

Theorem 3.1.3 with $l = 2L - 2R$ that

$$\begin{aligned}
|\bar{u}(t_2) - \bar{u}(t_1)| &= \left| \oint_{B_R} u(x + t_2 e_1) - u(x + t_1 e_1) \, dx \right| \\
&= \left| \oint_{B_R} \int_{t_1}^{t_2} \partial_{x_1} u(x + t e_1) \, dt \, dx \right| \\
&\leq \int_{-2L+2R}^{2L-2R} \oint_{B_R} |\partial_{x_1} u(x + t e_1)| \, dx \, dt \\
&\leq \frac{2R}{|B_R|} \int_{2\mathcal{C}} |\nabla u| \, dx \leq \frac{16RV}{|B_R|} \left(\oint_{2\mathcal{C}} |\nabla u|^2 \, dx \right)^{1/2} \lesssim L\Xi. \tag{3.7}
\end{aligned}$$

In particular, $|\bar{u}(t) - Ue_1| = |\bar{u}(t) - \bar{u}(0)| \lesssim L\Xi$ for every $t \in [-2L + 2R, 2L - 2R]$. $\square$

3.1.2 Velocity Decomposition

In the next subsection, to avoid the typical difficulties that arise when analyzing the nonlocal pressure field, we shall work instead on the vorticity equation, which does not explicitly depend on the pressure. As a preparation, we now introduce the velocity decomposition.

Fix a non-negative cut-off function $\phi \in C_c^\infty(2\mathcal{C})$ with $\phi \equiv 1$ in $\frac{3}{2}\mathcal{C}$ and satisfying the following pointwise bounds for some constant $C > 0$:

$$|\partial_{x_1} \phi| \leq \frac{C}{L}, \quad |\nabla \phi| \leq \frac{C}{R}, \quad |\nabla^2 \phi| \leq \frac{C}{R^2}. \tag{3.8}$$

We decompose $u = Ue_1 + v + h$, where

$$v = \operatorname{curl} (-\Delta)^{-1}(\phi\omega)$$

and $h = u - Ue_1 - v$. Note that v and h are both divergence-free. The vector field v represents the rotational part of the velocity because $\operatorname{curl} v = \omega$ in $\frac{3}{2}\mathcal{C}$. The vector field h is the potential part of the velocity because a simple computation reveals that h is harmonic in $\frac{3}{2}\mathcal{C}$: $\Delta h = \Delta u - \Delta v = -\operatorname{curl} \omega + \operatorname{curl}(\phi\omega) = -\operatorname{curl}((1 - \phi)\omega)$, which is zero on the set $\{\phi = 1\}$.

Note that, since $\phi\omega$ is a bounded, smooth, compactly supported function, and since the Biot-

Savart kernel decays at infinity, we have

$$\lim_{|x| \rightarrow \infty} v(x) = 0.$$

Moreover, by the Sobolev embedding and the Calderón–Zygmund estimate for singular integral operators, we have

$$\|v\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla v\|_{L^2(\mathbb{R}^3)} \lesssim \|\omega\|_{L^2(2\mathcal{C})} \lesssim \Xi V^{\frac{1}{2}}. \quad (3.9)$$

By the same argument as in the previous subsection, we have

$$|\bar{v}(t_2) - \bar{v}(t_1)| \lesssim L\Xi, \quad \forall t_1, t_2 \in [-2L + 2R, 2L - 2R].$$

We have bounded the oscillation of $\bar{v}$. To get a bound for its size we use the facts that $V \lesssim LR^2$, $\|v\|_{L^6(\mathbb{R}^3)} \lesssim \Xi V^{1/2}$, and Theorem 3.1.3 to control the average:

$$\left| \int_{-2L+2R}^{2L-2R} \bar{v}(t) dt \right| \leq \frac{R}{L|B_R|} \left| \int_{2\mathcal{C}} |v(x)| dx \right| \lesssim V^{-1} \|v\|_{L^6(2\mathcal{C})} V^{\frac{5}{6}} \leq \Xi V^{\frac{1}{3}}.$$

Since $L \geq R$, we know $V^{\frac{1}{3}} \lesssim L$, so

$$|\bar{v}(t)| \lesssim L\Xi + \Xi V^{\frac{1}{3}} \lesssim L\Xi, \quad \forall t \in [-2L + 2R, 2L - 2R].$$

Since $|\bar{u}(t) - Ue_1| \lesssim L\Xi$, we conclude

$$|\bar{h}(t)| \lesssim L\Xi, \quad \forall t \in [-2L + 2R, 2L - 2R].$$

In other words, $\bar{v}$ and $\bar{h}$ both enjoy the same bound for $\bar{u} - Ue_1$ that we obtained from the previous subsection.

Since h is harmonic inside $\frac{3}{2}\mathcal{C}$, we have by the mean-value-property that $\bar{h}(t) = h(te_1)$ for all

applicable t , whence it follows that

$$\sup_{t \in [-2L+2R, 2L-2R]} |h(te_1)| \lesssim L\Xi. \quad (3.10)$$

Moreover, from Equation (3.9) we also have

$$\|\nabla h\|_{L^2(2\mathcal{C})} \leq \|\nabla u\|_{L^2(2\mathcal{C})} + \|\nabla v\|_{L^2(2\mathcal{C})} \lesssim \Xi V^{\frac{1}{2}}. \quad (3.11)$$

In the following lemma, we show that since h is harmonic, we can in fact extend the same bound to the interior of $\frac{3}{2}\mathcal{C}$.

Lemma 3.1.4. *Let h be a harmonic function in a capsule $\frac{3}{2}\mathcal{C}$ satisfying Equations (3.10) and (3.11). Then $\|h\|_{L^\infty(\frac{4}{3}\mathcal{C})} \lesssim L\Xi$.*

Proof. Take $x \in B_{\frac{4}{3}R}$, $|t| \leq \frac{4}{3}(L - R)$, then $x + te_1 \in \frac{4}{3}\mathcal{C}$. By the mean-value property for ∇h in $\frac{3}{2}\mathcal{C}$, we have

$$\nabla h(x + te_1) = \oint_{B_{\frac{1}{6}R}} \nabla h(x + te_1 + y) \, dy,$$

which immediately implies that

$$|\nabla h(x + te_1)| \lesssim \frac{1}{|B_{\frac{1}{6}R}|} \int_{2\mathcal{C}} |\nabla h| \, dx \lesssim \frac{V}{|B_R|} \left(\int_{2\mathcal{C}} |\nabla h|^2 \right)^{1/2} \lesssim \frac{L}{R} \Xi.$$

This is true for any t, x , so h is $\frac{CL\Xi}{R}$ -Lipschitz in $\frac{4}{3}\mathcal{C}$. We now have

$$|h(x + te_1) - h(te_1)| \lesssim |x| \frac{L}{R} \Xi \lesssim L\Xi.$$

Together with Equation (3.10), we know $|h(x + te_1)| \lesssim L\Xi$. Since t and x were arbitrary, the claim is proven. $\square$

Since h is bounded in $\frac{4}{3}\mathcal{C}$ with bound given by $L\Xi$, we certainly have

$$\|h\|_{L^6(\frac{3}{2}\mathcal{C})} \lesssim L\Xi V^{\frac{1}{6}} + \Xi V^{\frac{1}{2}} \lesssim L\Xi V^{\frac{1}{6}}.$$

Here we used $V^{\frac{1}{3}} \lesssim L$. Combined with the L^6 bound on v in Equation (3.9), we have obtained an L^6 bound for $\zeta := u - Ue_1 = v + h$:

$$\|\zeta\|_{L^6(\frac{3}{2}C)} \lesssim L\Xi V^{\frac{1}{6}} + \Xi V^{\frac{1}{2}} \lesssim L\Xi V^{\frac{1}{6}}. \quad (3.12)$$

3.1.3 The Vorticity Equation

The vector field $\zeta = u - Ue_1$ satisfies the stationary Navier–Stokes equations with constant background drift $b = Ue_1$:

$$b \cdot \nabla \zeta + \zeta \cdot \nabla \zeta + \nabla p = \Delta \zeta. \quad (3.13)$$

We shall use the Bernoulli head pressure (in the moving frame) $H = \frac{|\zeta|^2}{2} + p$ and write Equation (3.13) in the following form using vorticity $\omega = \text{curl } u = \text{curl } \zeta$:

$$b \cdot \nabla \zeta + \omega \times \zeta + \nabla H = \Delta \zeta. \quad (3.14)$$

Taking the curl of Equation (3.14) yields

$$b \cdot \nabla \omega + \text{curl } (\omega \times \zeta) = \Delta \omega.$$

We use the local estimates proven for the Poisson equation in Theorem 3.4.3 of Section 3.4 (we recognize that $\text{curl } (\omega \times \zeta) = \text{div } T$, for the tensor $T_{ij} = \epsilon_{ijk}(\omega \times \zeta)_k$ and where ϵ_{ijk} is the antisymmetric symbol). We then have for all $1 < q < 3$ and $1/r = 1/q - 1/3$:

$$\|\omega\|_{L^r(\frac{3}{2}C)} \lesssim \|\omega \otimes \zeta\|_{L^q(\frac{3}{2}C)} + \left(\frac{UR}{L} + \frac{1}{R} \right) \|\omega\|_{L^q(\frac{3}{2}C)}. \quad (3.15)$$

However, we already know that (also using Equation (3.12))

$$\|\omega\|_{L^{\frac{3}{2}}(\frac{3}{2}C)} \leq \Xi V^{\frac{2}{3}}, \quad \|\omega \otimes \zeta\|_{L^{\frac{3}{2}}(\frac{3}{2}C)} \leq \|\omega\|_{L^2(\frac{3}{2}C)} \|\zeta\|_{L^6(\frac{3}{2}C)} \leq L\Xi^2 V^{\frac{2}{3}}.$$

Setting $q = \frac{3}{2}$ in the estimate from Equation (3.15), we get

$$\|\omega\|_{L^3(\frac{4}{3}\mathcal{C})} \lesssim \left(\frac{UR}{L} + \frac{1}{R} + L\Xi \right) \Xi V^{\frac{2}{3}}.$$

Vorticity in L^3 only guarantees that the velocity lies in BMO , not L^∞ . To get an L^∞ bound for ζ , we therefore need to refine our estimates. We first note that

$$\begin{aligned} \|\omega \otimes \zeta\|_{L^2(\frac{4}{3}\mathcal{C})} &\leq \|\omega \otimes v\|_{L^2(\frac{4}{3}\mathcal{C})} + \|\omega \otimes h\|_{L^2(\frac{4}{3}\mathcal{C})} \\ &\leq \|v\|_{L^6(\frac{4}{3}\mathcal{C})} \|\omega\|_{L^3(\frac{4}{3}\mathcal{C})} + \|\omega\|_{L^2(\frac{4}{3}\mathcal{C})} \|h\|_{L^\infty(\frac{4}{3}\mathcal{C})} \\ &\lesssim \Xi V^{\frac{1}{2}} \left[\left(\frac{UR}{L} + \frac{1}{R} + L\Xi \right) \Xi V^{\frac{2}{3}} + L\Xi \right]. \end{aligned}$$

Here we used Equation (3.9) and Theorem 3.1.4. Setting $q = 2$ in the estimate from Equation (3.15) and recalling that $\|\omega\|_{L^2} \lesssim \Xi V^{\frac{1}{2}}$, we get

$$\begin{aligned} \|\omega\|_{L^6(\frac{5}{4}\mathcal{C})} &\lesssim \Xi V^{\frac{1}{2}} \left[\left(\frac{UR}{L} + \frac{1}{R} + L\Xi \right) \Xi V^{\frac{2}{3}} + L\Xi \right] + \left(\frac{UR}{L} + \frac{1}{R} \right) \Xi V^{\frac{1}{2}} \\ &= \Xi V^{\frac{1}{2}} \left(\frac{UR}{L} + \frac{1}{R} + L\Xi \right) (\Xi V^{\frac{2}{3}} + 1). \end{aligned}$$

Now, if we fix another cut-off function ϕ_1 supported in $\frac{5}{4}\mathcal{C}$ and $\phi_1 \equiv 1$ in $\frac{6}{5}\mathcal{C}$ with derivative bounds as in Equation (3.8), then $v_1 := \text{curl}(-\Delta)^{-1}(\phi_1\omega)$ satisfies

$$\begin{aligned} |v_1(x)| &= C \left| \int_{\mathbb{R}^3} \frac{(\phi_1\omega)(x-y) \times y}{|y|^3} dy \right| \leq C \int_{3\mathcal{C}} |(\phi_1\omega)(x-y)| |y|^{-2} dy \\ &\lesssim \|\phi_1\omega\|_{L^6(\mathbb{R}^3)} \left(\int_{3\mathcal{C}} \frac{1}{|y|^{\frac{12}{5}}} dy \right)^{\frac{5}{6}} \end{aligned}$$

for every $x \in \mathbb{R}^3$, where we may additionally bound

$$\begin{aligned} \int_{3\mathcal{C}} \frac{1}{|y|^{\frac{12}{5}}} dy &\leq \int_{-R}^R \int_{-R}^R \int_{\mathbb{R}} \frac{1}{(y_1^2 + y_2^2 + y_3^2)^{\frac{6}{5}}} dy_1 dy_2 dy_3 \\ &= C \int_{-R}^R \int_{-R}^R \frac{1}{(y_2^2 + y_3^2)^{\frac{7}{10}}} dy_2 dy_3 = CR^{\frac{3}{5}}. \end{aligned}$$

Therefore

$$\|v_1\|_{L^\infty} \lesssim \Xi R^{\frac{1}{2}} V^{\frac{1}{2}} \left(\frac{UR}{L} + \frac{1}{R} + L\Xi \right) (\Xi V^{\frac{2}{3}} + 1).$$

Similarly, we can define $h_1 = \zeta - v_1$, which is harmonic, and use our previous argument to get $\|h_1\|_{L^\infty(\mathcal{C})} \lesssim L\Xi$, so

$$\|\zeta\|_{L^\infty(\mathcal{C})} \lesssim \Xi R^{\frac{1}{2}} V^{\frac{1}{2}} \left(\frac{UR}{L} + \frac{1}{R} + L\Xi \right) (\Xi V^{\frac{2}{3}} + 1) + L\Xi.$$

This completes the proof of Theorem 3.1.1. $\square$

3.1.4 Ansatz

Suppose that for some $\varepsilon_0 > 0$, $0 \leq \delta \leq \frac{5}{12}$ to be determined later, the following holds:

$$L\Xi \leq \frac{\varepsilon_0}{R} \left(\frac{L}{R} \right)^\delta \lesssim \varepsilon_0 \left(\frac{UR}{L} + \frac{1}{R} \right). \quad (3.16)$$

With such an ansatz, we have

$$\begin{aligned} \|u - Ue_1\|_{L^\infty(\mathcal{C})} &\lesssim \varepsilon_0 \left(\frac{L}{R} \right)^{\delta - \frac{1}{2}} \left(\frac{UR}{L} + \frac{1}{R} \right) \left(\frac{L}{R} \right)^{(\delta - \frac{1}{3})_+} + \varepsilon_0 \left(\frac{UR}{L} + \frac{1}{R} \right) \\ &\lesssim \varepsilon_0 \left(\frac{UR}{L} + \frac{1}{R} \right). \end{aligned} \quad (3.17)$$

This is the ansatz with which we can prove Liouville theorems in the following sections.

3.2 Capsules and a Covering Lemma

We introduce a way of putting a capsule satisfying the ansatz around every point in $\mathbb{R}^3$, and we prove a covering lemma for this collection of capsules. To begin with, recall that Theorem 3.1.1 controls the oscillation of u in $\mathcal{C}$ but it uses information of $|\nabla u|^2$ in $2\mathcal{C}$, which is not desirable. In the first subsection, we introduce the maximal functions, with which we can control $|\nabla u|^2$ in $2\mathcal{C}$

using only the information in $\mathcal{C}$.

3.2.1 Maximal Functions

Recall the definition of the maximal function $\mathcal{M}f$ of any $f \in L^1_{\text{loc}}(\mathbb{R}^3)$:

$$\mathcal{M}f(x) = \sup_{r>0} \int_{B_r(x)} |f(y)| \, dy.$$

We also introduce a streamwise maximal function $\mathcal{M}_\Phi f$ associated with the flow map Φ of an incompressible vector field.

Definition 3.2.1. *Let $\Phi_s : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $s \in \mathbb{R}$ be the flow map associated with the divergence free vector field u . In particular, Φ is given by the solution to the following initial value problem:*

$$\frac{\partial}{\partial s} \Phi_s(x) = u(\Phi_s(x)), \quad \Phi_0(x) = x.$$

For any continuous function $f \in C(\mathbb{R}^3)$, we define the streamwise maximal function $\mathcal{M}_\Phi f : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$(\mathcal{M}_\Phi f)(x) := \sup_{s>0} \frac{1}{2s} \int_{-s}^s |(f \circ \Phi_\tau)(x)| \, d\tau. \quad (3.18)$$

By continuity, $\mathcal{M}_\Phi f \geq f$ in the pointwise sense. In analogy with the classical maximal function, $\mathcal{M}_\Phi$ is also an operator of strong (p, p) -type for $p > 1$.

Lemma 3.2.2. *If $f \in C(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $1 < p < \infty$, then*

$$\|\mathcal{M}_\Phi f\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.$$

Proof. Without loss of generality, assume $f \geq 0$. For $T > 0$, denote

$$(\mathcal{M}_{\Phi, T} f)(x) := \sup_{0 < s < T} \frac{1}{2s} \int_{-s}^s |(f \circ \Phi_\tau)(x)| \, d\tau.$$

Since u is incompressible, Φ_s is a measure preserving map. Then for any $s' \in [-T, T]$ we have

$$\begin{aligned}
\|\mathcal{M}_{\Phi, T} f\|_{L^p(\mathbb{R}^3)}^p &= \int_{\mathbb{R}^3} \left(\sup_{0 < s < T} \frac{1}{2s} \int_{-s}^s f \circ \Phi_\tau \, d\tau \right)^p dx \\
&= \int_{\mathbb{R}^3} \left(\sup_{0 < s < T} \frac{1}{2s} \int_{-s}^s f \circ \Phi_{\tau+s'} \, d\tau \right)^p dx \\
&= \int_{\mathbb{R}^3} \left(\sup_{0 < s < T} \frac{1}{2s} \int_{s'-s}^{s'+s} f \circ \Phi_\tau \, d\tau \right)^p dx \\
&= \int_{\mathbb{R}^3} \left(\sup_{0 < s < T} \frac{1}{2s} \int_{s'-s}^{s'+s} F_T(\tau, x) \, d\tau \right)^p dx,
\end{aligned}$$

where we have denoted $F_T(\tau, x) := (f \circ \Phi_\tau(x)) \cdot \mathbf{1}_{(-2T, 2T)}(\tau)$. The previous equality holds for any $s' \in [-T, T]$, so we may take an average in s' :

$$\begin{aligned}
\|\mathcal{M}_{\Phi, T} f\|_{L^p(\mathbb{R}^3)}^p &= \frac{1}{2T} \int_{-T}^T \int_{\mathbb{R}^3} \left(\sup_{0 < s < T} \frac{1}{2s} \int_{s'-s}^{s'+s} F_T(\tau, x) \, d\tau \right)^p dx \, ds' \\
&\leq \frac{1}{2T} \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} \left(\sup_{s > 0} \frac{1}{2s} \int_{s'-s}^{s'+s} F_T(\tau, x) \, d\tau \right)^p ds' \, dx.
\end{aligned}$$

We know that the classical one-dimensional maximal function maps L^p to L^p , so we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\sup_{s > 0} \frac{1}{2s} \int_{s'-s}^{s'+s} F_T(\tau, x) \, d\tau \right)^p ds' &\leq C \int_{-\infty}^{\infty} F_T(\tau, x)^p \, d\tau \\
&= C \int_{-2T}^{2T} F_T(\tau, x)^p \, d\tau.
\end{aligned}$$

In the second step, we used that $F_T(\tau, x)$ is supported in $\{-2T \leq \tau \leq 2T\}$. Once again using the

fact that Φ_t is measure preserving, we have

$$\begin{aligned}
\|\mathcal{M}_{\Phi,T}f\|_{L^p(\mathbb{R}^3)}^p &\leq \frac{C}{2T} \int_{\mathbb{R}^3} \int_{-2T}^{2T} F_T(\tau, x)^p d\tau dx \\
&= \frac{C}{2T} \int_{-2T}^{2T} \int_{\mathbb{R}^3} (f \circ \Phi_\tau)^p dx d\tau \\
&= C\|f\|_{L^p(\mathbb{R}^3)}^p \cdot \frac{1}{2T} \int_{-2T}^{2T} d\tau \\
&\leq 2C\|f\|_{L^p(\mathbb{R}^3)}^p.
\end{aligned}$$

Since $\|\mathcal{M}_{\Phi,T}f\|_{L^p} \leq C\|f\|_{L^p}$ for a constant C independent of T , we may use the monotone convergence theorem to conclude $\|\mathcal{M}_\Phi f\|_{L^p} \leq C\|f\|_{L^p}$ as well. $\square$

Recall that u is a smooth vector field, so $|\nabla u|^2$ is globally Lipschitz continuous. Its maximal function

$$\mathcal{M}(|\nabla u|^2)(x) = \sup_{r>0} \int_{B_r(x)} |\nabla u(x')|^2 dx' = \sup_{r>0} \frac{1}{|B_r|} (\mathbf{1}_{B_r} * |\nabla u|^2)(x),$$

is also Lipschitz continuous with the same Lipschitz bound. Therefore, the composition of maximal functions $\mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)]$ in the following lemma is well-defined.

Lemma 3.2.3. *Given $\mathcal{C} = \mathcal{C}_{R,L,e_1}(0)$, assume there exist $U, \tilde{\Xi} > 0$, such that*

$$\int_{B_R} u dx = Ue_1, \quad \int_{\mathcal{C}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] dx = \tilde{\Xi}^2. \quad (3.19)$$

There exist absolute constants $C > 0$ and $\varepsilon_1 > 0$ such that if either $L \leq 2R$ or $L\tilde{\Xi} \leq \varepsilon_1 \frac{R}{L} \cdot U$, then

$$\int_{2\mathcal{C}} |\nabla u|^2 dx \leq C \int_{\mathcal{C}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] dx = C\tilde{\Xi}^2. \quad (3.20)$$

Proof. If $L \leq 2R$, we do not require use of the streamwise maximal function. Indeed, since $B_R(x) \subset$

$2\mathcal{C} \subset B_{6R}(x)$ for any $x \in \mathcal{C}$, we have

$$\int_{2\mathcal{C}} |\nabla u|^2 dx \lesssim \int_{B_{6R}(x)} |\nabla u|^2 dx \leq \mathcal{M}(|\nabla u|^2)(x) \leq \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)](x).$$

As this holds for any $x \in \mathcal{C}$, we can take average of the right-hand side and conclude (3.20).

From now on, we may assume $L > 2R$. Applying Theorem 3.1.3 twice, we have

$$\begin{aligned} \int_{\mathcal{C}_{2R,L+R}} |\nabla u|^2 dx &\leq \frac{C}{R} \int_{-L+R}^{L-R} \int_{B_{4R}} |\nabla u(x + te_1)|^2 dx dt \\ &\leq \frac{C}{R} \int_{-L+R}^{L-R} \int_{B_R} \mathcal{M}(|\nabla u|^2)(x + te_1) dx dt \\ &\leq C \int_{\mathcal{C}_{R,L}} \mathcal{M}(|\nabla u|^2) dx \leq C \int_{\mathcal{C}_{R,L}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] dx. \end{aligned}$$

Since the volume of $\mathcal{C}_{2R,L+R}$ is comparable to $\mathcal{C}_{R,L}$, we conclude

$$\int_{\mathcal{C}_{2R,L+R}} |\nabla u|^2 dx \leq C\tilde{\Xi}^2.$$

The length $L + R$ is smaller than the desired $2L$ length of $2\mathcal{C}$ in Equation (3.20). To improve it, we use a bootstrap argument.

Step 1: The Dirichlet Integral Controls the Proximity of Streamlines We claim that for

any $L' \in [L + R, 2L]$, if

$$\int_{\mathcal{C}_{2R,L'}} |\nabla u|^2 dx \leq C\tilde{\Xi}^2, \tag{3.21}$$

then

$$|\Phi_{\frac{t_1-t_0}{U}}(B_R(t_0e_1)) \cap B_R(t_1e_1)| \geq \mu|B_R|, \quad \forall |t_0|, |t_1| \leq L' - 2R \tag{3.22}$$

with $\mu = 1 - C\varepsilon_1$. Indeed, the measure on the left-hand side of (3.22) is given by

$$\int_{B_R(t_1 e_1)} \rho\left(\frac{t_1}{U}, x\right) dx,$$

where $\rho(s, x)$ is the solution to the following transport equation:

$$\partial_s \rho + \operatorname{div}(\rho u) = 0, \quad \rho\left(\frac{t_0}{U}, x\right) = \mathbf{1}_{B_R(t_0 e_1)}(x).$$

Note that $|\rho| \leq 1$ everywhere in space and time. We observe that

$$\begin{aligned} \frac{d}{ds} \int_{B_R(sU e_1)} \rho(s, x) dx &= \frac{d}{ds} \int_{B_R(0)} \rho(s, x + sU e_1) dx \\ &= \int_{B_R(sU e_1)} \partial_s \rho(s, x) dx + \int_{B_R(sU e_1)} U \partial_{x_1} \rho(s, x) dx \\ &= \int_{B_R(sU e_1)} \operatorname{div}(\rho(U e_1 - u)) dx \\ &= \int_{\partial B_R(sU e_1)} \rho(U e_1 - u) \cdot n d\sigma. \end{aligned}$$

Here $|\rho| \leq 1$ and n is the unit outward normal vector. By the Trace Theorem, we can estimate:

$$\left| \frac{d}{ds} \int_{B_R(sU e_1)} \rho(s, x) dx \right| \lesssim \int_{B_R(sU e_1)} |\nabla u| dx + |\partial B_R| \cdot |U e_1 - \bar{u}(sU)|. \quad (3.23)$$

By (3.21), we can apply Theorem 3.1.2 to $\mathcal{C}_{2R, L'}$ instead of $\mathcal{C}_{2R, 2L}$ and deduce for all $|sU| \leq L' - 2R$ that

$$|U e_1 - \bar{u}(sU)| \lesssim CL\tilde{\Xi}.$$

Without loss of generality, assume $t_0 < t_1$. Integrate Equation (3.23) from $s = \frac{t_0}{U}$ to $s = \frac{t_1}{U}$, we

get

$$\begin{aligned} \int_{B_R(t_1 e_1)} \rho\left(\frac{t_1}{U}, x\right) dx &\geq \int_{B_R(t_0 e_1)} \rho\left(\frac{t_0}{U}, x\right) dx - C \int_{\frac{t_0}{U}}^{\frac{t_1}{U}} \int_{B_R(s U e_1)} |\nabla u| dx ds \\ &\quad - |\partial B_R| \cdot C L \tilde{\Xi} \cdot \frac{t_1 - t_0}{U}. \end{aligned}$$

For the first term on the right, using the initial condition of ρ we get

$$\int_{B_R(t_0 e_1)} \rho\left(\frac{t}{U}, x\right) dx = |B_R|.$$

The second term is estimated using Theorem 3.1.3:

$$\begin{aligned} \int_{\frac{t_0}{U}}^{\frac{t_1}{U}} \int_{B_R(s U e_1)} |\nabla u| dx ds &= \frac{1}{U} \int_{t_0}^{t_1} \int_{B_R(t e_1)} |\nabla u| dx dt \\ &\leq \frac{2R^3 L}{U} \left(\int_{\mathcal{C}_{R, L'}} |\nabla u|^2 dx \right)^{1/2} \leq \frac{2R^3}{U} L \tilde{\Xi}. \end{aligned}$$

The third term is bounded by

$$|\partial B_R| \cdot C L \tilde{\Xi} \cdot \frac{t_1 - t_0}{U} \lesssim R^2 L \tilde{\Xi} \frac{L'}{U} \lesssim \frac{R^2 L}{U} L \tilde{\Xi}.$$

By assumption, $L \tilde{\Xi} \lesssim \varepsilon_1 \frac{R}{L} \cdot U$, so

$$\int_{B_R(t_1 e_1)} \rho\left(\frac{t_1}{U}, x\right) dx \geq |B_R| - \frac{C R^3}{U} L \tilde{\Xi} - \frac{C R^2 L}{U} L \tilde{\Xi} \geq |B_R| - C \varepsilon_1 R^3.$$

This completes the proof for Equation (3.22).

Step 2: The Proximity of Streamlines controls the Dirichlet Integral We claim if (3.22)

holds at $\mu = \frac{1}{2}$, then

$$\int_{\mathcal{C}_{2R, L'+2R}} |\nabla u|^2 dx \leq C \tilde{\Xi}^2 \tag{3.24}$$

So we can improve (3.21) from length L' to a longer length $L' + 2R$. To show (3.24), we use Theorem 3.1.3 and the definition of $\mathcal{M}_\Phi$:

$$\begin{aligned}
\int_{C_{R,L,e_1}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] dx &\geq \frac{1}{R} \int_{-L+R}^{L-R} \int_{B_R} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)](x + te_1) dx dt \\
&= \frac{1}{R} \int_{-L+R}^{L-R} \int_{B_R(te_1)} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] dx dt \\
&\geq \frac{1}{R} \int_{-L+R}^{L-R} \int_{B_R(te_1)} \int_{-\frac{4L}{U}}^{\frac{4L}{U}} \mathcal{M}(|\nabla u|^2) \circ \Phi_s ds dx dt \\
&\gtrsim \frac{1}{RL} \int_{-L+R}^{L-R} \int_{-4L}^{4L} \int_{\Phi_{\frac{t'}{U}}(B_R(te_1))} \mathcal{M}(|\nabla u|^2) dx dt' dt.
\end{aligned}$$

For $t' \in (-4L, 4L)$, $t \in (-L + R, L - R)$, we have $t_1 := t' + t \in (-4L + L - R, 4L - L + R) \supset (-L' + 2R, L' - 2R)$. So

$$\begin{aligned}
&\int_{-4L}^{4L} \int_{\Phi_{\frac{t'}{U}}(B_R(te_1))} \mathcal{M}(|\nabla u|^2) dx dt' \\
&\geq \int_{-L'+2R}^{L'-2R} \int_{\Phi_{\frac{t_1-t}{U}}(B_R(te_1))} \mathcal{M}(|\nabla u|^2) dx dt_1 \\
&\geq \int_{-L'+2R}^{L'-2R} \int_{\Phi_{\frac{t_1-t}{U}}(B_R(te_1)) \cap B_R(t_1 e_1)} \mathcal{M}(|\nabla u|^2) dx dt_1.
\end{aligned}$$

Since for each $x \in B_R(t_1 e_1)$,

$$\mathcal{M}(|\nabla u|^2)(x) \gtrsim \int_{B_{8R}(t_1 e_1)} |\nabla u|^2 dx,$$

so we can use the assumption in Equation (3.22) with $\mu = \frac{1}{2}$, to deduce

$$\begin{aligned}
\int_{\Phi_{\frac{t_1-t}{U}}(B_R(te_1)) \cap B_R(t_1 e_1)} \mathcal{M}(|\nabla u|^2) dx &\gtrsim \frac{1}{2} |B_R| \int_{B_{8R}(t_1 e_1)} |\nabla u|^2 dx \\
&\approx \int_{B_{8R}(t_1 e_1)} |\nabla u|^2 dx.
\end{aligned}$$

Using Theorem 3.1.3 we have

$$\begin{aligned} & \int_{-L'+2R}^{L'-2R} \int_{\Phi_{\frac{t_1-t}{U}}(B_R(te_1)) \cap B_R(t_1 e_1)} \mathcal{M}(|\nabla u|^2) dx dt_1 dt \\ & \gtrsim \int_{-L'+2R}^{L'-2R} \int_{B_{8R}(t_1 e_1)} |\nabla u|^2 dx dt_1 \gtrsim R \int_{\mathcal{C}_{4R, L'+2R, e_1}} |\nabla u|^2 dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathcal{C}_{R, L, e_1}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] dx & \gtrsim \frac{1}{LR} \cdot (2L - 2R) \cdot R \int_{\mathcal{C}_{4R, L'+2R, e_1}} |\nabla u|^2 dx \\ & \geq \int_{\mathcal{C}_{2R, L'+2R, e_1}} |\nabla u|^2 dx. \end{aligned}$$

This proves (3.24), as $\mathcal{C}_{R, L}$ and $\mathcal{C}_{2R, L'+2R}$ have comparable volumes.

Step 3: Bootstrap Combining the previous two steps, we know by choosing $\varepsilon_1 \leq \frac{1}{2C}$, (3.21) implies (3.24) for every $L' \in [L + R, 2L]$. Therefore, after finitely many iterations, we conclude (3.21) is true for $L' = 2L$, which proves (3.20). $\square$

3.2.2 Capsule Construction

For every point $x \in \mathbb{R}^3$, we find a capsule that satisfies our ansatz.

Proposition 3.2.4. *Fix $0 < \varepsilon_0 < 1$, $0 \leq \delta \leq \frac{5}{12}$ and $\sigma \geq \delta$. Let $\varepsilon_1 \ll \varepsilon_0$. For every $x \in \mathbb{R}^3$, there exists a capsule $\mathcal{C}_x = \mathcal{C}_{R(x), L(x), e(x)}(x)$ with radius $R(x)$ and length $L(x)$ pointing in the direction $e(x) \in \mathbb{S}^2$ such that*

$$\oint_{B_{R(x)}(x)} u dx = b(x) = U(x)e(x), \quad \oint_{\mathcal{C}_x} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] dx = \tilde{\Xi}(x)^2, \quad (3.25)$$

where $U(x)$ and $\tilde{\Xi}(x)$ satisfy

$$0 \leq U(x) \leq \frac{L(x)}{R(x)^2} \left(\frac{L(x)}{R(x)} \right)^\sigma, \quad \tilde{\Xi}(x) = \frac{\varepsilon_1}{L(x)R(x)} \left(\frac{L(x)}{R(x)} \right)^\delta. \quad (3.26)$$

The right-hand-side inequality for $U(x)$ becomes an equality when $L(x) > R(x)$. Moreover, it holds that

$$\begin{aligned} \|u - b(x)\|_{L^\infty(\mathcal{C}_x)} &\lesssim \frac{1}{R(x)}, & \text{if } L(x) = R(x), \\ \|u - b(x)\|_{L^\infty(\mathcal{C}_x)} &\lesssim \varepsilon_0 \frac{R(x)}{L(x)} \cdot U(x), & \text{if } L(x) > R(x). \end{aligned}$$

In particular

$$|u(x)| \lesssim \frac{L(x)}{R(x)^2} \left(\frac{L(x)}{R(x)} \right)^\sigma.$$

Proof. We first describe an algorithm for constructing the capsule $\mathcal{C}_x$. To begin with, for each $x \in \mathbb{R}^3$, $R > 0$, define

$$\mathcal{C}_R(x) := \mathcal{C}_{R, L^*, e^*}(x)$$

with $L^* \geq R$ and $e^* \in \mathbb{S}^2$ determined by

$$\oint_{B_R(x)} u \, dx = b^* = U^* e^*, \quad L^* = \max\{U^* R, 1\}^{\frac{1}{1+\sigma}} R.$$

Here $U^* = |b^*| \geq 0$, so U^* and e^* are uniquely determined whenever the average velocity b^* is nonzero. If the average velocity is zero, $U^* = 0$, but the definition of e^* is ambiguous. However, in this case we have $L^* = R$, so the definition of $\mathcal{C}_R(x)$ is the same for whichever e^* one wishes to pick.

From the above construction, we see that U^* and L^* change continuously in R and x , so let us denote $U^* =: U_R(x)$ and $L^* =: L_R(x)$. The capsule $\mathcal{C}_R(x)$ also changes continuously in R and x , because e^* is only discontinuous in the closed set $\{(x, R) \in \mathbb{R}^3 \times [0, \infty) : U_R(x) = 0\}$, but $\mathcal{C}_R(x)$ is

a ball whenever $U_R(x) < \frac{1}{R}$. Therefore, $\tilde{\Xi}_R(x)$ defined below is also continuous in R and x :

$$\tilde{\Xi}_R(x) := \left(\int_{\mathcal{C}_x(R)} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] \, dx \right)^{\frac{1}{2}}.$$

It remains to find a critical value $R > 0$ for each x such that

$$\tilde{\Xi}_R(x) L_R^{1-\delta}(x) R^{1+\delta} = \varepsilon_0.$$

For a fixed x , we have shown that $\tilde{\Xi}_R(x) L_R^{1-\delta}(x) R^{1+\delta}$ is a continuous function of $R \in [0, \infty]$. Note the following limits:

$$\begin{aligned} \lim_{R \rightarrow 0} U_R(x) &= |u(x)|, & \lim_{R \rightarrow 0} L_R(x) &= 0, & \lim_{R \rightarrow 0} \tilde{\Xi}_R(x) &= (\mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)](x))^{1/2}, \\ \lim_{R \rightarrow \infty} U_R(x) &= 0, & \lim_{R \rightarrow \infty} L_R(x) &= \infty, & \lim_{R \rightarrow \infty} \tilde{\Xi}_R(x) &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{R \rightarrow 0} \tilde{\Xi}_R(x) L_R^{1-\delta}(x) R^{1+\delta} &= 0, \\ \lim_{R \rightarrow \infty} \tilde{\Xi}_R(x) L_R^{1-\delta}(x) R^{1+\delta} &= \lim_{R \rightarrow \infty} \left(\int_{\mathcal{C}_x(R)} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] \, dx \right)^{\frac{1}{2}} L_R^{1-\delta}(x) R^{1+\delta} \\ &\geq \lim_{R \rightarrow \infty} \left(L_R^{1-2\delta}(x) R^{2\delta} \int_{\mathcal{C}_x(R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \\ &= \lim_{R \rightarrow \infty} \left(L_R^{1-2\delta}(x) R^{2\delta} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} = +\infty. \end{aligned}$$

By continuity, equality $\tilde{\Xi}_R L_R^{1-\delta} R^{1+\delta} = \varepsilon_0$ must be achieved at some $R^* > 0$. We pick the smallest R^* for which the equality is achieved. We can now finally define

$$\mathcal{C}_x := \mathcal{C}_{R^*}(x).$$

We remark again that even though e is not uniquely defined when $U = 0$, the capsule $\mathcal{C}_x$

degenerates to a ball in this case, so the choice of e is unimportant. Now each $x \in \mathbb{R}^3$ is associated with a cylinder $\mathcal{C}_x$. Moreover, we have chosen $R(x)$, $b(x)$, $U(x)$, $e(x)$, $L(x)$, $\tilde{\Xi}(x)$, with the equalities in Equation (3.25) holding precisely. As functions of x , these functions are not necessarily continuous due to the way we pick R , however, in the next subsection we shall show they are locally comparable.

Let us verify the ansatz (3.16) is satisfied with $\Xi(x)^2 = \int_{2\mathcal{C}_x} |\nabla u|^2 dx$. The conclusions in Equation (3.26) follow from our construction. If $L = R$, then

$$\tilde{\Xi}(x) = \frac{\varepsilon_0}{L(x)R(x)} \left(\frac{L(x)}{R(x)} \right)^\delta = \frac{\varepsilon_1}{L(x)R(x)}.$$

By Theorem 3.2.3, if we choose $\varepsilon_1 \leq \frac{1}{C}\varepsilon_0$ then $\Xi(x) \leq \frac{\varepsilon_0}{L(x)R(x)}$, and (3.16) is true. If $L > R$, then since $\sigma \geq \delta$

$$U(x) = \frac{L(x)}{R(x)^2} \left(\frac{L(x)}{R(x)} \right)^\sigma \geq \frac{L(x)}{R(x)^2} \left(\frac{L(x)}{R(x)} \right)^\delta,$$

and (3.16) is still true.

By our choice of $L(x)$, we see that if $L(x) = R(x)$, then $U(x)R(x) \leq 1$, so $U(x) \leq \frac{1}{R(x)}$, then Equation (3.17) gives the desired estimate. If $L(x) > R(x)$, then since $\sigma \geq 0$:

$$L = (UR)^{\frac{1}{1+\sigma}} R \implies U = \frac{1}{R} \left(\frac{L}{R} \right)^{1+\sigma} \geq \frac{L}{R^2}.$$

Equation (3.17) once again yields the desired estimate. □

Remark 3.2.5. *Since u and ∇u are globally bounded from above, we can see that $R(x)$ and $L(x)$ are globally bounded from below. Equivalently we have that $1/R(x)$, $1/L(x)$ are globally bounded from above.*

It will be useful to distinguish the two cases $L = R$ and $L > R$.

Definition 3.2.6. *We call $x \in \mathbb{R}^3$ a round point if $L(x) = R(x)$, and we call $x \in \mathbb{R}^3$ a long point otherwise. The set of round points is denoted by $\mathcal{R}$, and the set of long points is denoted by $\mathcal{L}$.*

3.2.3 Comparability and a Vitali Covering Lemma

Let $x \in \mathcal{L}$ be a long point, then by Theorem 3.2.4 we know

$$\|u - b(x)\|_{L^\infty(\mathcal{C}_x)} \leq \varepsilon_0 \frac{R(x)}{L(x)} \cdot U(x) \leq \varepsilon_0 U(x).$$

In particular, $|u(y) - b(x)| \leq \varepsilon_0 U(x)$ for every $y \in \mathcal{C}_x$. Therefore

$$(1 - \varepsilon_0)U(x) \leq |u(y)| \leq (1 + \varepsilon_0)U(x), \quad |u(x) - u(y)| \leq 2\varepsilon_0 U(x).$$

This means that $u(x) \approx u(y) \approx b(x)$ up to an order ε_0 relative error.

Lemma 3.2.7. *Suppose x and z are long points such that $\mathcal{C}_x \cap \mathcal{C}_z \neq \emptyset$. If $R(z) \leq 2R(x)$, then $\mathcal{C}_z \subset K\mathcal{C}_x$ for some absolute constant K .*

Proof. If $x, z \in \mathcal{L}$ such that $\mathcal{C}_x$ and $\mathcal{C}_z$ intersect at some point y , then we may conclude that $b(x) \approx u(y) \approx b(z)$ and consequently $U(x) \approx U(z)$, $e(x) \approx e(z)$. Since $x, z \in \mathcal{L}$, we know

$$\frac{L(x)^{1+\sigma}}{R(x)^{2+\sigma}} = U(x) \approx U(z) = \frac{L(z)^{1+\sigma}}{R(z)^{2+\sigma}} \quad \implies \quad L(z) \lesssim 2^{\frac{2+\sigma}{1+\sigma}} L(x) \lesssim L(x),$$

where we have used that $\sigma \geq 0$. Moreover, since

$$|b(x) - b(z)| \leq |b(x) - u(y)| + |u(y) - b(z)| \leq \varepsilon_0 \frac{R(x)}{L(x)} U(x) + \varepsilon_0 \frac{R(z)}{L(z)} U(z),$$

we know that

$$|e(x) - e(z)| \lesssim \varepsilon_0 \left(\frac{R(x)}{L(x)} + \frac{R(z)}{L(z)} \right).$$

Without loss of generality, assume x is at the origin and $e(x) = e_1$ is the first basis vector. Then

$e(z) = e_z^1 e_1 + e_z^2 e_2 + e_z^3 e_3$ satisfies

$$|1 - e_z^1| + |e_z^2| + |e_z^3| \lesssim \varepsilon_0 \left(\frac{R(x)}{L(x)} + \frac{R(z)}{L(z)} \right).$$

Taking any $\tilde{y} \in \mathcal{C}_z$, we compute its distance from y . For the first coordinate, we control it directly using the diameter:

$$|y_1 - \tilde{y}_1| \leq 2L(z) \leq KL(x).$$

For the second coordinate, we use the fact that the direction $e(z)$ is close to e_1 . Since $y \in z + te(z) + B_{R(z)}$ and $\tilde{y} \in z + \tilde{t}e(z) + B_{R(z)}$ for some $t, \tilde{t} \in [-L(z) + R(z), L(z) - R(z)]$, we have

$$\begin{aligned} |y_2 - \tilde{y}_2| &\lesssim |t - \tilde{t}| |e_z^2| + 2R(z) \\ &\leq 2L(z) \cdot \varepsilon_0 \left(\frac{R(x)}{L(x)} + \frac{R(z)}{L(z)} \right) + 2R(z) \\ &= 2\varepsilon_0 \frac{L(z)}{L(x)} \cdot R(x) + 2\varepsilon_0 R(z) + 2R(z) \\ &\leq \varepsilon_0 CR(x) + 4R(x) \leq KR(x), \end{aligned}$$

if ε_0 is chosen small. By a similar estimate for the third coordinate, we conclude that $y - \tilde{y} \in (K+1)\mathcal{C}_x$. Because $y \in \mathcal{C}_x$, we have $\tilde{y} \in (K+2)\mathcal{C}_x$, so $\mathcal{C}_z \subset (K+2)\mathcal{C}_x$. $\square$

We now show that the property proven in Theorem 3.2.7 guarantees that we have the following Vitali-type covering lemma:

Lemma 3.2.8. *Let $A \subset \mathcal{L}$ be a set such that $\sup_{x \in A} R(x) < \infty$. Then there exists a countable collection of pairwise disjoint capsules $\{\mathcal{C}_{x_i}\}_i$ such that $A \subset \bigcup_i K\mathcal{C}_{x_i}$ and*

$$\left| \bigcup_{x \in A} \mathcal{C}_x \right| \leq \sum_i |K\mathcal{C}_{x_i}|.$$

Proof. Let $\mathcal{Q}^{(0)} := \{\mathcal{C}_x\}_{x \in A}$ be the whole collection of capsules. For any $j \geq 1$, we first let x_j be

any point such that

$$R(x_j) > \frac{1}{2} \sup_{\mathcal{C}_x \in \mathcal{Q}^{(j-1)}} R(x).$$

Then we eliminate from the collection $\mathcal{Q}^{(j-1)}$ any capsule $\mathcal{C}_x$ that intersects $\mathcal{C}_{x_j}$ to create the collection $\mathcal{Q}^{(j)}$. This algorithm may cease after finitely many iterations if the collection $\mathcal{Q}^{(j+1)} = \emptyset$ for some $j \geq 0$, or it may never cease. In any case, we have constructed a countable collection of capsules $\{\mathcal{C}_{x_j}\}_{j \geq 0}$ that are pairwise disjoint by construction.

Suppose that $\sum_j |K\mathcal{C}_{x_j}| < \infty$, otherwise the final claim is already true. It follows that either $\{\mathcal{C}_{x_j}\}_{j \geq 0}$ is a finite collection, or it is an infinite collection with $R(x_j) \rightarrow 0$ as $j \rightarrow \infty$. In either case, every capsule $\mathcal{C}_x$ must be removed from $\mathcal{Q}^{(j)}$ at some iteration step. Otherwise, we would have $\mathcal{C}_x \in \mathcal{Q}^{(j-1)}$ for all j , and the first step in the algorithm would then imply that $R(x_j) > \frac{1}{2}R(x)$ for all j . However, the sequence $R(x_j)$ would then not converge to zero. Now suppose $\mathcal{C}_x \in \mathcal{Q}^{(j-1)} \setminus \mathcal{Q}^{(j)}$. Then we have $\mathcal{C}_x \cap \mathcal{C}_{x_j} \neq \emptyset$ and $R(x) < 2R(x_j)$. By Theorem 3.2.7 we have

$$\mathcal{C}_x \subset K\mathcal{C}_{x_j}.$$

Thus $\bigcup_{x \in A} \mathcal{C}_x \subset \bigcup_{j \geq 0} K\mathcal{C}_{x_j}$ and we have:

$$\left| \bigcup_{x \in A} \mathcal{C}_x \right| \leq \left| \bigcup_{j \geq 0} K\mathcal{C}_{x_j} \right| \leq \sum_{j \geq 0} |K\mathcal{C}_{x_j}|.$$

This proves the covering lemma. □

A corollary of the covering lemma is that $\tilde{\Xi} \in L^{2+, \infty}(\mathbb{R}^3)$.

Corollary 3.2.9. *For any $\epsilon > 0$, we have that $\tilde{\Xi}$ is in $L^{2+\epsilon, \infty}(\mathbb{R}^3)$ with bound*

$$\|\tilde{\Xi}\|_{L^{2+\epsilon, \infty}(\mathbb{R}^3)} \lesssim_{\epsilon} \|\nabla u\|_{L^{2+\epsilon}(\mathbb{R}^3)}.$$

Proof. We denote the set

$$S_\alpha = \{x \in \mathbb{R}^3 : \tilde{\Xi}(x)^2 > \alpha\},$$

then $|S_\alpha| = |S_\alpha \cap \mathcal{R}| + |S_\alpha \cap \mathcal{L}|$. Note that $\tilde{\Xi}(x)^2 > \alpha$ implies $R^2 \leq \frac{\varepsilon_0}{\Xi}$ is bounded in S_α . For the long points, we have by Theorem 3.2.8 that

$$|S_\alpha \cap \mathcal{L}| \leq \left| \bigcup_{x \in S_\alpha \cap \mathcal{L}} \mathcal{C}_x \right| \leq \sum_i |K\mathcal{C}_{x_i}|.$$

In the last inequality, a Vitali covering is chosen so that the capsules $\mathcal{C}_{x_i}$ are pairwise disjoint, and $K\mathcal{C}_{x_i}$ covers $S_\alpha \cap \mathcal{L}$. By Jensen's inequality, we have for any $\epsilon > 0$:

$$\int_{\mathcal{C}_x} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)] \, dx \leq \left(\int_{\mathcal{C}_x} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)]^{1+\epsilon} \, dx \right)^{\frac{1}{1+\epsilon}}.$$

Moreover, $\tilde{\Xi}(x_i)^2 > \alpha$ now means

$$\begin{aligned} & \left(\int_{\mathcal{C}_{x_i}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)]^{1+\epsilon} \, dx \right)^{\frac{1}{1+\epsilon}} > \alpha \\ \iff & \int_{\mathcal{C}_{x_i}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)]^{1+\epsilon} \, dx > \alpha^{1+\epsilon} |\mathcal{C}_{x_i}| \\ \iff & |\mathcal{C}_{x_i}| \leq \frac{1}{\alpha^{1+\epsilon}} \int_{\mathcal{C}_{x_i}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)]^{1+\epsilon} \, dx. \end{aligned}$$

Hence

$$\begin{aligned} |S_\alpha \cap \mathcal{L}| & \leq \frac{C}{\alpha^{1+\epsilon}} \sum_i \int_{\mathcal{C}_{x_i}} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)]^{1+\epsilon} \, dx \\ & \leq \frac{C}{\alpha^{1+\epsilon}} \int_{\mathbb{R}^3} \mathcal{M}_\Phi[\mathcal{M}(|\nabla u|^2)]^{1+\epsilon} \, dx \\ & \leq \frac{C}{\alpha^{1+\epsilon}} \int_{\mathbb{R}^3} |\nabla u|^{2(1+\epsilon)} \, dx. \end{aligned}$$

In the last inequality, we used that $\mathcal{M}_\Phi \circ \mathcal{M}$ is a bounded map from $L^{1+\epsilon}$ to $L^{1+\epsilon}$. For the round

points, we can use the usual Vitali covering for balls. Combining the inequalities we have

$$|S_\alpha| \leq \frac{C}{\alpha^{1+\epsilon}} \int_{\mathbb{R}^3} |\nabla u|^{2(1+\epsilon)} dx.$$

Recall $\nabla u \in L^2 \cap L^\infty$, so the right-hand side is finite. Therefore $\tilde{\Xi}^2$ is in $L^{1+\epsilon, \infty}$, so $\tilde{\Xi}$ is in $L^{2(1+\epsilon), \infty}$.

Replacing ϵ with $\epsilon/2$ in the course of the proof gets us our claimed result. $\square$

From this, we deduce that

Proposition 3.2.10. *If L/R is bounded then u is trivial.*

Proof. If L/R is bounded, then L and R are comparable. Then by Theorem 3.2.4

$$|u(x)| \leq \frac{L(x)}{R(x)^2} \left(\frac{L(x)}{R(x)} \right)^\sigma \approx \frac{C}{R(x)} \approx \tilde{\Xi}(x)^{\frac{1}{2}},$$

Therefore $u \in L^{4^+, \infty}(\mathbb{R}^3)$. Together with $u \in L^\infty$, we know that $u \in L^{\frac{9}{2}}$ and thus $u \equiv 0$ by Galdi's result. $\square$

3.3 Conditional Liouville Theorems

Here we give some conditions under which u is trivial, proving the theorems claimed in our introduction.

3.3.1 Bounded Line Integral Hypothesis

The stretching ratio L/R is comparable in size to a certain line integral, as the following lemma will demonstrate.

Lemma 3.3.1. *Let $x \in \mathcal{L}$ be a long point. Then*

$$\int_{x-L(x)e(x)}^{x+L(x)e(x)} u \cdot d\ell \approx L(x)U(x). \quad (3.27)$$

Here $L(x)$, $U(x)$ and $e(x)$ are constructed in Theorem 3.2.4.

Proof. According to Theorem 3.2.4, we know that in $\mathcal{C}_x$ we have $u(y) = U(x)e(x) + \zeta(y)$ with $|\zeta| \leq C\varepsilon_0 U(x) < \frac{1}{2}U(x)$. We thus have

$$\begin{aligned} \left| \int_{x-L(x)e(x)}^{x+L(x)e(x)} \zeta \cdot d\ell \right| &\leq 2L(x) \cdot \frac{1}{2}U(x) \\ &= \frac{1}{2} \int_{x-L(x)e(x)}^{x+L(x)e(x)} U(x)e(x) \cdot d\ell. \end{aligned}$$

We can thus bound the line integral from above and below by:

$$\frac{1}{2} \int_{x-L(x)e(x)}^{x+L(x)e(x)} U(x)e(x) \cdot d\ell \leq \int_{x-L(x)e(x)}^{x+L(x)e(x)} u \cdot d\ell \leq \frac{3}{2} \int_{x-L(x)e(x)}^{x+L(x)e(x)} U(x)e(x) \cdot d\ell.$$

The upper and lower bounds are both of order $L(x)U(x)$, which is exactly the right-hand side of Equation (3.27). $\square$

Suppose that the line integral of u is globally bounded:

$$\int_{x_0}^x u \cdot d\ell \leq C, \quad \forall x_0, x \in \mathbb{R}^3,$$

then the stretching ratio $LU \approx \left(\frac{L}{R}\right)^{2+\sigma}$ is bounded by Theorem 3.3.1. Theorem 3.2.10 would then imply that the solution is trivial. Since there is still a gap between $L^{4^+, \infty}$ and $L^{\frac{9}{2}}$, we can relax the boundedness assumption and allow an assumption like Equation (1.12) in Theorem 1.3.2.

Proof of Theorem 1.3.2. Combining the assumption in Equation (1.12) with Equation (3.27), we know that at any $x \in \mathcal{L}$:

$$LU \approx \left(\frac{L}{R}\right)^{2+\sigma} \lesssim L^\beta.$$

As discussed in Theorem 3.2.5, L is bounded from below, so the above inequality is also true for

round points. Now by Theorem 3.2.4

$$\begin{aligned}
|u| &\lesssim \frac{1}{L} \left(\frac{L}{R} \right)^{2+\sigma} \implies \frac{|u|^p}{\tilde{\Xi}^2} \lesssim L^{2-p} R^2 \left(\frac{L}{R} \right)^{(2+\sigma)p-2\delta} \\
&= L^{4-p} \left(\frac{L}{R} \right)^{(2+\sigma)p-2\delta-2} \\
&\lesssim L^{4-p+\beta(p-\frac{2(\delta+1)}{2+\sigma})}.
\end{aligned}$$

If the power of L satisfies

$$4 - p + \beta \left(p - \frac{2(\delta+1)}{2+\sigma} \right) = 0,$$

then $|u|^p/\tilde{\Xi}^2$ is bounded. Because $\tilde{\Xi}^2 \in L^{1+, \infty}$, we conclude that $u \in L^{p+, \infty}$. We get the optimal $p = \frac{4-34\beta/29}{1-\beta}$ when we set $\sigma = \frac{5}{12}$ and $\delta = \frac{5}{12}$, If $\beta < \frac{29}{193}$ then $p < \frac{9}{2}$, and $u \in L^{p, \infty} \cap L^\infty$ implies $u \in L^{\frac{9}{2}}$. Galdi's criterion then asserts that $u \equiv 0$. $\square$

3.3.2 Mean Oscillation Hypothesis

Previous results showed that if $u \in BMO^{-1}$, then u is trivial. We make some refinements in this direction. Suppose $u = \text{curl } \psi$ with $\psi \in L^1_{loc}$.

Lemma 3.3.2. *Let $x \in \mathcal{L}$ be a long point. Then*

$$\int_{B_{R(x)}(x)} (\psi(y) - (\psi)_{B_{R(x)}}) \cdot (e(x) \times (y - x)) \, dy \approx U(x) R(x)^5. \quad (3.28)$$

Here $L(x)$, $R(x)$ and $e(x)$ are constructed in Theorem 3.2.4.

Proof. For simplicity, assume $x = 0$ is the origin and $e(x) = e_1$, and let us abbreviate $R = R(x)$,

$L = L(x)$, and $U = U(x)$. We observe that

$$\begin{aligned} \int_{B_R} (\psi(y) - (\psi)_{B_R}) \cdot (e_1 \times y) \, dy &= \int_{B_R} (\psi(y) - (\psi)_{B_R}) \cdot \operatorname{curl} (y_1 y) \, dx \\ &= \int_{B_R} \operatorname{curl} \psi \cdot y_1 y \, dy + \int_{\partial B_R} (y_1 y) \times \psi \cdot n \, d\sigma \\ &= \int_{B_R} u \cdot y_1 y \, dy. \end{aligned}$$

We used that y is parallel to n on ∂B_R . We want to use the fact that $u \approx U e_1$ with small relative error. Note that

$$\int_{B_R} U e_1 \cdot y_1 y \, dy = U \int_{B_R} y_1^2 \, dy = \frac{U}{3} \int_{B_R} |y|^2 \, dy = \frac{U}{15} R^5.$$

By Theorem 3.2.4, we know

$$\left| \int_{B_R} (u(y) - U e_1) \cdot y_1 y \, dy \right| \leq \varepsilon_0 U(x) \int_{B_R} |y|^2 \, dy \leq \frac{U}{30} R^5.$$

Therefore, we have

$$\int_{B_R} u \cdot y_1 y \, dy \approx U R^5.$$

This finishes the proof of the lemma. □

Write $\bar{\psi} = \oint_{B_R} \psi$. Notice that for any $s \geq 1$:

$$\int_{B_R} (\psi - \bar{\psi}) \cdot (e_1 \times y) \, dy \leq R^4 \left(\oint_{B_R} |\psi - \bar{\psi}|^s \, dx \right)^{1/s}. \quad (3.29)$$

If $\|\psi\|_{BMO} \leq C$, then $UR \approx \left(\frac{L}{R}\right)^{1+\sigma} \lesssim \|\psi\|_{BMO}$, and we conclude that the ratio L/R is globally bounded, so $u \equiv 0$ by Theorem 3.2.10. Again, there is some room for improvement.

Proof of Theorem 1.3.1. Combining Equation (3.28), Equation (3.29), and the assumption in Equa-

tion (1.11), we know that for $x \in \mathcal{L}$ with $R(x) > 1$ it holds that

$$\left(\frac{L}{R}\right)^{1+\sigma} \lesssim R^\alpha.$$

The set $R(x) \leq 1$ is a bounded set, and R is bounded from below, so the above holds for all $x \in \mathbb{R}^3$.

Again by Theorem 3.2.4

$$\begin{aligned} |u| \lesssim \frac{1}{R} \left(\frac{L}{R}\right)^{1+\sigma} &\implies \frac{|u|^p}{\tilde{\Xi}^2} \lesssim L^2 R^{2-p} \left(\frac{L}{R}\right)^{(1+\sigma)p-2\delta} \\ &= R^{4-p} \left(\frac{L}{R}\right)^{(1+\sigma)p-2\delta+2} \\ &\lesssim R^{4-p+\alpha \cdot (p-\frac{2(\delta-1)}{1+\sigma})}. \end{aligned}$$

If the power of R satisfies

$$4 - p + \alpha \left(p + \frac{2(1-\delta)}{1+\sigma}\right) = 0,$$

then $|u|^p/\tilde{\Xi}^2$ is bounded, and we conclude $u \in L^{p^+, \infty}$. By sending $\sigma \rightarrow +\infty$, we get any $p > \frac{4}{1-\alpha}$.

If $\alpha < \frac{1}{9}$ then we can find some $p < \frac{9}{2}$, and again $u \equiv 0$. $\square$

3.3.3 Alternative Proofs

In the previous proof, we can see our construction in Theorem 3.2.4 and Theorem 3.2.10, u is in $L^{4^+, \infty}(\mathcal{R})$ for the round points, and additional assumptions added in Theorem 1.3.1 and 1.3.2 are used to show $L^{p, \infty}(\mathcal{L})$ for the long points. We can also use an alternative ansatz to weaken what we can obtain for the round points, and strengthen the conclusions for the long points. We briefly illustrate the ideas here.

New Ansatz and Construction

In Section 3.1.4, instead of assuming (3.16), we replace it by the following alternative assumption for $\lambda \geq 1$:

$$L\Xi = \frac{\varepsilon_0}{R^\lambda}. \quad (3.30)$$

This would lead to the following bounds instead of (3.17), provided $R \geq 1$:

$$\|u - Ue_1\|_{L^\infty(\mathcal{C})} \lesssim \Xi R^{\frac{1}{2}} V^{\frac{1}{2}} \left(\frac{UR}{L} + \frac{1}{R} \right) + L\Xi \lesssim \varepsilon_0 \left(\frac{UR}{L} + \frac{1}{R^\lambda} \right).$$

In the proof of Theorem 3.2.4, instead of setting $L^* = \max\{U^*R, 1\}^{\frac{1}{1+\sigma}}R$, we fix some $\gamma \in [0, \lambda]$ and set

$$L^* = \max\{U^*R^\gamma, 1\}R.$$

We would end up with $U(x)$ satisfying

$$U(x) \leq \frac{L(x)}{R(x)^{1+\gamma}}$$

rather than (3.26). Again, equality is achieved for the long points. The following conclusions will hold:

$$\begin{aligned} \|u - b(x)\|_{L^\infty(\mathcal{C}_x)} &\lesssim \varepsilon_0 \frac{1}{R(x)^\gamma}, & \text{if } L(x) = R(x), \\ \|u - b(x)\|_{L^\infty(\mathcal{C}_x)} &\lesssim \varepsilon_0 \frac{R(x)}{L(x)} \cdot U(x), & \text{if } L(x) > R(x), \\ |u(x)| &\lesssim \frac{L(x)}{R(x)^{1+\gamma}}. \end{aligned}$$

New Proof of Theorem 1.1

By Theorem 3.3.2, we conclude for a long point x it holds that

$$\int_{B_{R(x)}(x)} (\psi(y) - (\psi)_{B_{R(x)}}) \cdot (e(x) \times (y - x)) \, dy \approx UR^5 \approx LR^{4-\gamma}.$$

Therefore

$$LR^{4-\gamma} \lesssim R^4 \left(\int_{B_R} |\psi - \bar{\psi}|^s \, dx \right)^{1/s} \lesssim R^{4+\alpha} \implies L \lesssim R^{\alpha+\gamma}.$$

By setting $\gamma = 1 - \alpha$, we know L/R is globally bounded. Therefore

$$|u| \lesssim \frac{L}{R^{1+\gamma}} \approx R^{-\gamma}, \Xi \approx \frac{\varepsilon_0}{LR^\lambda} \approx R^{-1-\lambda} \implies \frac{|u|^p}{\Xi^2} \lesssim R^{-p\gamma+2+2\lambda}.$$

Setting $\lambda = 1$, we have the following consequence in the region $\{R \geq 1\}$:

$$|u|^p \lesssim \Xi^2 \iff -p\gamma + 2 + 2\lambda \leq 0 \iff p(1 - \alpha) \geq 4 \iff p \geq \frac{4}{1 - \alpha}.$$

New Proof of Theorem 1.2

By Theorem 3.3.1, we conclude for a long point x it holds that

$$\int_{x-L(x)e(x)}^{x+L(x)e(x)} u \cdot d\ell \approx L(x)U(x) \approx \frac{L(x)^2}{R(x)^{1+\gamma}}.$$

Therefore

$$\frac{L^2}{R^{1+\gamma}} \lesssim L^\beta \implies L^{2-\beta} \lesssim R^{1+\gamma}.$$

Setting $\gamma = 1 - \beta$, we know L/R is globally bounded. Proceeding as in the proof for Theorem 1.3.1, we show $u \in L^{p^+, \infty}$ with $p \geq \frac{4}{1-\beta}$. Unfortunately, this method cannot cover the entire $p \geq \frac{4-34\beta/29}{1-\beta}$ range.

3.4 Estimates for the Poisson Equation with Drift

In this section, we discuss the Poisson equation with constant drift $b = Ue_1$:

$$b \cdot \nabla \theta - \Delta \theta = f + \operatorname{div} g. \quad (3.31)$$

We find the fundamental solution and derive L^p estimates.

3.4.1 The Fundamental Solution

We first find the fundamental solution to Equation (3.31) in three dimensions.

Lemma 3.4.1. *The fundamental solution Γ that solves $b \cdot \nabla \Gamma - \Delta \Gamma = \delta_0$ is*

$$\Gamma(x) = \frac{1}{4\pi r} e^{-\lambda(r-x_1)}, \quad \lambda = \frac{U}{2}.$$

Here $x = (x_1, x_2, x_3)$, $b = Ue_1$, $r = |x|$, and $U \geq 0$.

Proof. To verify its validity, we first compute away from $r = 0$ the gradient $\nabla \Gamma = \Gamma \nabla \log \Gamma$ with

$$\nabla \log \Gamma = -\nabla \log r - \lambda \nabla(r - x_1) = -\frac{\hat{x}}{r} - \lambda(\hat{x} - e_1) = -\frac{1}{r}\hat{x} - \lambda\hat{x} + \lambda e_1.$$

Here we used the notation $\hat{x} = x/r$. Next we compute the Laplacian:

$$\Delta \Gamma = \operatorname{div} \nabla \Gamma = \operatorname{div}(\Gamma \nabla \log \Gamma) = \Gamma |\nabla \log \Gamma|^2 + \Gamma \Delta \log \Gamma,$$

where

$$\Delta \log \Gamma = \operatorname{div} \left(-\frac{1}{r}\hat{x} - \lambda\hat{x} \right) = -\left(\frac{1}{r} + \lambda \right) \frac{2}{r} + \frac{1}{r^2} = -\left(\frac{1}{r} + \lambda \right)^2 + \lambda^2.$$

Substituting this into $\Delta\Gamma$, we get

$$\begin{aligned} |\nabla \log \Gamma|^2 + \Delta \log \Gamma &= \left| \left(\frac{1}{r} + \lambda \right) \hat{x} - \lambda e_1 \right|^2 - \left(\frac{1}{r} + \lambda \right)^2 + \lambda^2 \\ &= -2\lambda e_1 \cdot \left(\left(\frac{1}{r} + \lambda \right) \hat{x} - \lambda e_1 \right) = 2\lambda e_1 \cdot \nabla \log \Gamma. \end{aligned}$$

Hence, $\Delta\Gamma = (2\lambda e_1 \cdot \nabla \log \Gamma)\Gamma = Ue_1 \cdot \nabla\Gamma = b \cdot \nabla\Gamma$.

We have verified that $b \cdot \nabla\Gamma - \Delta\Gamma = 0$ in the open set $\mathbb{R}^3 \setminus \{0\}$. It remains to check the behavior near $r = 0$. Taking a smooth test function φ supported in B_r , we compute

$$\begin{aligned} \int_{B_r} (b \cdot \nabla\Gamma - \Delta\Gamma)\varphi \, dx &= \int_{B_r} (-b \cdot \nabla\varphi - \Delta\varphi)\Gamma \, dx \\ &\quad + \int_{\partial B_r} (\Gamma b - \varphi \nabla\Gamma + \Gamma \nabla\varphi) \cdot n \, d\sigma. \end{aligned}$$

Fixing φ , the left-hand side is independent on the value of r , so we may let $r \rightarrow 0$. We notice that Γ has the following trivial bound:

$$|\Gamma(x)| \leq \frac{1}{4\pi r}.$$

All the integrals on the right-hand-side except the one with $\nabla\Gamma$ will vanish as $r \rightarrow 0$. Moreover,

$$\int_{\partial B_r} \nabla\Gamma \cdot \varphi n \, d\sigma = \int_{\partial B_r} \Gamma \nabla \log \Gamma \cdot \varphi n \, d\sigma = - \int_{\partial B_r} \Gamma \left(\frac{1}{r} \hat{x} + \lambda \hat{x} - \lambda e_1 \right) \cdot \varphi n \, d\sigma.$$

Only the term with $\frac{1}{r}$ will survive in the limit $r \rightarrow 0$. Note that $n = \hat{x}$, so we have

$$\lim_{r \rightarrow 0} \int_{\partial B_r} (-\nabla\Gamma) \cdot \varphi n \, d\sigma = \lim_{r \rightarrow 0} \int_{\partial B_r} \frac{\Gamma}{r} \varphi \, d\sigma = \varphi(0).$$

This finishes the proof. □

From the form of the fundamental solution, we see immediately that $|\Gamma| \leq \frac{1}{r}$ and $\Gamma \in L^{3,\infty}$. We

would like to establish an upper bound for $|\nabla\Gamma|$ that is uniform in the size of λ . Note that

$$|r(\hat{x} - e_1)|^2 = (x_1 - r)^2 + x_2^2 + x_3^2 = (r - x_1)^2 + r^2 - x_1^2 = 2r(r - x_1).$$

Therefore

$$|\nabla \log \Gamma| \leq \frac{1}{r} + \lambda \sqrt{\frac{2(r - x_1)}{r}} \leq (\lambda(r - x_1) + 1) \sqrt{\frac{2}{r(r - x_1)}}.$$

We used the inequality $0 \leq r - x_1 \leq 2r$. It follows that

$$|\nabla \Gamma| \leq \frac{1}{4\pi r} e^{-\lambda(r - x_1)} (\lambda(r - x_1) + 1) \sqrt{\frac{2}{r(r - x_1)}} \leq C r^{-\frac{3}{2}} (r - x_1)^{-\frac{1}{2}}.$$

We used that $e^{-\lambda(r - x_1)}(1 + \lambda(r - x_1))$ is bounded uniformly in x_1 , r , and λ .

Lemma 3.4.2. $\nabla\Gamma$ is bounded in $L_{x_1}^{\frac{3}{2},\infty} L_{x_2,x_3}^{\frac{3}{2}}$ uniformly in λ .

Proof. By direct integration:

$$\begin{aligned} & \int_{\mathbb{R}^2} \left[r^{-\frac{3}{2}} (r - x_1)^{-\frac{1}{2}} \right]^{\frac{3}{2}} dx_2 dx_3 \\ &= 2\pi \int_0^\infty (\rho^2 + x_1^2)^{-\frac{9}{8}} \left[(\rho^2 + x_1^2)^{\frac{1}{2}} - x_1 \right]^{-\frac{3}{4}} \rho d\rho \leq \frac{4}{|x_1|}. \end{aligned}$$

Since $\frac{1}{x_1} \in L_{x_1}^{1,\infty}$, we conclude that $\nabla\Gamma \in L_{x_1}^{\frac{3}{2},\infty} L_{x_2,x_3}^{\frac{3}{2}}$. □

3.4.2 Local L^p Estimates

We have found $\Gamma \in L^{3,\infty}$ and $\nabla\Gamma \in L_{x_1}^{\frac{3}{2},\infty} L_{x_2,x_3}^{\frac{3}{2}}$, with bounds uniform in the size of the drift.

Therefore, if $f \in L^p(\mathbb{R}^3)$, $1 < p < \frac{3}{2}$, it holds that

$$\|\Gamma * f\|_{L^r} \lesssim \|f\|_{L^p}$$

with $\frac{1}{r} = \frac{1}{p} - \frac{2}{3}$. If $g \in L^q(\mathbb{R}^3)$, $1 < q < 3$, then

$$\|\Gamma * \operatorname{div} g\|_{L^r} \lesssim \|g\|_{L^q}$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{3}$. A proof of these convolution inequalities is given in [72].

Recall $b = Ue_1$. Suppose θ solves Equation (3.31) in a *capsule* $\mathcal{C}$ of radius R , length $L \geq R$. We want to derive a local estimate on θ .

Lemma 3.4.3. *If $\theta \in L^q(\mathcal{C})$ solves Equation (3.31) in $\mathcal{C}$ with $f, g \in L^q(\mathcal{C})$ for some $1 < q < 3$, then*

$$\|\theta\|_{L^r(\frac{1}{2}\mathcal{C})} \lesssim R \|f\|_{L^q(\mathcal{C})} + \|g\|_{L^q(\mathcal{C})} + \left(\frac{UR}{L} + \frac{1}{R} \right) \|\theta\|_{L^q(\mathcal{C})} := \text{RHS},$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{3}$.

Proof. Fix a non-negative cut-off function $\phi \in C_c^\infty(\mathcal{C})$ with $\phi \equiv 1$ inside $\frac{1}{2}\mathcal{C}$, satisfying bounds as in Equation (3.8). Then

$$b \cdot \nabla(\phi\theta) - \Delta(\phi\theta) = \phi f + \operatorname{div}(\phi g) + \theta b \cdot \nabla\phi - 2 \operatorname{div}(\theta \nabla\phi) + \theta \Delta\phi - g \cdot \nabla\phi.$$

Denote

$$\tilde{f} = \phi f + \theta b \cdot \nabla\phi + \theta \Delta\phi - g \cdot \nabla\phi, \quad \tilde{g} = \phi g - 2\theta \nabla\phi.$$

Then $\phi\theta$ solves

$$b \cdot \nabla(\phi\theta) - \Delta(\phi\theta) = \tilde{f} + \operatorname{div} \tilde{g}.$$

The function $\phi\theta$ is the unique compactly supported solution, so $\phi\theta = \Gamma * (\tilde{f} + \operatorname{div} \tilde{g})$. First, for the forcing term in divergence form, we note that

$$\|\tilde{g}\|_{L^q(\mathcal{C})} \lesssim \|g\|_{L^q(\mathcal{C})} + \frac{1}{R} \|\theta\|_{L^q(\mathcal{C})} \leq \text{RHS}.$$

Therefore,

$$\|\Gamma * \operatorname{div} \tilde{g}\|_{L^r(\mathcal{C})} \lesssim \|\tilde{g}\|_{L^q(\mathcal{C})} \lesssim \text{RHS}.$$

For the forcing term not in divergence form, we notice that

$$\|\tilde{f}\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^q(\mathcal{C})} + \frac{1}{R} \|g\|_{L^q(\mathcal{C})} + \left(\frac{U}{L} + \frac{1}{R^2} \right) \|\theta\|_{L^q(\mathcal{C})}.$$

Here we used that $|b \cdot \nabla \phi| \lesssim \frac{U}{L}$. Moreover, we have the elementary bound:

$$\|\Gamma\|_{L^{\frac{3}{2}}(\mathcal{C})}^{\frac{3}{2}} \lesssim \int_{3\mathcal{C}} \frac{1}{r^{\frac{3}{2}}} dx \leq CR^{\frac{3}{2}},$$

where the constant C does not depend on L . Suppose $\tilde{\Gamma} = \phi_1 \Gamma$, where ϕ_1 is a smooth function supported on $2\mathcal{C}$ and vanishing off $3\mathcal{C}$. Since $\tilde{f}$ is supported in $\mathcal{C}$, we have that $(\tilde{\Gamma} * \tilde{f})(x) = (\Gamma * \tilde{f})(x)$ for all $x \in \mathcal{C}$. In addition, we have by Young's convolution inequality:

$$\begin{aligned} \|\Gamma * \tilde{f}\|_{L^r(\mathcal{C})} &= \|\tilde{\Gamma} * \tilde{f}\|_{L^r(\mathcal{C})} \leq \|\tilde{\Gamma} * \tilde{f}\|_{L^r(\mathbb{R}^3)} \leq \|\Gamma\|_{L^{\frac{3}{2}}(\mathcal{C})} \|\tilde{f}\|_{L^q(\mathcal{C})} \\ &\lesssim R \|f\|_{L^q(\mathcal{C})} + \|g\|_{L^q(\mathcal{C})} + \left(\frac{UR}{L} + \frac{1}{R} \right) \|\theta\|_{L^q(\mathcal{C})}. \end{aligned}$$

Since $\phi \equiv 1$ in $\frac{1}{2}\mathcal{C}$, we have completed the proof. $\square$

Chapter 4

Non-Uniqueness

4.1 Preliminaries

4.1.1 Notation

We use the convention $\mathbb{N} = \{0, 1, 2, \dots\}$. Capital letters such as N , M , etc. are reserved for parameters taking values in the dyadic natural numbers $2^{\mathbb{N}}$.

We write $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ for $x \in \mathbb{R}^3$. For any positive expression Y , $O(Y)$ stands in for any quantity X satisfying $X \leq CY$. The expressions $X \lesssim Y$, $X \gtrsim Y$, and $X \sim Y$ take on their usual meanings: $X \leq CY$, $X \geq CY$, and $C^{-1}X \leq Y \leq CX$ respectively. We add subscripts to O , $\lesssim$, etc. to indicate that the constants can depend on the variables in the subscript. See §4.1.4 for more precise details on the dependencies of the constants.

We write $a \vee b$ and $a \wedge b$ for $\max\{a, b\}$ and $\min\{a, b\}$, respectively.

For $a, b \in \mathbb{R}^3$, define the symmetric tensor product $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$. Let $S^{3 \times 3}(\mathbb{R})$ be the space of real symmetric 3×3 matrices equipped with the Frobenius inner product, which we denote by $A : B := A_{ij}B_{ij}$. Here and elsewhere, repeated indices are summed according to the Einstein convention.

We use the following Fourier series convention: if $f \in C^\infty(\mathbb{T}^3; \mathbb{R})$,

$$\hat{f}(\xi) = \oint_{\mathbb{T}^3} f(x) e^{-ix \cdot \xi} dx, \quad f(x) = \sum_{\xi \in \mathbb{Z}^3} \hat{f}(\xi) e^{ix \cdot \xi}.$$

Occasionally we write $\mathcal{F}f$ for $\hat{f}$. Here and elsewhere, $\mathbb{T}^3$ denotes the length- 2π torus $(\mathbb{R}/2\pi\mathbb{Z})^3$ and $\oint_{\mathbb{T}^3}$ is the normalized integral $(2\pi)^{-3} \int_{\mathbb{T}^3}$.

4.1.2 Littlewood–Paley Estimates

Let us define a dyadic partition of unity of $\mathbb{R}^3$,

$$\sigma(\xi) + \sum_{N \geq 1} \pi(N^{-1}\xi) = 1$$

for $\sigma \in C_c^\infty(\{\xi \in \mathbb{R}^3 : |\xi| < 1\})$ and $\pi \in C_c^\infty(\{\xi \in \mathbb{R}^3 : \frac{1}{2} < |\xi| < 2\})$, where N takes values in the dyadic natural numbers $2^{\mathbb{N}}$. The details are standard and we refer the reader to, for instance, [5]. For f a distribution on $\mathbb{T}^3$, one can define the Littlewood–Paley projections as Fourier multipliers according to

$$\begin{aligned} P_N f(x) &= \sum_{\xi \in \mathbb{Z}^3} \pi(N^{-1}\xi) \hat{f}(\xi) e^{ix \cdot \xi}, \\ P_{\leq N} f(x) &= \sum_{\xi \in \mathbb{Z}^3} (\sigma(\xi) + \sum_{M \leq N} \pi(M^{-1}\xi)) \hat{f}(\xi) e^{ix \cdot \xi}, \end{aligned}$$

with the obvious interpretation for related notation such as $P_{>N}$, etc.

More generally, for $m \in C^\infty(\mathbb{R}^3 \setminus 0; \mathbb{C})$ and $f \in \mathcal{S}(\mathbb{T}^3)$, we define the Fourier multiplier with symbol m by

$$m(\nabla/i) f(x) := \sum_{\xi \in \mathbb{Z}^3} m(\xi) \hat{f}(\xi) e^{ix \cdot \xi}.$$

Important examples for our purpose are constant coefficient differential operators with m a polynomial, the heat propagator with $m(\xi) = e^{-t|\xi|^2}$, and the Leray projection $\mathbb{P}$ with $m(\xi) = \text{Id} -$

$|\xi|^{-2}\xi \otimes \xi$. Recall that these operators are all mutually commuting.

Let us state some basic estimates on the interaction of Littlewood–Paley projection with these multipliers. Note that by applying them component-wise, they extend easily to vector- and matrix-valued multipliers.

Lemma 4.1.1 (Bernstein inequalities). *Let $m \in C^\infty(\mathbb{R}^3 \setminus 0; \mathbb{C})$ obeying*

$$|\nabla^i m(\xi)| \leq A|\xi|^{\alpha-i} \quad (4.1)$$

for $i = 0, 1, \dots, 10$. For all $1 \leq p \leq q \leq \infty$ and $f \in L^p(\mathbb{T}^3; \mathbb{R})$,

$$\|P_N m(\nabla/i) f\|_{L^q(\mathbb{T}^3)} \lesssim_{p,q} A N^{\alpha + \frac{3}{p} - \frac{3}{q}} \|f\|_{L^p(\mathbb{T}^3)}. \quad (4.2)$$

In particular, one obtains the usual Bernstein inequalities for $P_N \nabla^n$.

Lemma 4.1.2 (Estimates on the heat propagator). *Let $f \in L^p(\mathbb{T}^3; \mathbb{R})$ and $p \in [1, \infty]$. Then for all $t \geq 0$,*

$$\|e^{t\Delta} P_N f\|_{L^p(\mathbb{T}^3)} \lesssim_p \exp(-N^2 t/O(1)) \|f\|_{L^p(\mathbb{T}^3)}. \quad (4.3)$$

More generally,

$$\|e^{t\Delta} P_N b(\nabla/i) f\|_{L^p(\mathbb{T}^3)} \lesssim_{p,b} N^\alpha \exp(-N^2 t/O(1)) \|f\|_{L^p(\mathbb{T}^3)} \quad (4.4)$$

where $b \in C^\infty(\mathbb{R}^3 \setminus 0; \mathbb{C})$ is any α -homogeneous symbol. In other words, $b(\mu\xi) = \mu^\alpha b(\xi)$ for all $\mu > 0$.

Moreover, if b is as above with $\alpha > 0$,

$$\|e^{t\Delta} f\|_{L^p(\mathbb{T}^3)} \lesssim_p \exp(-t/O(1)) \|f\|_{L^p(\mathbb{T}^3)}, \quad (4.5)$$

$$\|e^{t\Delta} b(\nabla/i) f\|_{L^p(\mathbb{T}^3)} \lesssim_p t^{-\frac{\alpha}{2}} \exp(-t/O(1)) \|f\|_{L^p(\mathbb{T}^3)} \quad (4.6)$$

where, in (4.5), one must assume additionally that $f_{\mathbb{T}^3} f = 0$.

In (4.4), one should think of $b(\xi)$ as being a component of ∇ , $\nabla\mathbb{P}$, etc. The proofs are deferred to the final sections 4.7.

4.1.3 Function Spaces

In the sequel we make use of an equivalent definition of BMO^{-1} : $u \in BMO^{-1}$ if $u = \operatorname{div} \varphi$ for some vector field $\varphi \in (BMO)^3$. Recall that $\varphi \in L^1_{loc}$ is in BMO if

$$\|\varphi\|_{BMO} := \sup_Q \int_Q |\varphi(x) - \varphi_Q| dx < \infty$$

where the supremum runs over cubes $Q \subset \mathbb{R}^3$ and $\varphi_Q = f_Q \varphi$. BMO thus defined is obviously only a norm when taken modulo constants, but this issue vanishes for BMO^{-1} . These definitions are adapted without issue to functions on $\mathbb{T}^3$, considering them as periodic functions on $\mathbb{R}^3$. It is easy to see that averages on large cubes are comparable to averages over a single period. Thus, for instance, the spaces BMO^{-1} and bmo^{-1} from [59] coincide in the periodic setting.

We also use the homogeneous Hölder–Zygmund spaces, which are the spaces of distributions satisfying

$$\|u\|_{\mathcal{C}^s} := \sup_{N \in 2^{\mathbb{N}}} N^s \|P_N u\|_{L^\infty} < \infty$$

where $s \in \mathbb{R}$. It is well-known that for $s \in (0, 1)$, $\mathcal{C}^s$ coincides with the standard homogeneous Hölder space C^s . More generally, when $s = m + \alpha$ with $m \in \mathbb{N}$ and $\alpha \in (0, 1)$, the space $\mathcal{C}^s$ coincides with $C^{m, \alpha}$. For all $s \in \mathbb{R}$, $\mathcal{C}^s$ coincides with the Besov space $\dot{B}^s_{\infty, \infty}$. In particular, $\mathcal{C}^{-1} = \dot{B}^{-1}_{\infty, \infty}$ is a space discussed previously that is strictly weaker than BMO^{-1} . We shall use the fact that for all $s > 0$, $\mathcal{C}^s \cap L^\infty$ is a multiplicative algebra. Namely, for $f, g \in \mathcal{C}^s \cap L^\infty$ we have

$$\|fg\|_{\mathcal{C}^s} \lesssim_s \|f\|_{\mathcal{C}^s} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\mathcal{C}^s}.$$

We remark that $\mathcal{C}^s$ and C^s defined as above are homogeneous and do not control the function at frequency zero. This is as intended, and typically does not make a difference, since most of the fields we shall measure have zero average on $\mathbb{T}^3$. For such fields, it is clearly the case that $\|f\|_{L^\infty} \lesssim \|f\|_{\mathcal{C}^s}$ for all $s > 0$.

Let us also record the following heat estimate in the Hölder–Zygmund scale, which is an easy consequence of (4.4): for all $m \geq 0$, $s_1, s_2 \in \mathbb{R}$ satisfying $m + s_1 - s_2 \geq 0$, and $f \in \mathcal{C}^{s_2}$, we have

$$\|e^{t\Delta} \nabla^m f\|_{\mathcal{C}^{s_1}} \lesssim_{m, s_1, s_2} t^{-\frac{m+s_1-s_2}{2}} \|f\|_{\mathcal{C}^{s_2}}. \quad (4.7)$$

The estimate holds just as well if f on the left-hand side is replaced by $\mathbb{P}f$. Furthermore, one can replace $\mathcal{C}^0$ by the stronger norm L^∞ , with the exception of the endpoint case $\|e^{t\Delta} \mathbb{P}f\|_{L^\infty} \lesssim \|f\|_{L^\infty}$ which fails.

4.1.4 Parameters

We summarize the dependencies of the parameters in the construction and their relation to the various constants that appear in the inequalities.

We introduce a parameter $\alpha \in (0, \frac{1}{8})$ that measures the subcriticality of the perturbations $w^{(1)}$ and $w^{(2)}$. The data and solutions we construct are active on two sequences of scales M_k and N_k which we take to be of the form

$$M_k = \lfloor A^{b^k} \rfloor, \quad N_k = M_k \lfloor M_k^{\gamma-1} \rfloor, \quad k = 0, 1, 2, \dots$$

where $b > 5 \vee \frac{1}{1-8\alpha}$ and $\frac{1}{1-4\alpha} < \gamma < \frac{1}{4\alpha+b-1}$ can be chosen freely. Here $\lfloor x \rfloor$ is the largest integer $\geq x$. It is easy to see that the condition on b makes the condition on γ satisfiable. One should think of the length scales M_k^{-1} and N_k^{-1} as the period of the Mikado flow and the oscillation wavelength, respectively.

We record the following elementary estimate which will be useful:

$$4\alpha < 1 - \gamma^{-1} < 1 - b^{-1}. \quad (4.8)$$

Here we provide some detail on the dependencies of the parameters and constants. Having fixed b , γ , and α to satisfy the above relations, one can choose δ smaller than an absolute constant specified in the course of the proof of Lemma 4.2.3. Then δ_0 is chosen small depending on δ to satisfy (4.9), and depending on an absolute constant to satisfy (4.12). The additional parameters chosen throughout Sections 4.2–4.3 (e.g., ϵ , β , β_1 , c_1 , κ) are chosen to lie in explicit intervals depending on b , α , and γ . Throughout the proof (with the exception of Lemma 4.2.3 where δ is chosen), the implicit constants implied by the notation $\lesssim$, $O(\cdot)$, etc. may depend on α , β , γ , δ , δ_0 , ϵ , κ , b , and c_1 but *not* A . The named constants $C_1, C_2, \dots$ are chosen large depending on implicit constants and all parameters other than A . Finally, throughout the construction and proofs, the scale parameter A in the definition of M_k is taken to be as large as needed, depending on all the other parameters.

4.2 The Principal Part

4.2.1 Mikado Flow Potentials

The starting point for the construction of our initial data is the Mikado pipe flow introduced by Daneri and Székelyhidi [37]. Fix a small $\delta > 0$ to be determined which will be the total volume of the Mikado flow support in each period of $\mathbb{T}^3$. For $j \in \{1, 2, \dots, 6\}$, let $\theta_j \in \mathbb{Z}^3$ as in Lemma 4.5.1 and $\eta_j \in \mathbb{Z}^3 \setminus 0$ such that $\theta_j \cdot \eta_j = 0$, which will give the orientation of the shear flows. Let ℓ_j be the 2π -periodic line $x_j + \theta_j \mathbb{R} \bmod (2\pi\mathbb{Z})^3$ for some $x_j \in \mathbb{T}^3$ to be specified in Lemma 4.2.2. Note that since each $\theta_j \in \mathbb{Z}^3$, each ℓ_j has finite winding number.

We fix the radial profile of the pipes $\varphi \in C_c^\infty([0, 1])$ to be even and satisfy $\varphi(0) = 1$. Let δ_0 be

the pipe width, chosen small enough to keep the total volume of the pipes below δ :

$$\pi\delta_0^2 \sum_{j=1}^6 |\ell_j| \leq \delta, \quad (4.9)$$

where $|\ell_j|$ is the length of $\ell_j \cap [0, 2\pi]^3$. Then we let

$$\tilde{\varphi}_j(x) = \varphi(\delta_0^{-1} d_{\mathbb{T}^3}(x, \ell_j)) \mathbf{1}_{d_{\mathbb{T}^3}(x, \ell_j) \leq \delta_0}$$

be the profile that defines the support of the pipes. Let us emphasize that $\tilde{\varphi}_j$ should be interpreted as a function on $\mathbb{T}^3$, i.e., as a 2π -periodic function on $\mathbb{R}^3$.

For technical reasons related to localization in space, we define the Mikado flows in terms of vector potentials rather than velocity fields directly.

Definition 4.2.1. We define the Mikado flow vector potential

$$\Psi_{j,k}^0(x) = N_k^{-2} \tilde{\varphi}_j(M_k x) \sin(N_k(x - x_j) \cdot \eta_j) \theta_j$$

as well as the (approximately) heat evolved potential

$$\Psi_{j,k}(x, t) = \Psi_{j,k}^0(x) \exp(-|\eta_j|^2 N_k^2 t).$$

Further, we define slight expansions of the regions on which they are supported:

- $\Omega_{j,k}$ is the δ_0 -neighborhood of $\text{supp } \Psi_{j,k}$,
- $\tilde{\Omega}_{j,k}$ is the $(2\delta_0)$ -neighborhood of $\text{supp } \Psi_{j,k}$, and
- Ω_k and $\tilde{\Omega}_k$ are regions containing the intersection of the pipes up to and including those at scale k ,

$$\Omega_k := \bigcap_{m \leq k} \bigcup_{j=1}^6 \Omega_{j,m}, \quad \tilde{\Omega}_k := \bigcap_{m \leq k} \bigcup_{j=1}^6 \tilde{\Omega}_{j,m}.$$

Defined in this way, $\Psi_{j,k}$ solve the heat equation up to a small error with data $\Psi_{j,k}^0$. Let us record several basic properties of the Mikado potentials.

Lemma 4.2.2. *With $\Psi_{j,k}^0$ and $\Psi_{j,k}$ as in Definition 4.2.4, we have the following:*

1. $\Psi_{j,k}^0$ and $\Psi_{j,k}$ are smooth, 2π -periodic, and not identically zero. Moreover

$$\|\nabla^m \Psi_{j,k}^0\|_{L^\infty(\mathbb{T}^3)} \lesssim_m N_k^{-2+m}. \quad (4.10)$$

2. The vector fields $\Psi_{j,k}^0$ and $\Psi_{j,k}(t)$ at a fixed time solve the steady pressureless Euler equations:

$$\operatorname{div}(\Psi_{j,k} \otimes \Psi_{j,k}) = 0, \quad \operatorname{div} \Psi_{j,k} = 0. \quad (4.11)$$

3. Each $\Psi_{j,k}^0$ and $\Psi_{j,k}$ is supported in the periodic pipe region $\{x \in \mathbb{T}^3 : \operatorname{dist}(M_k x, \ell_j) \leq \delta_0\}$. Taking $\delta_0 > 0$ smaller if necessary, there exist $(x_j)_{j=1,\dots,6}$ such that the supports satisfy

$$\operatorname{dist}(\operatorname{supp} \tilde{\varphi}_{j_1}, \operatorname{supp} \tilde{\varphi}_{j_2}) > 10\delta_0 \quad \forall j_1 \neq j_2. \quad (4.12)$$

In particular, it follows that $\operatorname{dist}(\tilde{\Omega}_{j_1,k}, \tilde{\Omega}_{j_2,k}) > 5\delta_0 M_k^{-1}$.

4. With

$$A_{j,k} := \oint_{\mathbb{T}^3} \tilde{\varphi}_j^2(M_k x) \sin^2(N_k(x - x_j) \cdot \eta_j) dx, \quad (4.13)$$

we have $A_{j,k} \sim \delta$.

Proof. Properties 1–3 are elementary consequences of Definition 4.2.4. We emphasize that the choices of $(x_j)_{j=1}^6$ and δ_0 made to achieve (4.12) do not depend on k . Property 4 follows from an easy application of Lemma 4.5.5, using the fact that $\oint_{\mathbb{T}^3} \tilde{\varphi}_j^2 \sim \delta$, and that N_k/M_k can be made larger than any absolute constant by choosing a sufficiently large A . $\square$

Next we prove sharp estimates on the volume of $\tilde{\Omega}_k$. A useful tool will be the “improved Hölder’s inequality” from [70] following [14] which we quote in Section 4.5 for the reader’s convenience. Note

that for $X \subset \mathbb{T}^3$, the volume $|X|$ is defined in the natural way, namely the volume of X restricted to a single period.

Lemma 4.2.3. *We have*

$$|\tilde{\Omega}_k| \leq 2^{-k} |\mathbb{T}^3|.$$

Moreover, there is a constant $C_0 > 1$ such that for all cubes $Q \subset \mathbb{R}^3$,

$$|Q \cap \tilde{\Omega}_k| \leq 2^{-(k-k(Q))} |Q| \quad \forall k \geq k(Q) \quad (4.14)$$

where we define $k(Q) := \inf\{k \in \mathbb{N} : M_k \geq C_0 \ell(Q)^{-1}\}$.

Proof. We emphasize that in this proof, the constants (implicit and otherwise) do not depend on δ or δ_0 . Recall the functions $\tilde{\varphi}_j(M_k \cdot)$ that define the supports of the $\Psi_{j,k}$. Let $\phi \in C^\infty(\mathbb{T}^3; [0, 1])$ be such that

- $\phi \equiv 1$ on $\bigcup_j \text{supp } \tilde{\varphi}_j(x)$,
- $\phi \equiv 0$ outside of the δ_0 -neighborhood of $\bigcup_j \text{supp } \tilde{\varphi}_j(x)$, and
- $|\nabla \phi| \leq C' \delta_0^{-1}$ for an absolute constant C' .

Clearly, by (4.9), we have

$$\|\phi\|_{L^1} \leq \sum_j |\text{supp } \tilde{\varphi}_j| \leq 4\delta. \quad (4.15)$$

Then we define $\phi_k(x) = \phi(M_k x)$ leading to the pointwise inequality $\mathbf{1}_{\tilde{\Omega}_k} \leq \phi_0 \phi_1 \cdots \phi_k \leq \mathbf{1}_{\mathbb{T}^3}$. The lemma will follow from the stronger claim

$$\|\phi_0 \phi_1 \cdots \phi_k\|_{L^1} \leq (8\delta)^k |\mathbb{T}^3|,$$

and it suffices to take, say, $\delta = \frac{1}{20}$. We proceed by induction. There is nothing to prove for $k = 0$.

For $k \geq 1$, we apply Lemma 4.5.5 with $f = \phi_0 \cdots \phi_{k-1}$, $g = \phi$, and $\lambda = M_k$ to obtain

$$\begin{aligned} \|\phi_0 \phi_1 \cdots \phi_k\|_{L^1} &\leq \|\phi_0 \phi_1 \cdots \phi_{k-1}\|_{L^1} \|\phi\|_{L^1} + O(M_k^{-1} \|\phi_0 \phi_1 \cdots \phi_{k-1}\|_{C^1} \|\phi\|_{L^1}) \\ &\leq 4\delta(8\delta)^{k-1} |\mathbb{T}^3| + O(M_k^{-1} \sum_{j=0}^{k-1} \|\phi_j\|_{C^1}), \end{aligned}$$

in the last line using the inductive hypothesis and (4.15). Note that the constants implied by $O()$ are absolute, coming from Lemma 4.5.5. The first term is precisely $\frac{1}{2}(8\delta)^k |\mathbb{T}^3|$, so it remains to show the second has the same upper bound. We compute

$$\sum_{j=0}^{k-1} \|\phi_j\|_{C^1} \leq \sum_{j=0}^{k-1} C' M_j \leq C' k M_{k-1}$$

so it suffices to show

$$O(C' k M_{k-1} M_k^{-1}) \leq \frac{1}{2}(8\delta)^k |\mathbb{T}^3|. \quad (4.16)$$

One finds by differentiation that $k \mapsto k(8\delta)^{-k} M_{k-1} M_k^{-1}$ is decreasing for $k \geq 1$ as long as $1 + \log(8\delta)^{-1} - (b-1)(\log A)(\log b) < 0$, which can be arranged by taking A sufficiently large depending on b and δ . Thus (4.16) reduces to $O(C' M_0 M_1^{-1}) \leq \delta |\mathbb{T}^3|$ which indeed holds, using that $M_0 M_1^{-1} \leq 2A^{1-b}$ and again taking A sufficiently large depending on δ , b , and the absolute constants that appeared.

Next, (4.14) can be proven analogously, letting f be the product of $\phi_0 \cdots \phi_{k-1}$ and a bump function localized around Q . $\square$

Now, with δ and δ_0 fixed, the constants in the remainder of the chapter are allowed to depend on them, and we shall not make this dependence explicit.

4.2.2 Construction of the Data

Having defined suitable potentials for the Mikado flows, we proceed to fix the remaining elements that go into the construction of the initial data.

Definition 4.2.4. We define:

- The differential operator that takes vector fields to fields of symmetric rank-2 tensors,

$$\mathcal{D}f := -\nabla f - \nabla f^T + 2(\operatorname{div} f) \operatorname{Id};$$

- A cutoff function χ_k which is identically 1 in Ω_{k-1} , identically 0 outside of $\tilde{\Omega}_{k-1}$, and obeys

$$\|\nabla^m \chi_k\|_{L^\infty(\mathbb{T}^3)} \lesssim_m M_{k-1}^m; \quad (4.17)$$

- A standard mollifier ϕ_k at length scale $N_{k+1}^{-\frac{1}{3}} N_k^{-\frac{2}{3}}$, supported in $B(0, 1)$ with $\int_{\mathbb{T}^3} \phi_k = 1$;
- The Nash operators $(\Gamma_j)_{j=1}^6$ which obey

$$M = \sum_{j=1}^6 \Gamma_j(M)^2 \theta_j \otimes \theta_j$$

for all $M \in S^{3 \times 3}$ within a radius $c_0 = \frac{1}{1000}$ of the identity; see Lemma 4.5.1 for details.

One readily sees that $\mathcal{D}$ has the property $\operatorname{div} \mathcal{D}f = \operatorname{curl} \operatorname{curl} f$; see Remark 4.2.3 for further explanation of this choice.

Finally we can build the velocity potential for the initial data. The construction is carried out inductively on scales.

Definition 4.2.5. Let $\psi_0^0 := N_0 \phi_0 * \Psi_{1,0}^0$. Given any ψ_{k-1}^0 and $k \geq 1$, we inductively define

$$\psi_k^0 := \phi_k * \sum_j a_{j,k}(x) \Psi_{j,k}^0(x) \quad (4.18)$$

where

$$a_{j,k}(x) = N_k \left(\frac{2 \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}}{c_0 |\eta_j|^2 A_{j,k}} \right)^{\frac{1}{2}} \chi_k(x) \Gamma_j \left(\operatorname{Id} + c_0 \frac{\mathcal{D}\psi_{k-1}^0(x)}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}} \right)$$

with $\mathcal{D}$, χ_k , ϕ_k , and Γ_j as in Definition 4.2.4.

When $k = 0$, (4.18) holds just as well upon setting $a_{j,0} = N_0$ when $j = 1$ and $a_{j,0} = 0$ otherwise.

It is important to point out that $a_{j,k}$ is well-defined because the tensor at which we evaluate Γ_j lies in the domain as specified in Section 4.5. Here and elsewhere, the notation $\sum_j$ denotes a sum over $j \in \{1, 2, \dots, 6\}$.

Let us comment on the inclusion of the mollifier in (4.18). Because $\mathcal{D}\psi_{k-1}^0$ appears in the definition of ψ_k^0 (via $a_{j,k}$), there is a potential loss-of-derivatives problem in the proofs of Lemma 4.2.6 and Proposition 4.2.1. Similar to many convex integration constructions, mollification at a sequence of length scales shrinking to zero plays a Nash–Moser-type role in the iteration.

We also point out that the coefficients $a_{j,k}$ have amplitude $\sim N_k$, as the factors besides the N_k are of unit size. Furthermore, the frequency support of $a_{j,k}$ is essentially supported (neglecting a small error) at and below $|\xi| \sim M_k$. Thus, $a_{j,k}$ is effectively constant compared to $\Psi_{j,k}^0$ which oscillates near the higher frequency N_k .

From Definition 4.2.5, one can extract the following inductive estimates.

Lemma 4.2.6. *There is an absolute constant $C_1 > 0$ such that*

$$\|\psi_k^0\|_{L^\infty} \leq C_1 N_k^{-1} \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{\frac{1}{2}} \quad (4.19)$$

and

$$\|\mathcal{D}\psi_k^0\|_{L^\infty} \leq \frac{1}{2} C_1 \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{\frac{1}{2}} \left(1 + N_k^{-1} \frac{\|\nabla \mathcal{D}\psi_{k-1}^0\|_{L^\infty}}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}} \right) \quad (4.20)$$

for all $k \geq 1$.

Proof. We begin by estimating the coefficient functions $a_{j,k}$. First, by (4.45) and part 4 of Lemma 4.2.2 one has

$$\|a_{j,k}\|_{L^\infty} \lesssim N_k \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{1/2}. \quad (4.21)$$

Then, for the gradient,

$$\|\nabla a_{j,k}\|_{L^\infty} \lesssim N_k \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{1/2} \left(\|\nabla \chi_k\|_{L^\infty} + \|\nabla(\Gamma_j(\text{Id} + c_0 \frac{\mathcal{D}\psi_{k-1}^0(x)}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}}))\|_{L^\infty} \right).$$

The first term is easily estimated with (4.17). For the second we have

$$|\nabla(\Gamma_j(\text{Id} + c_0 \frac{\mathcal{D}\psi_{k-1}^0(x)}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}}))| \lesssim \frac{\|\nabla \mathcal{D}\psi_{k-1}^0\|_{L^\infty}}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}}$$

using (4.45). We conclude

$$\|\nabla a_{j,k}\|_{L^\infty} \lesssim N_k \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{\frac{1}{2}} \left(M_k + \frac{\|\nabla \mathcal{D}\psi_{k-1}^0\|_{L^\infty}}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}} \right). \quad (4.22)$$

Now we can proceed toward proving (4.19)–(4.20). From the definition of ψ_k^0 , the L^∞ boundness of the mollifier ϕ_k* , (4.21), and (4.10), we have

$$\|\psi_k^0\|_{L^\infty} \leq \sum_j (\|a_{j,k}(x)\|_{L^\infty} \|\Psi_{j,k}^0\|_{L^\infty}) \lesssim N_k^{-1} \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{1/2}.$$

Choosing C_1 large enough to absorb the constants, we immediately have (4.19). Towards (4.20), recall that $\mathcal{D}$ is a first-order differential operator so we may apply the Leibnitz rule to find

$$\|\mathcal{D}\psi_k^0\|_{L^\infty} \lesssim \sum_j (\|a_{j,k}\|_{L^\infty} \|\nabla \Psi_{j,k}^0\|_{L^\infty} + \|\nabla a_{j,k}\|_{L^\infty} \|\Psi_{j,k}^0\|_{L^\infty}).$$

Then (4.10), (4.21), and (4.22) immediately yield

$$\|\mathcal{D}\psi_k^0\|_{L^\infty} \lesssim \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{\frac{1}{2}} + N_k^{-1} \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{\frac{1}{2}} \left(M_k + \frac{\|\nabla \mathcal{D}\psi_{k-1}^0\|_{L^\infty}}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}} \right).$$

Taking C_1 larger if needed and using the fact that $M_k \lesssim N_k$, we arrive at (4.20). $\square$

Iteratively combining the estimates from Lemma 4.2.6 yields the following. Note that lower bounds will be crucial, both because $\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}$ appears in the denominator in Definition 4.2.5,

and because establishing non-uniqueness will require precise asymptotics.

Proposition 4.2.1. *With C_2 an absolute constant, we have*

$$C_2^{-2}(C_2^2\|\mathcal{D}\psi_0^0\|_{L^\infty})^{2^{-k}} \leq \|\mathcal{D}\psi_k^0\|_{L^\infty} \leq C_2^2(C_2^2\|\mathcal{D}\psi_0^0\|_{L^\infty})^{2^{-k}} \quad (4.23)$$

for all $k \geq 0$. Furthermore

$$\|\nabla^m \psi_k^0\|_{L^\infty} \lesssim_m N_k^{-1+m} \quad (4.24)$$

for all $k, m \geq 0$.

Proof. First, we establish (4.23) by induction. Suppose the claim holds for ψ_m^0 for all $m = 1, 2, \dots, k-1$. We begin with the lower bound in (4.23) and decompose $\mathcal{D}\psi_k^0 = I + II + III + IV$ depending on where the derivative falls: I when it falls on the sine within $\Psi_{j,k}^0$; II when it falls on $\tilde{\varphi}_j$ within $\Psi_{j,k}^0$; III when it falls on χ_k within $a_{j,k}$; and IV when it falls on Γ_j within $a_{j,k}$. We start with the dominant contribution I , and further decompose to extract the unmollified part:

$$\begin{aligned} I &= I_1 + I_2; \quad I_1 = N_k^{-2} \sum_j a_{j,k} \tilde{\varphi}_j(M_k x) \mathcal{D}(\sin(N_k(x - x_j) \cdot \eta_j) \theta_j) \\ I_2 &= \varphi_k * I_1 - I_1. \end{aligned}$$

We compute

$$I_1 = -2N_k^{-1} \sum_j a_{j,k} \tilde{\varphi}_j(M_k x) (\theta_j \odot \eta_j) \cos(N_k(x - x_j) \cdot \eta_j).$$

Note that the terms in the sum have disjoint supports $\tilde{\Omega}_{j,k}$, so let us restrict to a particular one.

Recall that within $\Omega_{j,k}$, the cutoff χ_k is identically 1. Thus

$$\begin{aligned} I_1|_{\Omega_{j,k}} &= -2^{\frac{3}{2}} |\eta_j| (c_0^{-1} A_{j,k}^{-1} \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty})^{\frac{1}{2}} \Gamma_j (\text{Id} + c_0 \frac{\mathcal{D}\psi_{k-1}^0}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}}) \\ &\quad \times \tilde{\varphi}_j(M_k x) (\theta_j \odot \eta_j) \cos(N_k(x - x_j) \cdot \eta_j). \end{aligned} \quad (4.25)$$

Recall from (4.45) that Γ_j is bounded from below; meanwhile the η_j are fixed vectors specified to be orthogonal to θ_j from Lemma 4.5.1. For $x = x_j$ on the line ℓ_j , both the cosine and $\tilde{\varphi}$ factors are exactly 1. We conclude that

$$\|I_1\|_{L^\infty} \gtrsim \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{\frac{1}{2}} \geq C_2^{-1} (C_2^2 \|\mathcal{D}\psi_0^0\|_{L^\infty})^{2^{-k}}$$

by the induction hypothesis. With a large enough choice of C_2 , this implies

$$\|I_1\|_{L^\infty} \geq 100C_2^{-2} (C_2^2 \|\mathcal{D}\psi_0^0\|_{L^\infty})^{2^{-k}}.$$

It remains to show that the other terms are uniformly much smaller.

To estimate I_2 , we need to bound the gradient of I_1 :

$$\|\nabla I_1\|_{L^\infty} \lesssim \frac{\|\nabla \mathcal{D}\psi_{k-1}^0\|_\infty}{\|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{1/2}} + \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{1/2} N_k.$$

The numerator in the first term is $O(N_k^{\frac{2}{3}} N_{k-1}^{\frac{1}{3}})$, using the mollifier to absorb the derivatives. The denominator is $1/O(1)$ by the inductive hypothesis. Similarly the second term is $O(N_k)$. Thus, by a standard mollifier estimate,

$$\|I_2\|_{L^\infty} \lesssim N_{k+1}^{-\frac{1}{3}} N_k^{-\frac{2}{3}} \|\nabla I_1\|_{L^\infty} \lesssim (N_k/N_{k+1})^{\frac{1}{3}} = A^{-\frac{1}{3}\gamma(b-1)b^k}.$$

Next, consider II . Recall that $\tilde{\varphi}_j$ is a fixed smooth function, so clearly $\nabla(\tilde{\varphi}_j(M_k x))$ is $O(M_k)$. From this and the inductive hypothesis, we have

$$\|II\|_{L^\infty} \lesssim N_k^{-1} M_k \|\mathcal{D}\psi_{k-1}^0\|_{L^\infty}^{1/2} \lesssim_{C_2, \psi_0^0} N_k^{-1} M_k = A^{-(\gamma-1)b^k}.$$

The argument for III is identical: when the derivative lands on χ_k , it yields a factor of M_{k-1} due to (4.17), which is even better than the factor M_k that appeared in II .

We finish by estimating IV . When $\mathcal{D}$ falls on Γ_j , the chain rule produces a factor $\nabla \mathcal{D}\psi_{k-1}^0$. As in

the analysis of ∇I_1 above, one places derivatives on the mollifier to obtain $\|\nabla \mathcal{D}\psi_{k-1}^0\|_{L^\infty} \lesssim N_k^{\frac{2}{3}} N_{k-1}^{\frac{1}{3}}$ which yields

$$\|IV\|_{L^\infty} \lesssim (N_{k-1}/N_k)^{\frac{1}{3}} = A^{-\gamma(1-b^{-1})b^k}.$$

To summarize, we have shown

$$\begin{aligned} \|\mathcal{D}\psi_k^0\|_{L^\infty} &\geq \|I_1\|_{L^\infty} - \|I_2\|_{L^\infty} - \|II\|_{L^\infty} - \|III\|_{L^\infty} - \|IV\|_{L^\infty} \\ &\geq 100C_2^{-2}(C_2^2\|\mathcal{D}\psi_0^0\|_{L^\infty})^{2^{-k}} - O_{C_2, \psi_0^0}(A^{-c(b, \gamma)b^k}) \end{aligned}$$

for some $c(b, \gamma) > 0$. Notice that the first term stays bounded from below as $k \rightarrow \infty$. Thus, choosing A sufficiently large depending on ψ_0^0 , C_2 , b , and γ , the second term becomes negligible and we arrive at the desired lower bound.

To prove the upper bound in (4.23), the same upper bounds on I_2 , II , III , and IV still suffice. One need only recognize that the estimate on I_1 is reversible and one can proceed from (4.25) to obtain $\|I_1\|_{L^\infty} \lesssim C_2(C_2^2\|\mathcal{D}\psi_0^0\|_{L^\infty})^{2^{-k}}$.

To see (4.24), one proceeds directly by differentiating the definition. When all the derivatives fall onto the sine, the upper bound is as claimed (analogously to I_1). When derivatives fall onto any other part, the upper bound is strictly smaller, which can be easily seen by arguing as above for II – IV . For instance, when derivatives fall onto Γ_j , the resulting higher derivatives of ψ_{k-1}^0 can be adequately estimated using the mollifier and the inductive hypothesis (4.23) as was done above for $\nabla \mathcal{D}\psi_{k-1}^0$. $\square$

The velocity potentials ψ_k^0 will be the building blocks for the initial velocity U^0 . The following proposition paves the way to properly defining U^0 as a distribution and proving that it lies in BMO^{-1} .

Proposition 4.2.2. *With ψ_k^0 as in Definition 4.2.5, the sum $\sum_{k=0}^\infty \text{curl } \psi_k^0$ converges in $L^1(\mathbb{T}^3)$.*

Furthermore,

$$\sum_{k=0}^{\infty} \operatorname{curl} \psi_k^0 \in BMO.$$

Proof. For brevity we write $\zeta_k = \operatorname{curl} \psi_k^0$. By (4.24), we have $\|\zeta_k\|_{L^\infty} = O(1)$. It follows from this and Lemma 4.2.3 that $\|\zeta_k\|_{L^1} \lesssim |\operatorname{supp} \zeta_k| \lesssim 2^{-k} |\mathbb{T}^3|$. Convergence in L^1 then follows because of the absolute convergence of the series

$$\sum_k \|\zeta_k\|_{L^1(\mathbb{T}^3)} < \infty.$$

To show $\zeta = \sum_k \zeta_k$ lies in BMO , we recall the definition from §4.1.3,

$$\|\zeta\|_{BMO} = \sup_{Q \subset \mathbb{R}^3} \int_Q |\zeta - \zeta_Q|.$$

By periodicity it is easy to see that it suffices to consider cubes $Q \subset \mathbb{R}^3$ with side length $\ell(Q) \in (0, M_0^{-1})$, say. Fix any such Q and recall $k(Q) = \inf\{k \in \mathbb{N} : M_k \geq C_0 \ell(Q)^{-1}\}$ from Lemma 4.2.3. We estimate

$$\int_Q |\zeta - \zeta_Q| \leq 2 \int_Q \sum_{k \geq k(Q)-1} |\zeta_k| + \sum_{k \leq k(Q)-2} \|\zeta_k - (\zeta_k)_Q\|_{L^\infty(Q)} = I + II.$$

To bound I , observe that the amplitude of ζ_k is $O(1)$ from (4.24), while the volume of the support can be estimated with (4.14) in Lemma 4.2.3. We arrive at

$$2 \int_Q \sum_{k \geq k(Q)-1} |\zeta_k| \lesssim \sum_{k \geq k(Q)-1} \frac{|Q \cap \tilde{\Omega}_k|}{|Q|} \leq \sum_{k \geq k(Q)-1} 2^{-(k-k(Q))} \lesssim 1.$$

Now consider *II*. For any $x \in Q$ and $k \leq k(Q) - 2$,

$$\begin{aligned} |\zeta_k(x) - (\zeta_k)_Q| &= \left| \oint_Q (\zeta_k(x) - \zeta_k(y)) dy \right| \\ &\leq \oint_Q \|\nabla \zeta_k\|_{L^\infty(Q)} dy \\ &\lesssim N_k \ell(Q) \end{aligned}$$

using (4.24). Summing over $k \leq k(Q) - 2$, we obtain the upper bound $O(N_{k(Q)-2} \ell(Q))$. By definition of $k(Q)$, in particular the minimality, we have $M_{k(Q)-1} < C_0 \ell(Q)^{-1}$ and conclude

$$II \lesssim N_{k(Q)-2} \ell(Q) \lesssim N_{k(Q)-2} M_{k(Q)-1}^{-1} = N_{k(Q)-2}^{1-b/\gamma}$$

which is bounded, recalling from §4.1.4 that $\gamma < b$. □

Finally we may define the velocity initial data.

Definition 4.2.7. With ψ_k^0 as in Definition 4.2.5, we define the data

$$U^0 = \operatorname{curl} \sum_{k=0}^{\infty} \operatorname{curl} \psi_k^0. \tag{4.26}$$

This is well-defined and in BMO^{-1} because of Proposition 4.2.2, with the outer curl taken in the distributional sense.

4.2.3 Construction of the Principal Parts of the Solutions

The initial data constructed in Definition 4.2.7 has the essential property that at each frequency level k , there are two different continuations consistent with the Navier–Stokes equations up to a small error: one with dissipation dominating and another with the inverse cascade dominating. (See §1.4.3 for further discussion.) We shall call these two alternatives v_k and $\bar{v}_k$ respectively. By interposing them in different ways we can construct two approximate Navier–Stokes solutions $v^{(1)}$ and $v^{(2)}$ with data U^0 .

Definition 4.2.8. Recalling the definitions from §4.2, we define for $k \geq 0$ the heat-dominated evolution

$$v_k(t, x) = \phi_k * \operatorname{curl} \operatorname{curl} \sum_j a_{j,k}(x) \Psi_{j,k}(x, t)$$

as well as the inverse cascade-dominated evolution

$$\bar{v}_k(t, x) = \frac{1}{2} N_{k+1}^{-2} \mathbb{P} \operatorname{div} \sum_j A_{j,k+1} |\eta_j|^2 e^{-2|\eta_j|^2 N_{k+1}^2 t} a_{j,k+1}^2(x) \theta_j \otimes \theta_j.$$

We occasionally write $\psi_k := \phi_k * \sum_j a_{j,k} \Psi_{j,k}$, which are the corresponding continuations forward in time from data ψ_k^0 . Then we define the pair of vector fields

$$v^{(1)} = \sum_{k \geq 0 \text{ even}} v_k + \sum_{k \geq 0 \text{ odd}} \bar{v}_k$$

and

$$v^{(2)} = \sum_{k \geq 0 \text{ odd}} v_k + \sum_{k \geq 0 \text{ even}} \bar{v}_k.$$

Remark 4.2.3. *The operator $\operatorname{curl} \operatorname{curl} = -\Delta + \nabla \operatorname{div}$ has the advantage that $v_k = \operatorname{curl} \operatorname{curl} \psi_k$ is both divergence-free and of the form $v_k = \operatorname{div} \mathcal{D} \psi_k$ where $\mathcal{D} \psi_k$ is a symmetric tensor. The latter property is essential for our eventual use of Lemma 4.5.1.*

Remark 4.2.4. *In §4.4, we give a precise proof that the two solutions constructed have identical data. Nonetheless, it is enlightening to observe at this point that this is true formally for the*

principal parts $v^{(1)}$ and $v^{(2)}$. Indeed, we compute

$$\begin{aligned}
\bar{v}_k(0, x) &= \frac{1}{2} N_{k+1}^{-2} \mathbb{P} \operatorname{div} \sum_j A_{j,k+1} |\eta_j|^2 a_{j,k+1}^2(x) \theta_j \otimes \theta_j \\
&= c_0^{-1} \|\mathcal{D}\psi_k^0\|_{L^\infty} \mathbb{P} \operatorname{div} \left(\chi_{k+1} \sum_j \Gamma_j^2 (\operatorname{Id} + c_0 \frac{\mathcal{D}\psi_k^0}{\|\mathcal{D}\psi_k^0\|_{L^\infty}}) \theta_j \otimes \theta_j \right) \\
&= c_0^{-1} \|\mathcal{D}\psi_k^0\|_{L^\infty} \mathbb{P} \operatorname{div} (\chi_{k+1} (\operatorname{Id} + c_0 \frac{\mathcal{D}\psi_k^0}{\|\mathcal{D}\psi_k^0\|_{L^\infty}})) \\
&= \mathbb{P} \operatorname{div} (\chi_{k+1} \mathcal{D}\psi_k^0) \\
&= \operatorname{curl} \operatorname{curl} \psi_k^0
\end{aligned}$$

where we have used Lemmas 4.5.1 and 4.5.3, the fact that $\mathbb{P} \operatorname{div}$ annihilates scalar function multiples of the identity, and that χ_{k+1} is identically equal to 1 on $\operatorname{supp} \mathcal{D}\psi_k^0$. Meanwhile, the fact that $v_k(0, x) = \operatorname{curl} \operatorname{curl} \psi_k^0$ is immediate from the construction in Definitions 4.2.5 and 4.2.8. It follows that formally we have the equality

$$v^{(i)}|_{t=0} = \sum_{k \geq 0} \operatorname{curl} \operatorname{curl} \psi_k^0 = U^0$$

for both $i = 1$ and 2 .

Later in §4.3, we shall see that $v^{(1)}$ and $v^{(2)}$ solve the Navier–Stokes up to a small subcritical error and that the error can be corrected by a small perturbation. Before doing so, we need some basic estimates on their sizes.

Proposition 4.2.5. *The two vector fields $v^{(1)}$ and $v^{(2)}$ are divergence-free and obey the estimates*

$$\|\nabla^m v^{(i)}(t)\|_{L^\infty(\mathbb{T}^3)} \lesssim_m t^{-\frac{(m+1)}{2}} e^{-N_0^2 t / O_m(1)} \quad (4.27)$$

for all $m \geq 0$ and $t > 0$. Moreover

$$\|v^{(i)}\|_{L^2([t_1, t_2], dt; L^\infty)}^2 + \|v^{(i)}\|_{L^1([t_1, t_2], t^{-\frac{1}{2}} dt; L^\infty)} \lesssim 1 + (\log A)^{-1} \log(t_2/t_1) \quad (4.28)$$

for $0 < t_1 \leq t_2 \leq 1$, where one may recall the large parameter A from §4.1.4.

We remark that we almost never take advantage of the exponential factor in (4.27), since all of the nontrivial behavior of $v^{(i)}$ is confined to times $t \in [0, N_0^{-2}]$.

Proof. The divergence-free property is immediate from Definition 4.2.8 because the constituent parts of $v^{(i)}$, namely v_k and $\bar{v}_k$, appear inside of curl and $\mathbb{P}$, respectively.

Next we proceed to the estimates, suppressing the dependence of $v^{(i)}$ on i for conciseness. We recall the coefficient functions $a_{j,k}$ from Definition 4.2.5. With (4.24), (4.17), and (4.46), one can use the Leibnitz rule to estimate

$$\|\nabla^m a_{j,k}\|_{L^\infty} \lesssim_m N_{k-1}^m N_k \quad (4.29)$$

for $m \geq 0$. It follows from (4.29) and (4.10) that

$$\|\nabla^m v_k\|_{L^\infty} \lesssim N_k^{1+m} e^{-N_k^2 t}. \quad (4.30)$$

For $\bar{v}_k$, one runs into the difficulty that $\mathbb{P}$ is not bounded on L^∞ . Instead, we carry out the estimate in the Hölder space C^{β_1} where $\beta_1 = \frac{b-1}{4} \wedge \frac{1}{2}$. By (4.17), (4.29), and (4.24),

$$\|\nabla^m \bar{v}_k\|_{L^\infty} \lesssim \|\nabla^m \bar{v}_k\|_{C^{\beta_1}} \lesssim N_k^{1+\beta_1+m} e^{-N_{k+1}^2 t}. \quad (4.31)$$

Note that the homogeneous C^{β_1} norm controls L^∞ in this case because $\bar{v}_k$ has zero average on $\mathbb{T}^3$.

To see (4.27), we combine the bounds on v_k and $\bar{v}_k$ to find

$$\|\nabla^m v(t)\|_{L_x^\infty} \lesssim_m \sum_{k \geq 0} (N_k^{1+m} e^{-N_k^2 t} + N_k^{1+\beta_1+m} e^{-N_{k+1}^2 t}). \quad (4.32)$$

We analyze the two terms separately. Because N_k grows rapidly, the first term in the sum is controlled by the largest summand; when $t \leq N_0^{-2}$, the largest possible value occurs near $N_k = t^{-\frac{1}{2}}$, leading to a sum of $O(t^{-(1+m)/2})$. When $t > N_0^{-2}$, the summands decrease mono-

tonically leading to an upper bound of $N_0^{1+m}e^{-N_0^2 t}$. Using that $tN_0^2 > 1$, this is controlled by $O_m(t^{-(1+m)/2}e^{-N_0^2 t/O_m(1)})$ due to the exponential decay. This yields the desired bound (4.27) for the first term in (4.32). The second is estimated in the same way: the sum is controlled by its largest contribution which occurs when N_{k+1} is near $t^{-\frac{1}{2}}$. In total, one finds that (4.32) leads to

$$\|\nabla^m v(t)\|_{L^\infty} \lesssim_m t^{-\frac{1+m}{2}} + \min\left\{t^{-\frac{1+\beta_1+m}{2b}}, N_0^{1+\beta_1+m}e^{-N_1^2 t}\right\}.$$

By definition of β_1 , we have $\frac{1+\beta_1+m}{2b} \leq \frac{1+m}{2}$. It follows that when $t \leq N_0^{-2}$, $t^{-\frac{1+\beta_1+m}{2b}}$ can be absorbed into the first term. When $t > N_0^{-2}$, we have

$$N_0^{1+\beta_1+m}e^{-N_1^2 t} \lesssim_m N_0^{1+\beta_1+m}(N_1^2 t)^{-\frac{1+m}{2}}e^{-N_0^2 t} = \left(\frac{N_0}{N_1}\right)^m \frac{N_0^{1+\beta_1}}{N_1} t^{-\frac{1+m}{2}}e^{-N_0^2 t}.$$

Recall that $N_0 < N_1$ and, more strongly, $N_0^{1+\beta_1} < N_1$ because $\beta_1 < b-1$. We conclude that the second contribution in (4.32) can be absorbed into $t^{-\frac{1+m}{2}}$ as well. This completes the proof of (4.27).

Next we turn to (4.28). We remark that it suffices to prove the L^1 estimate because the L^2 estimate is immediate by Hölder's inequality, combining the L^1 bound with (4.27). We decompose

$$\begin{aligned} \|v\|_{L_t^1([t_1, t_2], t^{-\frac{1}{2}} dt; L_x^\infty)} &\leq \sum_k \int_{t_1}^{t_2} \left(\|v_k(t)\|_{L^\infty} \mathbf{1}_{t \leq N_k^{-2}} + \|v_k(t)\|_{L^\infty} \mathbf{1}_{t > N_k^{-2}} + \|\bar{v}_k(t)\|_{L^\infty} \right) t^{-\frac{1}{2}} dt \\ &\lesssim \sum_{k: N_k \leq t_1^{-1/2}} \int_{t_1}^{N_k^{-2} \wedge t_2} N_k t^{-\frac{1}{2}} dt + \sum_{k: N_k \geq t_2^{-1/2}} \int_{N_k^{-2} \vee t_1}^{t_2} N_k e^{-N_k^2 t} t^{-\frac{1}{2}} dt \\ &\quad + \sum_k \int_{t_1}^{t_2} N_k^{1+\beta_1} e^{-N_{k+1}^2 t} t^{-\frac{1}{2}} dt. \end{aligned}$$

Let I , II , and III be the three terms in the final lines. For I we simply use the fundamental theorem of calculus to find

$$I \leq \sum_{k: N_k \leq t_1^{-1/2}} N_k (N_k^{-2} \wedge t_2)^{1/2} \leq \sum_{k: N_k \leq t_2^{-1/2}} N_k t_2^{1/2} + \sum_{k: t_2^{-1/2} \leq N_k \leq t_1^{-1/2}} 1.$$

Clearly the first contribution is $O(1)$, while the second is the number of N_k found in the interval $(t_2^{-1/2}, t_1^{-1/2})$. We estimate this in a wasteful way: N_k is more lacunary than the geometric sequence with ratio A^{b-1} because $N_{k+1}/N_k = A^{\gamma(b^{k+1}-b^k)} \geq A^{\gamma(b-1)}$. Therefore,

$$\#\{k : N_k \in [t_2^{-\frac{1}{2}}, t_1^{-\frac{1}{2}}]\} \leq \log_{A^{\gamma(b-1)}} \left(\frac{t_1^{-1/2}}{t_2^{-1/2}} \right) = (2\gamma(b-1) \log A)^{-1} \log(t_2/t_1)$$

which yields the desired estimate on I . For II , we have

$$\begin{aligned} II &\leq \sum_{k: N_k \geq t_2^{-1/2}} N_k^{-1} (N_k^{-2} \vee t_1)^{-\frac{1}{2}} e^{-N_k^2 (N_k^{-2} \vee t_1)} \\ &\lesssim \sum_{k: t_2^{-\frac{1}{2}} \leq N_k \leq t_1^{-\frac{1}{2}}} 1 + \sum_{k: N_k > t_1^{-\frac{1}{2}}} N_k^{-1} t_1^{-\frac{1}{2}} e^{-N_k^2 t_1} \end{aligned}$$

which is once again $O(1 + \#\{k : M_k \in [t_2^{-\frac{1}{2}}, t_1^{-\frac{1}{2}}]\})$, and we conclude as in I . For III , we bound $t^{-\frac{1}{2}}$ in the trivial way and evaluate the integral to find

$$III \lesssim \sum_k N_k^{1+\beta_1} N_{k+1}^{-2} t_1^{-\frac{1}{2}} e^{-N_{k+1}^2 t_1} \lesssim t_1^{\frac{1}{2} - \frac{1+\beta_1}{2b}}.$$

Recall that $\beta_1 < b-1$ by definition so the power is positive. As we assumed $t_1 \leq 1$, we conclude that $III = O(1)$. $\square$

4.3 Construction of the Perturbation

To summarize our progress to this point, we have constructed two vector fields $v^{(1)}$ and $v^{(2)}$ with common initial data U^0 and reasonable estimates for $t > 0$. The purpose of this section is to show that they nearly solve the Navier–Stokes equations and can be perturbed into exact solutions.

4.3.1 Estimates on the Residual

For $i \in \{1, 2\}$, we shall define the residuals $F^{(i)} \in C^\infty((0, 1] \times \mathbb{T}^3; S^{3 \times 3})$ in divergence form which satisfy

$$-\mathbb{P} \operatorname{div} F^{(i)} = \partial_t v^{(i)} - \Delta v^{(i)} + \mathbb{P} \operatorname{div} v^{(i)} \otimes v^{(i)}. \quad (4.33)$$

Note that $\mathbb{P} \operatorname{div}$ does not have a unique right inverse because $F^{(i)}$ has too many degrees of freedom. In fact, an arbitrary choice of $F^{(i)}$ satisfying (4.33) would probably lack the required smallness to complete the construction. In the following proposition, we fix a suitable pair $F^{(1)}, F^{(2)}$.

In order to state the proposition, we introduce two important parameters: the subcriticality parameter $\alpha \in (0, \frac{1}{8})$ and the Hölder exponent $\kappa \in (0, \frac{1}{2} - \frac{1}{2\gamma} - 2\alpha)$. We also recall the growth rate A for the frequency scales N_k, M_k from §4.1.4. The objective is to find an $F^{(i)}$ that can be estimated in, say, L^∞ in such a way that the upper bound has slightly subcritical dependence on time. There are two types of terms that appear: those with subcritical *amplitude* with an upper bound of the form $N_k^{1-c} \exp(-N_k^2 t)$, and those with subcritical *decay* with an upper bound similar to $N_k \exp(-N_k^{2+c} t)$ for some $c > 0$. In either case, we can extract some smallness in $C^{1,\kappa}$ for $\kappa > 0$ as above with a subcritical weight in time.

Proposition 4.3.1. *For any $\epsilon_0 > 0$ and all sufficiently large $A > 1$ (depending on ϵ_0), there exist $F^{(1)}, F^{(2)} \in C^\infty((0, 1] \times \mathbb{T}^3; S^{3 \times 3})$ obeying (4.33) and*

$$\sup_{t \in (0, 1]} (t^{1-\alpha} \|F^{(i)}\|_{L^\infty} + t^{\frac{3}{2}-\alpha} \|\nabla F^{(i)}\|_{C^\kappa}) \leq \epsilon_0, \quad i \in \{1, 2\}.$$

Proof. Before beginning to define and estimate the residual, we extract the part of v_k that will not be perturbative, decomposing $v_k = v_k^p + v_k^e$ where

$$v_k^p = N_k^2 \sum_j |\eta_j|^2 a_{j,k} \Psi_{j,k},$$

$$v_k^e = v_k^{e,1} + v_k^{e,2} + v_k^{e,3} + v_k^{e,4},$$

having defined

$$\begin{aligned}
v_k^{e,1} &= (\phi_k - \delta) * \operatorname{curl} \operatorname{curl} \sum_j a_{j,k} \Psi_{j,k}, \\
v_k^{e,2} &= -N_k^{-2} \sum_j \Delta(a_{j,k} \tilde{\varphi}_j(M_k x)) \sin(N_k(x - x_j) \cdot \eta_j) \theta_j \exp(-|\eta_j|^2 N_k^2 t), \\
v_k^{e,3} &= -2N_k^{-1} \sum_j \eta_j \cdot \nabla(a_{j,k} \tilde{\varphi}_j(M_k x)) \cos(N_k(x - x_j) \cdot \eta_j) \theta_j \exp(-|\eta_j|^2 N_k^2 t) \\
v_k^{e,4} &= N_k^{-2} \nabla \sum_j \theta_j \cdot \nabla a_{j,k} \tilde{\varphi}_j(M_k x) \sin(N_k(x - x_j) \cdot \eta_j) \exp(-|\eta_j|^2 N_k^2 t).
\end{aligned}$$

Here, $v_k^{e,1}$ captures the effect of the mollifier, while the rest consists of the terms arising when $\operatorname{curl} \operatorname{curl} = -\Delta + \nabla \operatorname{div}$ acts on v_k . Note that for $v_k^{e,4}$, we have used (4.11). The dominant term is v_k^p which appears when the Laplacian falls on the sine in $\Psi_{j,k}$. Sharp estimates for v_k^p are immediate from (4.10) and (4.29),

$$\|\nabla^m v_k^p\|_{L^\infty} \lesssim_m N_k^{1+m} \exp(-N_k^2 t). \quad (4.34)$$

The “error part” v_k^e is slightly more involved to bound because we require smallness in a subcritical norm. By (4.29),

$$\begin{aligned}
\|\nabla^m v_k^{e,1}\|_{L^\infty} &\lesssim_m N_k^{\frac{4}{3}+m} N_{k+1}^{-\frac{1}{3}} \exp(-N_k^2 t) \\
\|\nabla^m v_k^{e,2}\|_{L^\infty} &\lesssim_m N_k^{-1+m} M_k^2 \exp(-N_k^2 t) \\
\|\nabla^m v_k^{e,3}\|_{L^\infty} &\lesssim_m N_k^m M_k \exp(-N_k^2 t) \\
\|\nabla^m v_k^{e,4}\|_{L^\infty} &\lesssim_m N_{k-1} N_k^m \exp(-N_k^2 t)
\end{aligned}$$

for all $m \geq 0$. Each of the right-hand sides is of the form $A^{-c(b,\gamma)b^k} M_k N_k^m \exp(-N_k^2 t)$ for some $c(b, \gamma) \geq 0$, and in fact $c = 0$ is achieved for $v_k^{e,3}$ (see §4.1.4 for the relevant parameter relations),

leading to

$$\|\nabla^m v_k^e\|_{L^\infty} \lesssim_m M_k N_k^m \exp(-N_k^2 t). \quad (4.35)$$

To define $F^{(1)}$ and $F^{(2)}$, we need to further decompose the interaction

$$v_k^p \otimes v_k^p = N_k^4 \sum_{j_1, j_2} |\eta_{j_1}|^2 |\eta_{j_2}|^2 a_{j_1, k} a_{j_2, k} \Psi_{j_1, k} \otimes \Psi_{j_2, k} = \mathcal{N}_{k,1} + \mathcal{N}_{k,2} + \mathcal{N}_{k,3}$$

where

$$\begin{aligned} \mathcal{N}_{k,1} &= \sum_j a_{j,k} |\eta_j|^4 e^{-2|\eta_j|^2 N_k^2 t} a_{j,k}^2 \theta_j \otimes \theta_j, \\ \mathcal{N}_{k,2} &= \sum_j |\eta_j|^4 e^{-2|\eta_j|^2 N_k^2 t} a_{j,k}^2 (\tilde{\varphi}_j^2(M_k x) \sin^2(N_k(x - x_j) \cdot \eta_j) - A_{j,k}) \theta_j \otimes \theta_j, \\ \mathcal{N}_{k,3} &= N_k^4 \sum_{j_1 \neq j_2} |\eta_{j_1}|^2 |\eta_{j_2}|^2 a_{j_1, k} a_{j_2, k} \Psi_{j_1, k} \otimes \Psi_{j_2, k} \end{aligned}$$

where $A_{j,k}$ was defined in (4.13). $\mathcal{N}_{k,1}$ is the leading part that acts as a “force” on $\bar{v}_{k-1}$, while $\mathcal{N}_{k,2}$ will be perturbative. Meanwhile, $\mathcal{N}_{k,3}$ consists of cross terms which vanish identically due to the pipe separation assumption (4.12).

Let us also define the tensor fields

$$\begin{aligned} R_k &= \phi_k * \mathcal{D} \sum_j a_{j,k} \Psi_{j,k} \\ \bar{R}_k &= 2^{-1} N_{k+1}^{-2} \sum_j A_{j,k+1} |\eta_j|^2 e^{-2|\eta_j|^2 N_{k+1}^2 t} a_{j,k+1}^2(x) \theta_j \otimes \theta_j \end{aligned}$$

which are constructed so that

$$\mathbb{P} \operatorname{div} R_k = v_k, \quad \mathbb{P} \operatorname{div} \bar{R}_k = \bar{v}_k. \quad (4.36)$$

At this point we specialize to the case $i = 1$ in which, we recall from Definition 4.2.8, $v^{(1)} =$

$v_0 + \bar{v}_1 + v_2 + \bar{v}_3 + \dots$. To simplify this expansion, we define the consolidated parts of the velocity

$$v^p := \sum_{k \geq 0 \text{ even}} v_k^p, \quad v^e := \sum_{k \geq 0 \text{ even}} v_k^e, \quad \bar{v} := \sum_{k \geq 1 \text{ odd}} \bar{v}_k$$

so that we may write $v^{(1)} = v^p + v^e + \bar{v}$. From (4.34) we easily have

$$\|\nabla^m v^p\|_{L^\infty} \lesssim_m \sum_k N_k^{1+m} \exp(-N_k^2 t) \lesssim_m t^{-\frac{1+m}{2}}. \quad (4.37)$$

Next, from (4.35),

$$\begin{aligned} \|\nabla^m v^e\|_{L^\infty} &\lesssim_m \sum_{k \geq 0} M_k N_k^m \exp(-N_k^2 t) \\ &\lesssim (\sup_k M_k / N_k)^{\frac{1}{2}} \sum_{k \geq 0} (M_k N_k)^{\frac{1}{2}} N_k^m \exp(-N_k^2 t). \end{aligned}$$

The factor in front is comparable to $\sup_k A^{-(\gamma-1)b^k/2} = A^{-(\gamma-1)/2}$. With a sufficiently large choice of A (depending on γ), this is majorized by ϵ_0/C_3 where C_3 can be chosen as needed to absorb incidental constants that appear. With $t > 0$ fixed, we estimate the sum. Observe that the summands increase rapidly until $N_k \sim t^{-\frac{1}{2}}$, then they decrease rapidly. As a result, the sum is controlled by the largest term. We can crudely provide an upper bound by letting N_k run over $(0, \infty)$ and maximizing; clearly the maximum is attained at $N_k \sim t^{-\frac{1}{2}}$. We arrive at

$$t^{\frac{1}{2}} \|v^e\|_{L^\infty} + t \|\nabla v^e\|_{C^\kappa} \lesssim C_0^{-1} \epsilon_0 t^{\frac{1}{4} - \frac{1}{4\gamma} - \frac{\kappa}{2}} \leq C_0^{-1} \epsilon_0 t^\alpha \quad (4.38)$$

by (4.8), since $M_k = N_k^{1/\gamma}$ and $t \leq 1$. Similarly, using (4.31),

$$\|\nabla^m \bar{v}\|_{L^\infty} \lesssim_m \sum_{k \geq 1} N_k^{1+\beta_1+m} e^{-N_{k+1}^2 t/O(1)} \lesssim (\sup_k N_k / N_{k+1})^{\frac{1}{4}} \sum_{k \geq 1} N_k^{\frac{3}{4}+\beta_1+m} N_{k+1}^{\frac{1}{4}} e^{-N_{k+1}^2 t/O(1)}$$

and therefore, by analogous reasoning and a sufficiently large choice of A depending on b ,

$$t^{\frac{1}{2}} \|\bar{v}\|_{L^\infty} + t \|\nabla \bar{v}\|_{C^\kappa} \lesssim C_3^{-1} \epsilon_0 (t^{\frac{3}{8} - \frac{3/4 + \beta_1}{2b}} + t^{\frac{1}{2} - \frac{1}{2b} - \frac{\kappa}{2}} t^{\frac{3}{8} - \frac{3/4 + \beta_1}{2b}}) \leq C_3^{-1} \epsilon_0 t^\alpha, \quad (4.39)$$

where $\beta_1 = \min\{\frac{b-1}{4}, \frac{1}{2}\}$ as before, and estimating using (4.8) that $\frac{3}{8} - \frac{3/4 + \beta_1}{2b} \geq \frac{1}{4}(1 - b^{-1}) > \alpha$ and that $\kappa < 1 - b^{-1}$. A nearly identical argument will be used for every contribution to extract some smallness in exchange for a bit of criticality so we shall not repeat the details each time.

Finally we can construct $F^{(1)}$:

$$F^{(1)} := F_1^{(1)} + F_2^{(1)} + F_3^{(1)} + F_4^{(1)}$$

where

$$\begin{aligned} F_1^{(1)} &= - \sum_{k \geq 0 \text{ even}} (\partial_t - \Delta) R_k \\ F_2^{(1)} &= \sum_{k \geq 1 \text{ odd}} \Delta \bar{R}_k \\ F_3^{(1)} &= -\mathcal{R} \operatorname{div} \mathcal{N}_{0,1} - \sum_{k \geq 1 \text{ odd}} (\partial_t \bar{R}_k + \mathcal{N}_{k+1,1}) \\ F_4^{(1)} &= -v^{(1)} \otimes v^{(1)} + \sum_{k \geq 0 \text{ even}} (v_k^p \otimes v_k^p - \mathcal{R} \operatorname{div} \mathcal{N}_{k,2}) \end{aligned}$$

where $\mathcal{R}$ is the anti-divergence operator defined in Lemma 4.5.2. Let us briefly verify that $F^{(1)}$

defined in this way satisfies (4.33). We compute

$$\begin{aligned}
-\mathbb{P} \operatorname{div} F^{(1)} &= \sum_{k \geq 0 \text{ even}} (\partial_t - \Delta) \mathbb{P} \operatorname{div} R_k - \sum_{k \geq 1 \text{ odd}} \Delta \mathbb{P} \operatorname{div} \bar{R}_k + \mathbb{P} \operatorname{div} \mathcal{N}_{0,1} \\
&+ \sum_{k \geq 1 \text{ odd}} (\partial_t \mathbb{P} \operatorname{div} \bar{R}_k + \mathbb{P} \operatorname{div} \mathcal{N}_{k+1,1}) + \mathbb{P} \operatorname{div} v^{(1)} \otimes v^{(1)} \\
&- \sum_{k \geq 0 \text{ even}} \mathbb{P} \operatorname{div} v_k^p \otimes v_k^p + \sum_{k \geq 0 \text{ even}} \mathbb{P} \operatorname{div} \mathcal{N}_{k,2} \\
&= \sum_{k \geq 0 \text{ even}} (\partial_t - \Delta) v_k + \sum_{k \geq 1 \text{ odd}} (\partial_t - \Delta) \bar{v}_k + \mathbb{P} \operatorname{div} v^{(1)} \otimes v^{(1)} \tag{4.40}
\end{aligned}$$

$$+ \mathbb{P} \operatorname{div} \left(\mathcal{N}_{0,1} + \sum_{k \geq 1 \text{ odd}} \mathcal{N}_{k+1,1} - \sum_{k \geq 0 \text{ even}} (v_k^p \otimes v_k^p - \mathcal{N}_{k,2}) \right) \tag{4.41}$$

using (4.36) and the fact that $\operatorname{div} \mathcal{R}$ acts as the identity on zero-average fields (which, indeed, is the case throughout by the divergence theorem). Clearly the expression on line (4.40) is precisely $\partial_t v^{(1)} - \Delta v^{(1)} + \mathbb{P} \operatorname{div} v^{(1)} \otimes v^{(1)}$. Meanwhile, the expression on the next line (4.41) identically vanishes because $v_k^p \otimes v_k^p = \mathcal{N}_{k,1} + \mathcal{N}_{k,2}$. Thus we conclude that (4.33) indeed holds for $F^{(1)}$.

Now we proceed to prove the claimed estimate on $F^{(1)}$. Consider the contribution to $F_1^{(1)}$ from a fixed $j \in \{1, 2, \dots, 6\}$ and k even. The key observation is that the heat operator exactly annihilates the $\sin(N_k(x - x_j) \cdot \eta_j) \exp(-|\eta_j|^2 N_k^2 t)$ factor within $\Psi_{j,k}$; thus one is left with the remaining terms from the Laplacian,

$$N_k^{-2} |\phi_k * \mathcal{D}(\Delta(\tilde{\varphi}_j a_{j,k}) \sin \Theta + 2\eta_j \cdot \nabla(\tilde{\varphi}_j a_{j,k}) N_k \cos \Theta)| e^{-|\eta_j|^2 N_k^2 t} \lesssim M_k N_k e^{-N_k^2 t}$$

by (4.29), having written Θ in place of $N_k(x - x_j) \cdot \eta_j$ for brevity. Once again by (4.29), additional derivatives cost a factor of N_k each and we obtain

$$\|\nabla^m F_1^{(1)}\|_{L^\infty} \lesssim \sum_{k \geq 0 \text{ even}} M_k N_k^{1+m} \exp(-N_k^2 t)$$

for $m \in \{0, 1\}$. This implies

$$t\|F_1^{(1)}\|_{L^\infty} + t^{\frac{3}{2}}\|\nabla F_1^{(1)}\|_{C^\kappa} \lesssim C_3^{-1}\epsilon_0 t^{\frac{1}{4} - \frac{1}{4\gamma}} \leq C_3^{-1}\epsilon_0 t^\alpha \quad (4.42)$$

by (4.8) and identical reasoning as above to extract the ϵ_0 factor.

For $F_2^{(1)}$, we use the crude upper bound coming from (4.29),

$$\frac{1}{2}N_{k+1}^{-2}|\Delta \sum_j A_{j,k+1}|\eta_j|^2 e^{-2|\eta_j|^2 N_{k+1}^2 t} a_{j,k+1}^2(x) \theta_j \otimes \theta_j| \lesssim N_k^2 e^{-N_{k+1}^2 t}.$$

Once again, additional derivatives cost N_k so

$$\|\nabla^m F_2\|_{L^\infty} \lesssim \sum_{k \geq 1 \text{ odd}} N_k^{2+m} \exp(-N_{k+1}^2 t).$$

By the same reasoning as before, we may extract some smallness from the fast time decay and obtain

$$t\|F_2^{(1)}\|_{L^\infty} + t^{\frac{3}{2}}\|\nabla F_2^{(1)}\|_{C^\kappa} \lesssim C_3^{-1}\epsilon_0 t^{\frac{1}{2} - \frac{1}{2b}} \leq C_3^{-1}\epsilon_0 t^\alpha$$

by (4.8) and a sufficiently large choice of A .

Next, we claim that $F_3^{(1)}$ vanishes identically. Indeed, $\mathcal{N}_{0,1}$ is a constant because $a_{j,0}$ are constants, so it is annihilated by div . Then, it is a trivial calculation from the definitions that $\partial_t \bar{R}_k + \mathcal{N}_{k+1,1}$ identically vanishes for all $k \geq 1$.

Moving to $F_4^{(1)}$, recall the decomposition $v^{(1)} = v^p + v^e + \bar{v}$ from which we can expand

$$\begin{aligned} F_4^{(1)} &= -\left(\sum_{k \neq m \geq 0 \text{ even}} v_k^p \otimes v_m^p \right) - v^{(1)} \otimes (\bar{v} + v^e) - (\bar{v} + v^e) \otimes v^p - \sum_{k \geq 0 \text{ even}} \mathcal{R} \text{div} \mathcal{N}_{k,2} \\ &=: F_{4,1} + F_{4,2} + F_{4,3} + F_{4,4}. \end{aligned}$$

We immediately have from (4.27), (4.37), (4.38), (4.39), and the product rule for $C^{1,\kappa} \cap L^\infty$ that $F_{4,2}$ and $F_{4,3}$ obey the desired estimates.

Next consider $F_{4,1}$. By (4.34),

$$\begin{aligned}
\|F_{4,1}(t)\|_{L^\infty} &\leq \sum_{k \neq m} \|v_k^p \otimes v_m^p\|_{L^\infty} \lesssim \sum_{k \neq m} N_k N_m \exp(-(N_k^2 + N_m^2)t) \\
&\lesssim \sum_m \left(N_m \exp(-N_m^2 t) \sum_{k < m} N_k \right) \\
&\lesssim \sum_m N_{m-1} N_m \exp(-N_m^2 t) \\
&\lesssim C_3^{-1} \epsilon_0 t^{-\frac{3}{4} - \frac{1}{4b}}
\end{aligned}$$

where, to obtain the second line from the first, we invoke the symmetry of k and m and sum over only the indices $k < m$ without losing of generality.

Similarly, for the derivative,

$$\begin{aligned}
\|\nabla F_{4,1}(t)\|_{C^\kappa} &\lesssim \sum_{k \neq m} (N_k + N_m)^{1+\kappa} N_k N_m \exp(-N_{k \vee m}^2 t) \\
&\lesssim \sum_m \left(N_m^{2+\kappa} \exp(-N_m^2 t) \sum_{k < m} N_k \right) \\
&\lesssim \sum_m N_{m-1} N_m^{2+\kappa} \exp(-N_m^2 t / O(1)) \\
&\lesssim C_3^{-1} \epsilon_0 t^{-\frac{5}{4} - \frac{1}{4b} - \frac{\kappa}{2}}.
\end{aligned}$$

With (4.8) and the fact that $\kappa < \frac{1}{2} - \frac{1}{2b} - 2\alpha$, we conclude that

$$t\|F_{4,1}(t)\|_{L^\infty} + t^{\frac{3}{2}}\|F_{4,1}(t)\|_{L^\infty} \lesssim C_3^{-1} \epsilon_0 t^\alpha.$$

Finally we estimate $F_{4,4}$:

$$-\mathcal{R} \operatorname{div} \sum_{k \geq 0 \text{ even}} \sum_j |\eta_j|^4 e^{-2|\eta_j|^2 N_k^2 t} a_{j,k}^2 (\tilde{\varphi}_j^2(M_k x) \sin^2(N_k(x - x_j) \cdot \eta_j) - A_{j,k}) \theta_j \otimes \theta_j.$$

Let us fix a particular j, k . The first observation is that when div falls on the expression in

parentheses, the result vanishes due to (4.11). What remains is

$$F_{4,4} = - \sum_{k \geq 0} \sum_{\text{even } j} |\eta_j|^4 e^{-2|\eta_j|^2 N_k^2 t} F_{4,4,j,k} : \theta_j \otimes \theta_j$$

where $F_{4,4,j,k}$ can be written in coordinates as

$$F_{4,4,j,k}^{j_1 j_2 j_3 j_4} := \mathcal{R}^{j_1 j_2 j_3} \left(\partial_{j_4} (a_{j,k}^2) (\tilde{\varphi}_j^2(M_k x) \sin^2(N_k(x - x_j) \cdot \eta_j) - A_{j,k}) \right).$$

We shall henceforth omit the coordinates because they play no role in the estimates. We observe that $F_{4,4,j,0}$ vanishes identically because $a_{j,0}$ is a constant.

Recall by definition that N_k is an integer multiple of M_k ; thus we may write

$$h_{j,k}(M_k x) = \tilde{\varphi}_j^2(M_k x) \sin^2(N_k(x - x_j) \cdot \eta_j) - A_{j,k}$$

for a 2π -periodic function $h_{j,k} \in C^\infty(\mathbb{T}^3; \mathbb{R})$. Most importantly, due to (4.13), $h_{j,k}$ has zero average on $\mathbb{T}^3$. Thus, taking a Fourier series of $h_{j,k}$, we obtain

$$F_{4,4,j,k}(x) = \sum_{\xi \in \mathbb{Z}^3 \setminus 0} \hat{h}_{j,k}(\xi) \mathcal{R} \left(\nabla(a_{j,k}^2)(x) e^{iM_k \xi \cdot x} \right). \quad (4.43)$$

We need to estimate the Fourier coefficients of $h_{j,k} = \tilde{\varphi}_j^2(x) \sin^2(N_k M_k^{-1}(x - M_k x_j) \cdot \eta_j) - A_{j,k}$. Obviously the $\sin^2$ factor has Fourier support at just $\xi = 0$ and $\xi = \pm 2N_k M_k^{-1} \eta_j$. Thus

$$\hat{h}_{j,k}(\xi) = \left(s_0 \mathcal{F}(\tilde{\varphi}_j^2)(\xi) + s_1 \mathcal{F}(\tilde{\varphi}_j^2)(\xi + 2N_k M_k^{-1} \eta_j) + s_1 \mathcal{F}(\tilde{\varphi}_j^2)(\xi - 2N_k M_k^{-1} \eta_j) \right) \mathbf{1}_{\xi \neq 0}$$

where s_0 and s_1 are the Fourier coefficients of $\sin^2(N_k M_k^{-1}(x - M_k x_j) \cdot \eta_j)$ at $\xi = 0$ and $\xi = \pm 2N_k M_k^{-1} \eta_j$, respectively. One readily computes $|s_0| = 4\pi^3$ and $|s_1| = 2\pi^3$. This implies

$$|\hat{h}_{j,k}(\xi)| \lesssim \langle \xi \rangle^{-10} + \langle \xi + 2N_k M_k^{-1} \eta_j \rangle^{-10} + \langle \xi - 2N_k M_k^{-1} \eta_j \rangle^{-10}.$$

Finally, we can estimate (4.43) with an application of Lemma 4.5.4 with Hölder parameter β freely chosen in the range

$$0 < \beta < 1 - \gamma(4\alpha + b^{-1})$$

where the coefficient function “ a ” in the lemma becomes $\nabla(a_{j,k}^2)$. Note that this range is non-empty by definition of γ . Using (4.29), we have, for all $\xi \in \mathbb{Z}^3 \setminus 0$,

$$\|\mathcal{R}(\nabla(a_{j,k}^2)(x)e^{iM_k\xi \cdot x})\|_{C^\beta} \lesssim_m N_{k-1}N_k^2(M_k^{-1+\beta} + M_k^{-m+\beta}N_{k-1}^m + M_k^{-m}N_{k-1}^{m+\beta}).$$

The third term is clearly dominated by the second and can be dropped. By choosing $m > b/(1-\gamma/b)$, we can make $(N_{k-1}/M_k)^m \ll M_k^{-1}$, allowing the second term to be dropped as well. This leaves us with

$$\|F_{4,4,j,k}\|_{C^\beta} \lesssim N_{k-1}M_k^{-1+\beta}N_k^2 \sum_{\xi \in \mathbb{Z}^3} \left(\langle \xi \rangle^{-10} + \langle \xi + 2N_kM_k^{-1}\eta_j \rangle^{-10} + \langle \xi - 2N_kM_k^{-1}\eta_j \rangle^{-10} \right).$$

Clearly this is summable with a uniform constant; thus

$$\begin{aligned} \|F_{4,4}(t)\|_{L^\infty} &\lesssim \sum_{k \geq 2 \text{ even}} N_{k-1}M_k^{-1+\beta}N_k^2 e^{-N_k^2 t} \\ &\lesssim \sup_k (N_{k-1}M_k^{-1+\beta})^{\frac{1}{2}} \sum_k N_{k-1}^{\frac{1}{2}} M_k^{-(1-\beta)/2} N_k^2 e^{-N_k^2 t} \\ &\lesssim \left(\sup_k A^{(\frac{\gamma}{b}-1+\beta)b^k} \right)^{\frac{1}{2}} t^{-\frac{1}{4b} + \frac{1-\beta}{4\gamma} - 1}. \end{aligned}$$

By definition of β , the power $\frac{\gamma}{b} - 1 + \beta < -4\alpha\gamma$ is negative so the prefactor can be made as small as needed with the choice of A , as usual.

Finally, we estimate the gradient. Recall that $\nabla\mathcal{R}$ is a Calderón–Zygmund operator so it is bounded on $C^\kappa(\mathbb{T}^3)$. Thus, by (4.29) and interpolation,

$$\|\nabla F_{4,4,j,k}\|_{C^\kappa} \lesssim \|\nabla(a_{j,k}^2)(\tilde{\varphi}_j^2(M_k x) \sin^2(N_k(x - x_j) \cdot \eta_j) - A_{j,k})\|_{C^\kappa} \lesssim N_{k-1}N_k^{2+\kappa}.$$

Let $c_1 \in (0, 1 - \frac{b(\kappa+2\alpha)}{b-1})$. Then

$$\|\nabla F_{4,4}(t)\|_{C^\kappa} \lesssim \sup_k (N_{k-1}/N_k)^{c_1} \sum_{k \geq 2 \text{ even}} e^{-2N_k^2 t/O(1)} N_{k-1}^{1-c_1} N_k^{2+\kappa+c_1} \lesssim t^{-\frac{1-c_1}{2b} - \frac{2+\kappa+c_1}{2}}.$$

We conclude

$$t\|F_{4,4}(t)\|_{L^\infty} + t^{\frac{3}{2}}\|\nabla F_{4,4}(t)\|_{L^\infty} \lesssim C_3^{-1} \epsilon_0 (t^{\frac{1-\beta}{4\gamma} - \frac{1}{4b}} + t^{\frac{1-\kappa-c_1}{2} - \frac{1-c_1}{2b}}).$$

Both powers are bounded from below by α : the first because of the definitions of β, b , and γ , and the second because of the definitions of c_1, κ , and b .

With $F^{(1)}$ fully estimated, let us now address $F^{(2)}$ which corresponds to $v^{(2)} = \bar{v}_0 + v_1 + \bar{v}_2 + v_3 + \dots$. We have an exactly analogous decomposition $v^{(2)} = v^p + v^e + \bar{v}$ where v^p , v^e , and $\bar{v}$ are as before, the only change being that “even” and “odd” are reversed in the definitions. The estimates (4.37)–(4.39) still hold by the same arguments. Now we can define $F^{(2)}$:

$$F^{(2)} := F_1^{(2)} + F_2^{(2)} + F_3^{(2)} + F_4^{(2)}$$

where

$$\begin{aligned} F_1^{(2)} &= - \sum_{k \geq 1 \text{ odd}} (\partial_t - \Delta) R_k \\ F_2^{(2)} &= \sum_{k \geq 0 \text{ even}} \Delta \bar{R}_k \\ F_3^{(2)} &= - \sum_{k \geq 0 \text{ even}} (\partial_t \bar{R}_k + \mathcal{N}_{k+1,1}) \\ F_4^{(2)} &= -v^{(2)} \otimes v^{(2)} + \sum_{k \geq 1 \text{ odd}} (v_k^p \otimes v_k^p - \mathcal{R} \operatorname{div} \mathcal{N}_{k,2}). \end{aligned}$$

From here, identical calculations to the ones for $F^{(1)}$ can be used to verify (4.33) and the claimed estimates for $F^{(2)}$. $\square$

4.3.2 Semigroup of the Linearization around the Principal Part

Recall the subcriticality parameter $\alpha \in (0, \frac{1}{8})$ and Hölder exponent $\kappa \in (0, \frac{1}{2} - \frac{1}{2\gamma} - 2\alpha)$. We consider the Banach space

$$Y = \{a \in C^0((0, 1]; C^{1,\kappa}(\mathbb{T}^3; S^{3 \times 3})) : \|a\|_Y := \sup_{t \in (0, 1]} (t^{1-\alpha} \|a\|_{L^\infty(\mathbb{T}^3)} + t^{\frac{3}{2}-\alpha} \|\nabla a\|_{C^\kappa(\mathbb{T}^3)}) < \infty\}$$

in which we shall measure, for instance, the $F^{(i)}$ and $w^{(i)} \otimes w^{(i)}$. This choice is natural in light of Proposition 4.3.1.

For $i \in \{1, 2\}$, $a \in Y$, and $0 < t' \leq t \leq 1$, we define the semigroup $S^{(i)}(t, t')a$ as the solution of

$$\begin{aligned} \partial_t S^{(i)}(t, t')a - \Delta S^{(i)}(t, t') + 2\mathbb{P} \operatorname{div}(v^{(i)}(t) \odot S^{(i)}(t, t')a) &= 0 \\ S^{(i)}(t', t') &= \mathbb{P} \operatorname{div} a(t'). \end{aligned}$$

There is no difficulty with $S^{(i)}(t, t')$ being well-defined and smooth because $v^{(i)}$ is smooth with uniform estimates on $[t', 1] \times \mathbb{T}^3$. For the reader's convenience, we give a proof of existence in Section 4.6.1, or see [61, Theorem 21.3]. What is not so clear is whether one can obtain bounds that degenerate only mildly near $t' = 0$, given that the smoothness of $v^{(i)}$ breaks down.

The effect of the following proposition is that $S^{(i)}(t, t')$ behaves comparably to $e^{(t-t')\Delta} \mathbb{P} \operatorname{div}$, with the slight loss $(t/t')^\epsilon$ ($\epsilon > 0$ as small as desired) because $v^{(i)}$ does not belong to $L_t^2(\mathbb{R}_+; L_x^\infty)$ or $L^1(\mathbb{R}_+, t^{-\frac{1}{2}} dt; L^\infty)$. If $v^{(i)}$ were an arbitrary vector field in, say, the Koch–Tataru space X_{KT} and additionally obeyed (4.27), we do not expect the loss would be so mild. It is essential that the loss is, in particular, smaller than the subcriticality window $(t')^\alpha$. Fortunately, due to $v^{(i)}$ being supported on a lacunary sequence of scales, quantities such as $\|v^{(i)}\|_{L_t^2([t', t]; L_x^\infty)}$ can be arranged to deteriorate as slowly as desired as $t'/t \rightarrow 0$; see the proof of Proposition 4.2.5.

Proposition 4.3.2. *For all $i \in \{1, 2\}$, $a \in Y$, and $0 < t' \leq t \leq 1$, we have*

$$\|S^{(i)}(t, t')a\|_{L^\infty(\mathbb{T}^3)} + (t - t')^{\frac{1}{2}} \|\nabla S^{(i)}(t, t')a\|_{C^\kappa(\mathbb{T}^3)} \lesssim (t')^{-1+\alpha-\epsilon} t^{-\frac{1}{2}+\epsilon} \|a\|_Y.$$

Proof. We suppress the dependence on i and write v for either $v^{(1)}$ or $v^{(2)}$. For a fixed $t' \in (0, 1)$, the semigroup obeys the Duhamel formula

$$\begin{aligned} S(t, t')a &= e^{(t-t')\Delta} \mathbb{P} \operatorname{div} a(t') - 2 \int_{t'}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(v(s) \odot S(s, t')a) ds \\ &=: I + II + III \end{aligned}$$

where I is the linear propagator and II and III are the pieces of the integral over $[t', t' \vee (t/2)]$ and $[t' \vee (t/2), t]$ respectively.

Consider I . There are two ways to estimate: either by moving to C^κ on which $e^{(t-t')\Delta} \mathbb{P}$ is uniformly bounded, or applying the $C^1 \rightarrow L^\infty$ boundedness of the heat kernel from Lemma 4.1.2. We use whatever upper bound is more favorable and obtain

$$\begin{aligned} \|I\|_{L^\infty} &\lesssim \|\nabla a(t')\|_{C^\kappa} \wedge ((t-t')^{-\frac{1}{2}} \|a(t')\|_{L^\infty}) \\ &\lesssim (t')^{-\frac{3}{2}+\alpha} \wedge ((t-t')^{-\frac{1}{2}} (t')^{-1+\alpha}) \|a\|_Y \\ &= (t')^{-1+\alpha} (t' \vee (t-t'))^{-\frac{1}{2}} \|a\|_Y \\ &\lesssim (t')^{-1+\alpha} t^{-\frac{1}{2}} \|a\|_Y. \end{aligned}$$

Next we have by (4.27)

$$\begin{aligned} \|II\|_{L^\infty} &\lesssim \int_{t'}^{t' \vee (t/2)} (t-s)^{-\frac{1}{2}} \|v(s)\|_{L^\infty} \|S(s, t')a\|_{L^\infty} ds \\ &\lesssim t^{-\frac{1}{2}} \int_{t'}^{t' \vee (t/2)} s^{-\frac{1}{2}} \|v(s)\|_{L^\infty} \|S(s, t')a\|_{L^\infty} s^{\frac{1}{2}} ds \end{aligned}$$

where we have used the fact that the integral vanishes identically unless $t > 2t'$, so $t/(t-s) \leq t/(t-t/2) = 2$. Also by (4.27),

$$\begin{aligned} \|III\|_{L^\infty} &\lesssim \int_{t' \vee (t/2)}^t (t-s)^{-\frac{1}{2}} \|v(s)\|_{L^\infty} \|S(s, t')a\|_{L^\infty} ds \\ &\lesssim t^{-\frac{1}{2}} \int_{t' \vee (t/2)}^t (t-s)^{-\frac{1}{2}} \|v(s)\|_{L^\infty} \|S(s, t')a\|_{L^\infty} s^{\frac{1}{2}} ds, \end{aligned}$$

since $(t/s)^{1/2} \leq \sqrt{2}$ on the interval $s \in [t' \vee (t/2), t]$. Letting $h(t) = t^{\frac{1}{2}} \|S(t, t')a\|_{L^\infty}$ and summing the estimates for *I*, *II*, and *III*, we have arrived at

$$h(t) \lesssim (t')^{-1+\alpha} \|a\|_Y + \int_{t'}^t (s^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}}) \|v(s)\|_{L^\infty} h(s) ds.$$

This is in a form where we can apply Lemma 4.6.2 with $p = 3$ (say), $g_1(s) = s^{-\frac{1}{2}} \|v(s)\|_{L^\infty}$ and $g_2(s) = \|v(s)\|_{L^\infty}$ which lead to

$$h(t) \lesssim (t')^{-1+\alpha} \|a\|_Y \exp\left(O(\|v\|_{L^1([t', t], t^{-\frac{1}{2}} dt; L^\infty)} + \|s^{\frac{1}{2}} v\|_{L_{t,x}^\infty([t', t])} \|v\|_{L^2([t', t]; L^\infty)}^2)\right).$$

By (4.27)–(4.28), we may estimate the norms in the exponent to obtain

$$h(t) \lesssim (t')^{-1+\alpha} \|a\|_Y \exp(O(1 + (\log A)^{-1} \log(t/t'))).$$

The exponential factor becomes $O((t/t')^{O(1/\log A)})$. Choosing A sufficiently large depending on ϵ , the power becomes smaller than $\epsilon/2$. Unpacking the definition of h thus yields

$$\|S(t, t')a\|_{L^\infty} \lesssim t^{-\frac{1}{2}} (t')^{-1+\alpha} (t/t')^{\frac{\epsilon}{2}} \|a\|_Y. \quad (4.44)$$

Next we address the $C^{1,\kappa}$ norm. To do so, we redefine the decomposition of $S(t, t')a$: *I* stays the same but *II* and *III* split the integral into $[t', (t+t')/2]$ and $[(t+t')/2, t]$ respectively. Then, by a similar calculation using (4.7),

$$\begin{aligned} \|\nabla I\|_{C^\kappa} &\lesssim ((t-t')^{-\frac{1}{2}} \|\nabla a(t')\|_{C^\kappa}) \wedge ((t-t')^{-1-\frac{\kappa}{2}} \|a(t')\|_{L^\infty}) \\ &\lesssim ((t-t')^{-\frac{1}{2}} (t')^{-\frac{3}{2}+\alpha}) \wedge ((t-t')^{-1-\frac{\kappa}{2}} (t')^{-1+\alpha}) \|a\|_Y \\ &\lesssim (t-t')^{-\frac{1}{2}} (t')^{-1+\alpha} t^{-\frac{1+\kappa}{2}} \|a\|_Y, \end{aligned}$$

having used the fact that $t' \vee (t-t')^{1+\kappa} \geq (t/2)^{1+\kappa}$ which can be seen by considering the cases

$t' > t/2$ and $t' \leq t/2$ and using $t \leq 1$. Next, by (4.7), (4.27), and (4.44), we have

$$\begin{aligned}\|\nabla II\|_{C^\kappa} &\lesssim \int_{t'}^{(t+t')/2} (t-s)^{-1-\frac{\kappa}{2}} \|v(s)\|_{L^\infty} \|S(s, t')a\|_{L^\infty} ds \\ &\lesssim (t')^{-1+\alpha-\frac{\kappa}{2}} \int_{t'}^{(t+t')/2} (t-s)^{-1-\frac{\kappa}{2}} s^{-1+\frac{\kappa}{2}} ds \|a\|_Y \\ &\lesssim (t-t')^{-\frac{\kappa}{2}} (t')^{-1+\alpha-\frac{\kappa}{2}} t^{-1+\frac{\kappa}{2}} \|a\|_Y\end{aligned}$$

and, additionally using (4.44) and that $C^{1,\kappa} \cap L^\infty$ is a multiplication algebra,

$$\begin{aligned}\|\nabla III\|_{C^\kappa} &\lesssim \int_{(t+t')/2}^t (t-s)^{-\frac{1}{2}} (\|\nabla v(s)\|_{C^\kappa} \|S(s, t')a\|_{L^\infty} + \|v(s)\|_{L^\infty} \|\nabla S(s, t')a\|_{C^\kappa}) ds \\ &\lesssim \int_{(t+t')/2}^t (t-s)^{-\frac{1}{2}} (s^{-\frac{3}{2}+\frac{\kappa}{2}-\frac{\kappa}{2}} (t')^{-1+\alpha-\frac{\kappa}{2}} \|a\|_Y + \|v(s)\|_{L^\infty} \|\nabla S(s, t')a\|_{C^\kappa}) ds.\end{aligned}$$

The first term in the integrand contributes $O((t')^{-1+\alpha-\frac{\kappa}{2}} t^{-1+\frac{\kappa}{2}-\frac{\kappa}{2}} \|a\|_Y)$. Thus, defining $h(t) = (t-t')^{\frac{1}{2}} t^{\frac{1+\kappa}{2}} \|\nabla S(t, t')\|_{C^\kappa}$, we have

$$h(t) \lesssim (t')^{-1+\alpha} (t/t')^{\frac{\kappa}{2}} \|a\|_Y + \int_{(t+t')/2}^t (t-s)^{-\frac{1}{2}} \|v(s)\|_{L^\infty} h(s) ds,$$

using that t and s are comparable on this time interval. We conclude once again by Lemma 4.6.2, (4.27), and (4.28). $\square$

4.3.3 Fixed Point Argument

We construct the perturbation $w^{(i)}$ by means of a fixed point argument on the Banach space

$$X = \{V \in C^0((0, 1]; C^{1,\kappa}(\mathbb{T}^3; \mathbb{R}^3)) : \|V\|_X := \sup_{t \in (0, 1]} (t^{\frac{1}{2}-\frac{\alpha}{2}} \|V\|_{L^\infty} + t^{1-\frac{\alpha}{2}} \|\nabla V\|_{C^\kappa}) < \infty\}.$$

Recall the parameter ϵ_0 from Proposition 4.3.1 which quantifies the smallness of the residuals $F^{(i)}$. This smallness, combined with the zero initial data, enables us to construct the perturbations $w^{(i)}$ by a standard fixed point argument in X .

Proposition 4.3.3. *There exists $C_4 > 0$ such that for all $\epsilon_1 \in (0, C_4^{-1})$ and sufficiently large $A > 1$, there exist $w^{(1)}, w^{(2)} \in B_X(0, \epsilon_1)$ such that $u^{(1)} = v^{(1)} + w^{(1)}$ and $u^{(2)} = v^{(2)} + w^{(2)}$ both satisfy (1.13). Furthermore, $w^{(i)}(t) \rightarrow 0$ in $\mathcal{C}^{-1+\frac{\alpha}{2}}$ as $t \rightarrow 0$.*

Note that at this point, we do not make any claim about the initial data of the $u^{(i)}$. The second statement will eventually play a role by, in effect, enforcing that the data is determined only by $v^{(i)}$. We point out that $w^{(i)} \in X$ alone does not imply any control near the initial time.

Proof. Once again in this proof, we suppress the dependence on i of $v^{(i)}, w^{(i)}, F^{(i)}$, and $S^{(i)}$.

In order for $u = v + w$ to solve (1.13), the perturbation w should solve

$$\begin{aligned} \partial_t w - \Delta w + 2\mathbb{P} \operatorname{div} v \odot w &= \mathbb{P} \operatorname{div}(F - w \otimes w) \\ w|_{t=0} &= 0 \end{aligned}$$

where F is as in Proposition 4.3.1, in particular satisfying (4.33). We solve for w as a fixed point of the map

$$T(w)(t) = \int_0^t S(t, t')(F - w \otimes w)(t') dt'$$

where $S(t, t')$ is the semigroup of the linearization around v defined in §4.3.2. Recall as well the function space Y defined in §4.3.2. From the following estimates, it will follow that T is a well-defined operator on X .

Based on the definitions of X and Y , we have the elementary product estimate

$$\|w \otimes w\|_Y \lesssim \sup_{0 < t \leq 1} (t^{\frac{3}{2}-\alpha} \|w(t)\|_{L^\infty} \|\nabla w(t)\|_{C^\kappa} + t^{1-\alpha} \|w(t)\|_{L^\infty}^2) \lesssim \|w\|_X^2.$$

Combining this with Propositions 4.3.1 and 4.3.2 and the fact that $\epsilon < \alpha$,

$$\begin{aligned}\|T(w)(t)\|_{L^\infty} &\lesssim t^{-\frac{1}{2}+\epsilon} \int_0^t (t')^{-1+\alpha-\epsilon} dt' \|F - w \otimes w\|_Y \\ &\lesssim t^{-\frac{1}{2}+\epsilon} \int_0^t (t')^{-1+\alpha-\epsilon} dt' (\|w\|_X^2 + \|F\|_Y) \\ &\lesssim t^{-\frac{1}{2}+\alpha} (\|w\|_X^2 + \epsilon_0)\end{aligned}$$

and

$$\begin{aligned}\|\nabla T(w)(t)\|_{C^\kappa} &\lesssim t^{-\frac{1}{2}+\epsilon} \int_0^t (t')^{-1+\alpha-\epsilon} (t-t')^{-\frac{1}{2}} dt' \|F - w \otimes w\|_Y \\ &\lesssim t^{-1+\alpha} (\|w\|_X^2 + \epsilon_0).\end{aligned}$$

It can be arranged via Proposition 4.3.1 that ϵ_0 is so small that

$$\|T(w)\|_X \leq O(\|w\|_X^2) + \frac{1}{2}\epsilon_1,$$

having also used $t \leq 1$. By a similar calculation,

$$\|T(w_1) - T(w_2)\|_X \lesssim \|w_1 - w_2\|_X (\|w_1\|_X + \|w_2\|_X).$$

Choosing C_4 sufficiently large to compensate for the implicit constants, it follows that T is a contraction on the ball $B_X(0, \epsilon_1)$ for all $\epsilon_1 \in (0, C_4^{-1})$, and we conclude by the Banach fixed point theorem.

Finally, we prove control of w near the initial time using the Duhamel formula

$$w(t) = \int_0^t e^{(t-t')\Delta} \mathbb{P} \operatorname{div}(-2v \odot w - w \otimes w + F)(t') dt'.$$

By (4.7), (4.27), Proposition 4.3.1, interpolation, and the fact that $\|w\|_X \leq \epsilon_1$, we have

$$\begin{aligned} \|w(t)\|_{\mathcal{C}^{-1+\alpha/2}} &\lesssim \int_0^t \| -2v \odot w(t') - w \otimes w(t') + F(t') \|_{\mathcal{C}^{\alpha/2}} dt' \\ &\lesssim \int_0^t ((t')^{-1+\frac{\alpha}{4}} + (t')^{-1+\frac{3\alpha}{4}}) dt' \\ &\lesssim t^{\alpha/4} \end{aligned}$$

for all $t \in (0, 1]$. □

4.4 Proof of the Non-Uniqueness Theorem

We have shown that for $i \in \{1, 2\}$, there exist $v^{(i)} \in C^\infty((0, 1] \times \mathbb{T}^3)$ and $w^{(i)} \in C^0((0, 1]; C^{1, \kappa})$ obeying suitable equations such that $u^{(i)} = v^{(i)} + w^{(i)}$ obey (1.13). By standard regularity theory, $u^{(i)}$ is in fact smooth on $(0, 1] \times \mathbb{T}^3$.

Next we argue that both solutions attain the data U^0 . We showed in Proposition 4.3.3 that $w^{(i)}(t) \rightarrow 0$ in $\mathcal{C}^{-1+\alpha/2}(\mathbb{T}^3)$. This subcritical topology is too strong for the principal part of the solutions; instead we claim that $v^{(i)}(t) \rightarrow U^0$ in $\dot{W}^{-1, p}(\mathbb{T}^3)$, for all $1 < p < \infty$. Recall from Definitions 4.2.7–4.2.8 that for $i \in \{1, 2\}$, we can write

$$v^{(i)}(t) = \operatorname{curl} \sum_{k \in \mathbb{N}_i} \operatorname{curl} \psi_k(t) + \sum_{k \in \mathbb{N} \setminus \mathbb{N}_i} \bar{v}_k(t), \quad U^0 = \operatorname{curl} \sum_{k \in \mathbb{N}} \operatorname{curl} \psi_k^0$$

where $\mathbb{N}_i := i - 1, i + 1, i + 3, \dots$ for $i \in \{1, 2\}$. For the heat-dominated part of the data, we use

Definition 4.2.8, (4.29), and (4.10) to estimate

$$\begin{aligned}
\left\| \sum_{k \in \mathbb{N}_i} (\operatorname{curl} \psi_k(t) - \operatorname{curl} \psi_k^0) \right\|_{L^p} &\lesssim \sum_k \|\operatorname{curl} (a_{j,k}(\Psi_{j,k} - \Psi_{j,k}^0))\|_{L^p} \\
&\lesssim \sum_{j,k} \|\nabla(a_{j,k} \Psi_{j,k}^0)\|_{L^p} (1 - e^{-|\eta_j|^2 N_k^2 t}) \\
&\lesssim \sum_k |\tilde{\Omega}_k|^{\frac{1}{p}} (1 \wedge (N_k^2 t)) \\
&\leq \sum_{k : N_k \leq t^{-1/4}} N_k^2 t + \sum_{k : N_k > t^{-1/4}} 2^{-\frac{k}{p}}.
\end{aligned}$$

In the last two lines, we used the fact that $\|\nabla(a_{j,k} \Psi_{j,k}^0)\|_{L^\infty} = O(1)$ and that $\operatorname{supp}(a_{j,k} \Psi_{j,k}^0) \subset \tilde{\Omega}_k$, whose volume is bounded by 2^{-k} as shown in Lemma 4.2.3. The terms in the first sum grow faster than exponentially so the sum is controlled by the largest term which is $O(t^{\frac{1}{2}})$. For the second sum, clearly the first index k satisfying $N_{k+1} > t^{-1/4}$ will grow to infinity as $t \rightarrow 0$. Since the terms themselves are independent of t , the sum is the shrinking tail of a convergent series so it goes to zero. We conclude that $\operatorname{curl} \sum_{k \in \mathbb{N}_i} \psi_k(t) \rightarrow \operatorname{curl} \sum_{k \in \mathbb{N}} \psi_k^0$ in $\dot{W}^{-1,p}$.

Likewise for the nonlinear evolution-dominated part, with Definition 4.2.8, we compute

$$\begin{aligned}
&\operatorname{curl} \operatorname{curl} \psi_k^0 - \bar{v}_k(t) \\
&= \operatorname{curl} \operatorname{curl} \psi_k^0 - \sum_j \mathbb{P} \operatorname{div} (e^{-2|\eta_j|^2 N_{k+1}^2 t} 2^{-1} N_{k+1}^{-2} A_{j,k+1} |\eta_j|^2 a_{j,k+1}^2(x) \theta_j \otimes \theta_j) \\
&= \mathbb{P} \operatorname{div} \sum_j (1 - e^{-2|\eta_j|^2 N_{k+1}^2 t}) 2^{-1} N_{k+1}^{-2} A_{j,k+1} |\eta_j|^2 a_{j,k+1}^2(x) \theta_j \otimes \theta_j \\
&\quad + \left(\operatorname{curl} \operatorname{curl} \psi_k^0 - 2^{-1} N_{k+1}^{-2} \mathbb{P} \operatorname{div} \sum_j a_{j,k+1} |\eta_j|^2 a_{j,k+1}^2(x) \theta_j \otimes \theta_j \right).
\end{aligned}$$

By the same computation from Remark 4.2.4, the quantity in the last line identically vanishes. For

the remaining part, we estimate using (4.29) and Lemma 4.2.3 that

$$\begin{aligned}
& \|\mathbb{P} \operatorname{div} \sum_j (1 - e^{-2|\eta_j|^2 N_{k+1}^2 t}) 2^{-1} N_{k+1}^{-2} A_{j,k+1} |\eta_j|^2 a_{j,k+1}^2(x) \theta_j \otimes \theta_j\|_{\dot{W}^{-1,p}} \\
& \lesssim \sum_j (1 - e^{-2|\eta_j|^2 N_{k+1}^2 t}) N_{k+1}^{-2} \|a_{j,k+1}\|_{L^{2p}}^2 \\
& \lesssim (1 \wedge (N_{k+1}^2 t)) |\tilde{\Omega}_k|^{\frac{1}{p}}
\end{aligned}$$

for all $p \in (1, \infty)$, since $\|a_{j,k+1}\|_{L^\infty} \lesssim N_{k+1}$ and $\operatorname{supp} a_{j,k+1} \subset \tilde{\Omega}_k$ by Definitions 4.2.4 and 4.2.5. Summing, we find for all $t < 1$ that $\sum_{k \in \mathbb{N} \setminus \mathbb{N}_i} (\bar{v}_k(t) - \operatorname{curl} \operatorname{curl} \psi_k^0)$ is absolutely summable in the $\dot{W}^{-1,p}$ norm and obeys the bound

$$\left\| \sum_{k \in \mathbb{N} \setminus \mathbb{N}_i} (\bar{v}_k(t) - \operatorname{curl} \operatorname{curl} \psi_k^0) \right\|_{\dot{W}^{-1,p}} \lesssim \sum_k |\tilde{\Omega}_k|^{\frac{1}{p}} (1 \wedge (N_{k+1}^2 t))$$

which converges to zero by the same reasoning as in the previous case. This completes the proof that the initial data is attained by both solutions.

Finally, we show that the two solutions are distinct. We consider a time scale when all but the lowest frequency mode should have dissipated away, for instance $t_0 = N_0^{-2}$. Recall from Proposition 4.3.3 that $w \in B_X(0, \epsilon_1)$. From the triangle inequality,

$$\begin{aligned}
\|v^{(1)}(t_0) - v^{(2)}(t_0)\|_{L^\infty} & \geq \|v_0(t_0)\|_{L^\infty} - \sum_{k \geq 1} \|v_k(t_0)\|_{L^\infty} - \sum_{k \geq 0} \|\bar{v}_k(t_0)\|_{L^\infty} \\
& \quad - \|w^{(1)}(t_0)\|_{L^\infty} - \|w^{(2)}(t_0)\|_{L^\infty}.
\end{aligned}$$

We estimate using (4.30) that

$$\begin{aligned}
\sum_{k \geq 1} \|v_k(t_0)\|_{L^\infty} & \lesssim \sum_{k \geq 1} N_k \exp(-N_k^2 N_0^{-2}) \lesssim N_1 \exp(-N_1^2 N_0^{-2}) \\
& \leq N_0 (N_1/N_0)^{-100}
\end{aligned}$$

and

$$\sum_{k \geq 0} \|\bar{v}_k(t_0)\|_{L^\infty} \lesssim \sum_{k \geq 0} N_k \exp(-N_{k+1}^2 N_0^{-2}) \lesssim N_0 \exp(-N_1^2 N_0^{-2})$$

which is even smaller than the previous bound. For the $w^{(i)}$, we have by Proposition 4.3.3

$$\|w^{(i)}(t_0)\|_{L^\infty} \lesssim \epsilon_1 t_0^{-\frac{1}{2} + \frac{\alpha}{2}} = \epsilon_1 N_0^{1-\alpha} = \epsilon_1 A^{-\alpha\gamma} N_0.$$

It remains only to prove a lower bound on $v_0(t_0)$. Recall the notation of the proof of Proposition 4.3.1, in particular the decomposition $v_0 = v_0^p + v_0^e$. By (4.35),

$$\|v_0^e(t_0)\|_{L^\infty} \lesssim M_0 \leq 2N_0 A^{-\gamma+1}.$$

We prove the lower bound on $v_0^p(x, t) = N_0^3 |\eta_1|^2 \Psi_{1,0}(x, t)$, recalling from Definition 4.2.5 that $a_{j,0}$ takes a particularly simple form when $k = 0$. By inspection of Definition 4.2.1, we easily have $\|\Psi_{1,0}(t_0)\|_{L^\infty} \gtrsim N_0^{-2}$ from which it follows $\|v_0^p(t_0)\|_{L^\infty} \geq C_5^{-1} N_0$ for a large enough constant C_5 . Collecting everything,

$$\|v^{(1)}(t_0) - v^{(2)}(t_0)\|_{L^\infty} \geq C_5^{-1} N_0 - A^{-c(b,\alpha,\gamma)} N_0$$

for a $c(b, \alpha, \gamma) > 0$. With a sufficiently large choice of A , we conclude that $v^{(1)}$ and $v^{(2)}$ are distinct.

Finally, let us extend our solutions on $[0, 1]$ to be global solutions. Recall from (4.27) that $\|\nabla^m v^{(i)}|_{t=1}\|_{L^\infty} \leq \exp(-N_0^2/O_m(1))$ for all $m \geq 0$. It follows that

$$\|P_N v^{(i)}|_{t=1}\|_{L^\infty} \lesssim \langle N \rangle^{-5} \exp(-A^{2\gamma})$$

and therefore $v^{(i)}|_{t=1}$ can be made arbitrarily small in, say, $H^{\frac{1}{2}}(\mathbb{T}^3)$ or $B_{\infty,2}^{-1} \subset BMO^{-1}$, and existence of a global-in-time solution follows. $\square$

4.5 Tools from Convex Integration

The construction of the initial data U^0 in §4.2 makes use of several tools appearing in works on convex integration. In the following sections we quote those results for the reader's convenience.

First, we have the following rank-1 decomposition of a symmetric tensor field. This idea appeared first in the work of Nash [71], and subsequently in many of the modern works on fluid equations.

Lemma 4.5.1 (Nash lemma). *There exist $c_0 > 0$, $\theta_1, \dots, \theta_6 \in \mathbb{Z}^3$, and $\Gamma_1, \dots, \Gamma_6 \in C^\infty(B; \mathbb{R})$ such that*

$$M = \sum_{j=1}^6 \Gamma_j(M)^2 \theta_j \otimes \theta_j$$

for all $M \in B$, where $B = B(\text{Id}, c_0)$ is the closed ball of radius c_0 centered at the identity in $S^{3 \times 3}$. The Γ_j obey the estimates

$$\frac{1}{100} \leq \Gamma_j \leq 1 \quad \forall M \in B \tag{4.45}$$

and

$$\|\nabla^m \Gamma_j\|_{L^\infty(B)} \lesssim_m 1 \tag{4.46}$$

for all $m \geq 1$.

Proof. Let us define the vectors

$$\begin{aligned} \theta_1 &= (0, 0, 1), & \theta_2 &= (2, 0, 1), & \theta_3 &= (1, 1, 1), \\ \theta_4 &= (-1, 1, 1), & \theta_5 &= (-2, 0, 1), & \theta_6 &= (0, -2, 1). \end{aligned}$$

For any $M \in B$, let $\epsilon = M - \text{Id}$. One can verify that the claim holds if

$$\begin{aligned}\Gamma_1(M)^2 &= \frac{1}{3} - \frac{\epsilon_{11}}{4} - \frac{5\epsilon_{22}}{12} - \frac{\epsilon_{23}}{3} + \epsilon_{33} \\ \Gamma_2(M)^2 &= \frac{1}{12} + \frac{\epsilon_{11}}{8} - \frac{\epsilon_{12}}{4} + \frac{\epsilon_{13}}{4} - \frac{\epsilon_{22}}{24} - \frac{\epsilon_{23}}{12} \\ \Gamma_3(M)^2 &= \frac{1}{6} + \frac{\epsilon_{12}}{2} + \frac{\epsilon_{22}}{6} + \frac{\epsilon_{23}}{3} \\ \Gamma_4(M)^2 &= \frac{1}{6} - \frac{\epsilon_{12}}{2} + \frac{\epsilon_{22}}{6} + \frac{\epsilon_{23}}{3} \\ \Gamma_5(M)^2 &= \frac{1}{12} + \frac{\epsilon_{11}}{8} + \frac{\epsilon_{12}}{4} - \frac{\epsilon_{13}}{4} - \frac{\epsilon_{22}}{24} - \frac{\epsilon_{23}}{12} \\ \Gamma_6(M)^2 &= \frac{1}{6} + \frac{\epsilon_{22}}{6} - \frac{\epsilon_{23}}{6}.\end{aligned}$$

By taking $c_0 = \frac{1}{1000}$ (say), one can arrange that each of these expressions is bigger than $1/100$. Thus there is a smooth positive choice of $(\Gamma_j)_{j=1,\dots,6}$. The estimates (4.45)–(4.46) are elementary. $\square$

Next, we define the following anti-divergence operator which first appeared in [38].

Definition 4.5.2. We define the operator $\mathcal{R} : C^\infty(\mathbb{T}^3; \mathbb{R}^d) \rightarrow C^\infty(\mathbb{T}^3; S^{3 \times 3})$ by $(\mathcal{R}V)_{ij} := \mathcal{R}_{ijk}V_k$ where

$$\mathcal{R}_{ijk} = -\frac{1}{2}\Delta^{-2}\partial_{ijk}^3 - \frac{1}{2}\Delta^{-1}\delta_{ij}\partial_k + \Delta^{-1}\delta_{jk}\partial_i + \Delta^{-1}\delta_{ik}\partial_j$$

where δ denotes the Kronecker delta.

The utility of Definition 4.5.2 is that $\mathcal{R}$ is a right inverse for the divergence operator acting on symmetric tensor fields. For a proof, see for instance Lemma 4.3 in [38].

Lemma 4.5.3. *For any $V \in C^\infty(\mathbb{T}^3; \mathbb{R}^d)$, we have $\mathcal{R}V \in C^\infty(\mathbb{T}^3; S^{3 \times 3}(\mathbb{R}))$ as well as the identity*

$$\text{div } \mathcal{R}V = V - \oint_{\mathbb{T}^3} V.$$

Variants of the following stationary phase lemma can be found in the papers of Buckmaster–De Lellis–Isett–Székelyhidi [13] and De Lellis–Székelyhidi [38].

Lemma 4.5.4. *With $\mathcal{R}$ as in Definition 4.5.2,*

$$\|\mathcal{R}(a(x)e^{i\lambda k \cdot x})\|_{C^\beta} \lesssim_{\beta,m} \lambda^{-1+\beta} \|a\|_{L^\infty} + \lambda^{-m+\beta} \|\nabla^m a\|_{L^\infty} + \lambda^{-m} \|\nabla^m a\|_{C^\beta}.$$

Finally, we quote the following “improved Hölder’s inequality” that first appeared in work of Modena–Székelyhidi [70, Lemma 2.1], inspired by a similar lemma of Buckmaster–Vicol [14]. The idea, roughly speaking, is that the gains due to intermittency at separated scales is multiplicative.

Lemma 4.5.5. *For $f, g \in C^\infty(\mathbb{T}^3)$ and $\lambda \in \mathbb{N}$,*

$$\|f(\cdot)g(\lambda \cdot)\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1} + O(\lambda^{-1} \|f\|_{C^1} \|g\|_{L^1}).$$

4.6 Tools for Constructing the Semigroup

4.6.1 Existence for the Perturbed Stokes System

In the interest of the argument being self-contained, we construct solutions to the Navier–Stokes linearized around the principal part of the solutions. This will be necessary for defining the semigroup $S(t; t')$ in §4.3.2. One can find a much more precise result in the book of Lemarie-Rieusset [61, Theorem 21.3] where the drift is allowed to be in a very refined critical space.

Proposition 4.6.1. *Fix $t_0 \in \mathbb{R}$. Let $v \in C^\infty([t_0, \infty) \times \mathbb{T}^3; \mathbb{R}^3)$ and $w^0 \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ be divergence-free vector fields. Then there exists a solution $w \in C^\infty([t_0, \infty) \times \mathbb{T}^3; \mathbb{R}^3)$ of*

$$\begin{aligned} \partial_t w - \Delta w + v \cdot \nabla w + w \cdot \nabla v + \nabla p &= 0 \\ \operatorname{div} w &= 0 \\ w(t_0, x) &= w^0(x). \end{aligned} \tag{4.47}$$

Proof. It suffices to prove the proposition on the finite interval $[t_0, T]$ for all $T > 0$. We begin by proving the existence of a weak solution in $L_t^\infty L_x^2 \cap L_t^2 H_x^1([t_0, T] \times \mathbb{T}^3; \mathbb{R}^3)$ by the Galerkin method as in [92, Theorem 3.5]. Fix $\{\phi_j\}_{j=1}^\infty \subset C^\infty(\mathbb{T}^3; \mathbb{R}^3)$, a divergence-free orthonormal basis

of $H^1(\mathbb{T}^d; \mathbb{R}^d)$. Consider the finite rank approximation of w_0 in this basis:

$$w_m^0(x) = \sum_{j=1}^m \phi_j(x) (\phi_j, w^0)_{L^2}.$$

We search for a solution of (4.47) in this basis, namely a vector field of the form

$$w_m(x, t) = \sum_{j=1}^m g_j(t) \phi_j(x) \quad (4.48)$$

and desire to solve a system of ODE's for the coefficient functions $g_j(t)$ with initial data given by $g_j(t_0) = (\phi_j, w^0)$.

We would like to choose the coefficients in the ansatz (4.48) so that w_m exactly obeys (4.47). Of course this is not possible because the partial basis is not closed under the operations in the equation; instead, we solve (4.47) projected onto the subspace. Namely,

$$(\phi_i, \sum_{j=1}^m (g_j' \phi_j - g_j \Delta \phi_j + g_j v \cdot \nabla \phi_j + g_j \phi_j \cdot \nabla v))_{L^2} = 0$$

which leads to the ODE system

$$g_i'(t) + a_{ij}(t) g_j = 0$$

$$g_i|_{t=t_0} = (\phi_i, w^0)_{L^2}$$

for $i = 1, 2, \dots, m$, where

$$a_{ij}(t) = (\varphi_i, -\Delta \varphi_j + v(t) \cdot \nabla \varphi_j + \varphi_j \cdot \nabla v(t))_{L^2}.$$

Because $v \in L^1([t_0, \infty); C^1(\mathbb{T}^3))$, the norm of a_{ij} is integrable and standard ODE theory gives a unique global solution $g_1(t), \dots, g_m(t)$ to the system. Having constructed the coefficients, we can build the function w_m as defined in (4.48). By construction, w_m obeys (4.47) projected onto any

of the $\phi_1, \dots, \phi_m$. In particular we may test against w_m to obtain

$$\frac{1}{2} \int_{\mathbb{T}^d} |w_m(t)|^2 dx + \int_{t_0}^t \int_{\mathbb{T}^d} |\nabla w_m|^2 dx dt = \frac{1}{2} \int_{\mathbb{T}^d} |w_m^0|^2 dx + \int_{t_0}^t \int_{\mathbb{T}^d} (w_m \cdot \nabla w_m) \cdot v dx dt,$$

having used that v and w_m are divergence-free. We estimate the last term using the Peter-Paul inequality:

$$\begin{aligned} \int_{t_0}^t \int_{\mathbb{T}^d} (w_m \cdot \nabla w_m) \cdot v dx dt &\leq \int_{t_0}^t \|w_m\|_{L^2} \|\nabla w_m\|_{L^2} \|v\|_{L^\infty} dt \\ &\leq \int_{t_0}^t \left(\frac{1}{4} \|\nabla w_m\|_{L^2}^2 + 4 \|w_m\|_{L^2}^2 \|v\|_{L^\infty}^2 \right) dt. \end{aligned}$$

Inserting this back in to the energy equality, we conclude that

$$\frac{1}{2} \int_{\mathbb{T}^d} |w_m(t)|^2 dx + \frac{1}{4} \int_{t_0}^t \int_{\mathbb{T}^d} |\nabla w_m|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{T}^d} |w_m^0|^2 dx + \int_{t_0}^t 4 \|w_m\|_{L^2}^2 \|v\|_{L^\infty}^2 dt.$$

By Grönwall's inequality, we have

$$\|w_m\|_{L_t^\infty L_x^2([t_0, T] \times \mathbb{T}^d)}^2 + \frac{1}{2} \|\nabla w_m\|_{L_t^2 L_x^2([t_0, T] \times \mathbb{T}^d)}^2 \leq \|w^0\|_{L^2(\mathbb{T}^3)}^2 + 8 \exp\left(\|v\|_{L_t^2 L_x^\infty([t_0, T] \times \mathbb{T}^d)}^2\right).$$

Thus w_m is a priori bounded in the energy space, uniformly in m . As a result, there exists a limiting function $w_\infty(x, t)$ such that w_m converges weak-* to w_∞ in $L_t^\infty L_x^2$ and w_m converges weakly to w_∞ in $L_t^2 L_x^2$. Moreover w_∞ satisfies the same a priori bound as the w_m . To show that w_∞ is a weak solution of (4.47), it suffices to test against functions of the form $\zeta(x, t) := \theta(t)\phi_j(x)$. By construction, for all $m \geq j$ we have

$$\int_{t_0}^T \int_{\mathbb{T}^d} -w_m \cdot \partial_t \zeta - w_m \cdot \Delta \zeta - (v \odot w_m) : \nabla \zeta dx = 0.$$

By the weak convergence of w_m , we can pass to the limit and conclude w_∞ satisfies (4.47). From the energy bounds on $w = w_\infty$, it is easy to see that $\partial_t w_\infty \in L_t^2 H_x^{-1}$ exists in the weak sense, so we in fact have $w_\infty \in C([t_0, T], L_x^2)$ and convergence to the initial data w^0 .

Finally, one can upgrade this weak solution to a strong one. One approach is to consider w as a solution of the forced Stokes equation

$$\partial_t w - \Delta w + \nabla p = \operatorname{div} f$$

$$\operatorname{div} w = 0$$

$$w(t_0, x) = w^0(x)$$

where $f = 2v \odot w \in L_t^\infty L_x^2$. The weak solution can be expressed in terms of the Stokes semigroup $e^{t\Delta}\mathbb{P}$ as

$$w(t) = e^{t\Delta}w^0 + \int_0^t e^{(t-t')\Delta}\mathbb{P} \operatorname{div} f(t') dt'.$$

From this formula, one can iteratively bootstrap regularity using the smoothing effect of the heat kernel via (4.6). For instance,

$$\|w(t)\|_{H^{\frac{1}{2}}} \lesssim \|w^0\|_{H^{\frac{1}{2}}} + \int_0^t (t-t')^{-\frac{3}{4}} \|f(t')\|_{L^2} dt' < \infty,$$

and so on. □

4.6.2 Grönwall Inequalities

The following nonlinear Grönwall inequality will be used to prove the fractional Grönwall inequality (Lemma 4.6.2). It is similar to some that have appeared before (e.g. [97]) but we could not locate the version necessary for our purposes in the literature.

Lemma 4.6.1. *Let $f(t)$, $h_i(t)$, $b_i(t)$, and $a(t)$ be non-negative continuous functions on $[t_0, \infty)$, with $b_i(t)$ and $a(t)$ non-decreasing. Suppose for some $p > 1$,*

$$f(t) \leq a(t) + b_1(t) \int_{t_0}^t h_1(s) f(s) ds + (b_2(t) \int_{t_0}^t h_2(s) f(s)^p ds)^{\frac{1}{p}}.$$

Then for any constant C_6 large enough so that $C_6^{-1} + (C_6 p)^{-\frac{1}{p}} < 1$, we have

$$f(t) \leq C_6 a(t) \exp \left(C_6 (b_1(t) \int_{t_0}^t h_1(s) ds + b_2(t) \int_{t_0}^t h_2(s) ds) \right). \quad (4.49)$$

Proof. First, assume a and b_i are constant. In this case the b_i can be absorbed into the h_i . The proof is by a continuity argument. Consider the set $I \subset [t_0, \infty)$ defined by

$$I = \{T \in [t_0, \infty) : (4.49) \text{ for all } t \in [t_0, T]\}.$$

Then I is clearly closed (by continuity) and non-empty (because of the trivial point $t = t_0$). The lemma is proved once we show that I is open in $[t_0, \infty)$. Suppose we have $T \in I$. Then

$$\begin{aligned} f(T) &\leq a + \int_{t_0}^T h_1(s) f(s) ds + \left(\int_{t_0}^T h_2(s) f(s)^p ds \right)^{\frac{1}{p}} \\ &\leq a + C_6 a \int_{t_0}^T h_1(s) \exp \left(C_6 \int_{t_0}^s (h_1 + h_2) \right) ds \\ &\quad + C_6 a \left(\int_{t_0}^T h_2(s) \exp \left(p C_6 \int_{t_0}^s (h_1 + h_2) \right) ds \right)^{\frac{1}{p}} \\ &\leq a + a \int_{t_0}^T \frac{d}{ds} \exp \left(C_6 \int_{t_0}^s (h_1 + h_2) \right) ds \\ &\quad + (C_6 p)^{-\frac{1}{p}} C_6 a \left(\int_{t_0}^T \frac{d}{ds} \exp \left(p C_6 \int_{t_0}^s (h_1 + h_2) \right) ds \right)^{\frac{1}{p}} \\ &= a \exp \left(C_6 \int_{t_0}^T (h_1 + h_2) \right) + (C_6 p)^{-\frac{1}{p}} C_6 a \left(\exp \left(p C_6 \int_{t_0}^T (h_1 + h_2) ds \right) - 1 \right)^{\frac{1}{p}} \\ &\leq a(1 + (C_6 p)^{-\frac{1}{p}} C_6) \exp \left(C_6 \int_{t_0}^T (h_1 + h_2) \right). \end{aligned}$$

By assumption, the coefficient is strictly less than C_6 ; thus by continuity, there is an $\epsilon > 0$ such that $[T, T + \epsilon) \subset I$. We conclude that $I = [t_0, \infty)$.

Now we consider the case of non-decreasing a and b_i . Fix any $T > 0$. We have

$$f(t) \leq a(T) + b_1(T) \int_{t_0}^t h_1(s) f(s) ds + (b_2(T) \int_{t_0}^t h_2(s) f(s)^p ds)^{\frac{1}{p}}$$

for all $t \in [t_0, T]$. We may apply the “constant” version of the lemma proved above to conclude

$$f(t) \leq C_6 a(T) \exp \left(C_6 (b_1(T) \int_{t_0}^t h_1(s) ds + b_2(T) \int_{t_0}^t h_2(s) ds) \right) \quad \forall t \in [t_0, T].$$

In particular, this holds at $t = T$ which proves the general claim. $\square$

Grönwall-type inequalities similar to the one below have appeared in the literature and are known as “fractional Grönwall” inequalities (e.g., [96]). The one required in §4.3.2 has an endpoint character. One can see, for instance, that the extra factor $\|s^{\frac{1}{2}} g_2\|_{L^\infty}$ in the exponential reflects the failure of the Hardy–Littlewood–Sobolev inequality at the L^∞ endpoint.

Lemma 4.6.2. *If $a(t)$, $f(t)$, $g_1(t)$, and $g_2(t)$ are positive continuous functions on $[t_0, \infty)$ with $a(t)$ non-decreasing and*

$$f(t) \leq a(t) + \int_{t_0}^t (g_1(s) + (t-s)^{-\frac{1}{2}} g_2(s)) f(s) ds \quad \forall t \geq t_0 \quad (4.50)$$

then for any $p > 2$,

$$f(t) \lesssim_p a(t) \exp \left(O_p \left(\int_{t_0}^t g_1(s) ds + \|s^{\frac{1}{2}} g_2\|_{L^\infty([t_0, t])}^{p-2} \int_{t_0}^t g_2(s)^2 ds \right) \right) \quad \forall t \geq t_0.$$

Proof. By Hölder’s inequality,

$$\int_{t_0}^t (t-s)^{-\frac{1}{2}} g_2(s) f(s) ds \leq \left(\int_{t_0}^t (t-s)^{-\frac{p'}{2}} g_2(s)^{(1-\frac{2}{p})p'} ds \right)^{\frac{1}{p'}} \left(\int_{t_0}^t g_2(s)^2 f(s)^p ds \right)^{\frac{1}{p}}.$$

The first factor on the right-hand side is controlled by

$$\|s^{\frac{1}{2}} g_2\|_{L^\infty}^{1-\frac{2}{p}} \left(\int_{t_0}^t (t-s)^{-\alpha} s^{-\beta} ds \right)^{1-\frac{1}{p}}$$

where $\alpha = \frac{p}{2(p-1)}$ and $\beta = \frac{p-2}{2(p-1)}$. In general we have that when $\alpha \in (0, 1)$ and $\alpha + \beta = 1$ then

$$\int_{t_0}^t (t-s)^{-\alpha} s^{-\beta} ds \leq \frac{\pi}{\sin \pi \alpha} \quad \forall t \geq t_0$$

with equality when $t_0 = 0$. In summary,

$$f(t) \leq a(t) + \int_{t_0}^t g_1(s)f(s)ds + O_p(\|s^{\frac{1}{2}}g_2\|_{L^\infty([t_0,t])}^{1-\frac{2}{p}}(\int_{t_0}^t g_2(s)^2 f(s)^p ds)^{\frac{1}{p}})$$

so we conclude by Lemma 4.6.1, using that the coefficient function $t \mapsto \|s^{\frac{1}{2}}g_2\|_{L^\infty([t_0,t])}^{1-\frac{2}{p}}$ is non-decreasing. $\square$

4.7 Multiplier Estimates

Proving the basic properties of Littlewood–Paley calculus is slightly more subtle on the torus than in the whole space due to the lack of scaling symmetry, so we include the proofs in the interest of being self-contained. The idea is that one can move between $\mathbb{T}^3$ and $\mathbb{R}^3$ using the Poisson summation formula; see for instance the book of Schmeisser and Triebel [79].

4.7.1 A General Estimate

Let $a : \mathbb{Z}^3 \rightarrow \mathbb{C}$. We reserve the notation $\mathcal{F}$ and $\mathcal{F}^{-1}$ for the forward and inverse Fourier transforms on $\mathbb{T}^3$; see §4.1.1. We also recall the forward and inverse Fourier transforms on $\mathbb{R}^3$: for Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^3; \mathbb{C})$, we define

$$\mathcal{F}_{\mathbb{R}^3} \varphi(x) := \int_{\mathbb{R}^3} \varphi(\xi) e^{-ix \cdot \xi} d\xi, \quad \mathcal{F}_{\mathbb{R}^3}^{-1} \varphi(x) := (2\pi)^{-3} \int_{\mathbb{R}^3} \varphi(\xi) e^{ix \cdot \xi} d\xi.$$

Let us briefly recall a version of the Poisson summation formula. Let $f \in \mathcal{S}(\mathbb{R}^3; \mathbb{C})$ be Schwartz and define the periodization

$$f_p : \mathbb{T}^3 \rightarrow \mathbb{C}; \quad f_p(x) := \sum_{n \in (2\pi\mathbb{Z})^3} f(x + n).$$

Then it is straightforward to compute the following relation between the Fourier series of f_p and the Fourier transform of f :

$$\hat{f}_p(\xi) = (2\pi)^{-3} \mathcal{F}_{\mathbb{R}^3} f(\xi) \quad \forall \xi \in \mathbb{Z}^3. \quad (4.51)$$

By setting $\xi = 0$ in (4.51), one obtains the standard version of the Poisson summation formula. More useful for us is the following observation: let $f = \mathcal{F}_{\mathbb{R}^3}^{-1} \varphi$ with $\varphi \in \mathcal{S}(\mathbb{R}^3; \mathbb{C})$. Then (4.51) implies that $\hat{f}_p(\xi) = (2\pi)^{-3} \varphi(\xi)$ for all $\xi \in \mathbb{Z}^3$. Viewing φ as a function $\mathbb{Z}^3 \rightarrow \mathbb{C}$, it follows that $\mathcal{F}^{-1} \varphi = (2\pi)^3 f_p$. In other words,

$$\mathcal{F}^{-1} \varphi(x) = (2\pi)^3 \sum_{\xi \in \mathbb{Z}^3} \mathcal{F}_{\mathbb{R}^3}^{-1} \varphi(x + 2\pi \xi) \quad \forall x \in \mathbb{T}^3. \quad (4.52)$$

Having established (4.52), we can proceed to prove the multiplier bounds.

Lemma 4.7.1 (General multiplier boundedness). *If $m \in \mathcal{S}(\mathbb{R}^3; \mathbb{C})$, $1 \leq p \leq q \leq \infty$, and $n \geq 4$, then*

$$\|m(\nabla/i)f\|_{L^q(\mathbb{T}^3)} \lesssim_{p,q,n} C_n(m) \|f\|_{L^p(\mathbb{T}^3)}$$

for all $f \in L^p(\mathbb{T}^3; \mathbb{R})$ where

$$C_n(m) = \|\mathcal{F}_{\mathbb{R}^3}^{-1} m\|_{L^1(\mathbb{R}^3)}^{1+1/q-1/p} \max(\|m\|_{L^1(\mathbb{R}^3)}, \|m\|_{L^1(\mathbb{R}^3)}^{1-3/n} \|\nabla^n m\|_{L^1(\mathbb{R}^3)}^{3/n})^{1/p-1/q}.$$

Proof. By density, we may assume without loss of generality that $f \in C^\infty(\mathbb{T}^3; \mathbb{R})$.

The operator $m(\nabla/i)$ can be expressed as convolution on $\mathbb{T}^3$ with the kernel

$$K(x) = \mathcal{F}^{-1} m(x) = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} \mathcal{F}_{\mathbb{R}^3}^{-1} m(x + 2\pi k)$$

using (4.52). Recalling that we identify $\mathbb{T}^3$ with $[0, 2\pi]^3$,

$$\|K\|_{L^1(\mathbb{T}^3)} \leq (2\pi)^3 \sum_{k \in \mathbb{Z}^3} \|\mathcal{F}_{\mathbb{R}^3}^{-1} m(x + 2\pi k)\|_{L^1(\mathbb{T}^3)} = (2\pi)^3 \|\mathcal{F}_{\mathbb{R}^3}^{-1} m\|_{L^1(\mathbb{R}^3)}.$$

Moreover, by the Riemann–Lebesgue lemma,

$$\begin{aligned} |K(x)| &\leq (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |\mathcal{F}_{\mathbb{R}^3}^{-1} m(x + 2\pi k)| \\ &\lesssim \sum_{k \in \mathbb{Z}^3} \min(\|m\|_{L^1(\mathbb{R}^3)}, |x + 2\pi k|^{-n} \|\nabla^n m\|_{L^1(\mathbb{R}^3)}). \end{aligned}$$

If $\|\nabla^n m\|_{L^1(\mathbb{R}^3)} > \|m\|_{L^1(\mathbb{R}^3)}$, then one splits the sum based on which item in the minimum dominates to obtain $\|m\|_{L^1(\mathbb{R}^3)}^{1-3/n} \|\nabla^n m\|_{L^1(\mathbb{R}^3)}^{3/n}$. Otherwise, one can estimate at $|k| \leq 10$ using $\|m\|_{L^1}$ and at $|k| > 10$ with the other bound to obtain $\|m\|_{L^1} + \|\nabla^n m\|_{L^1} \lesssim \|m\|_{L^1}$.

The lemma follows by Young’s inequality and interpolation,

$$\|K\|_{L^r} \leq \|K\|_{L^1}^{1/r} \|K\|_{L^\infty}^{1-1/r}$$

where $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$. □

We are now equipped to prove the Lemmas 4.2–4.1.2.

4.7.2 Proof of Lemma 4.1.1 and Lemma 4.1.2

Recall from §4.1.2 the symbol $\pi(\xi/N)$ for the Fourier multiplier P_N . Then we have

$$\|\mathcal{F}_{\mathbb{R}^3}^{-1}(\pi(\cdot/N)m)\|_{L^1(\mathbb{R}^3)} = N^d \|(\mathcal{F}_{\mathbb{R}^3}^{-1}(m(N\cdot)\pi))(N\cdot)\|_{L^1(\mathbb{R}^3)} = \|\mathcal{F}_{\mathbb{R}^3}^{-1}(m(N\cdot)\pi)\|_{L^1(\mathbb{R}^3)}$$

by homogeneity. Recall that by definition, $\pi(\cdot/N)$ is supported in an annulus at spatial scale $\sim N$.

By the Riemann–Lebesgue lemma, Hölder, and (4.1), we have both the L^∞ bound

$$\|\mathcal{F}_{\mathbb{R}^3}^{-1}(m(N\cdot)\pi)\|_{L^\infty(\mathbb{R}^3)} \lesssim \|m(N\cdot)\pi\|_{L^1} \lesssim AN^\alpha$$

and the weighted L^∞ bound

$$\| |x|^{10} \mathcal{F}_{\mathbb{R}^3}^{-1}(m(N\cdot)\pi) \|_{L^\infty(\mathbb{R}^3)} \lesssim \| \nabla^{10}(m(N\cdot)\pi) \|_{L^1} \lesssim AN^\alpha.$$

One may easily combine these pointwise estimates to control the L^1 norm and obtain

$$\| \mathcal{F}_{\mathbb{R}^3}^{-1}(\pi(\cdot/N)m) \|_{L^1(\mathbb{R}^3)} \lesssim AN^\alpha.$$

Finally, one estimates

$$\| \nabla^n(\pi(\cdot/N)m) \|_{L^1(\mathbb{R}^3)} = AN^{\alpha-n} \| 1 \|_{L^1(\text{supp } \pi(\cdot/N))} = O(AN^{3+\alpha-n})$$

and the claim follows by Lemma 4.7.1. □

With m the multiplier for $P_N e^{t\Delta}$, it is straightforward to obtain the bound

$$| \mathcal{F}_{\mathbb{R}^3}^{-1} m(x) | \lesssim \min(1, \left(\frac{1+N^2 t}{|x|} \right)^{10}) \exp(-N^2 t/O(1))$$

which implies $\| \mathcal{F}_{\mathbb{R}^3}^{-1} m \|_{L^1(\mathbb{R}^3)} \lesssim \exp(-N^2 t/O(1))$. Then (4.3) follows by Lemma 4.7.1.

Let us turn to (4.4), first assuming that $t \leq N^{-2}$. Define the multiplier

$$m(\xi) = e^{-t|\xi|^2} \pi_N(\xi) b(\xi).$$

We easily have

$$| \nabla^n m(\xi) | \lesssim |\xi|^{\alpha} t^{\frac{n}{2}} (t^{\frac{1}{2}} |\xi|)^{-n} \mathbf{1}_{|\xi| \sim N}$$

for all $n \geq 0$ which leads to

$$\| \nabla^n m(\xi) \|_{L^1(\mathbb{R}^3)} \lesssim N^{3+\alpha-n}$$

and thus

$$\|\mathcal{F}_{\mathbb{R}^3}^{-1}m\|_{L^1(\mathbb{R}^3)} \lesssim N^\alpha$$

for this set of times. On the other hand, when $t > N^{-2}$,

$$|\nabla^n m(\xi)| \lesssim |\xi|^\alpha t^{\frac{n}{2}} \exp(-N^2 t/O(1)) \mathbf{1}_{|\xi| \sim N}.$$

Note that we can remove all factors of $(t^{\frac{1}{2}}|\xi|)$ by enlarging the $O(1)$ in the exponential, as needed.

It follows that

$$\|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^3)} \lesssim t^{\frac{3}{2}} N^{3+\alpha} \exp(-N^2 t/O(1)) \lesssim N^\alpha \exp(-N^2 t/O(1)).$$

Thus we conclude the desired bounds for all times by Lemma 4.7.1.

Next, if $\hat{f}(0) = 0$ and $t > 1$, we can decompose $f = \sum_{N \geq 1} P_N f$ and directly obtain (4.5) from (4.3). If instead $t \leq 1$, that approach would not lead to the optimal estimate because there is a $O(1)$ contribution for each of the modes between $N = 1$ and $N \sim t^{-\frac{1}{2}}$, leading to a logarithmic loss. Instead we decompose

$$\|e^{t\Delta}f\|_{L^\infty(\mathbb{T}^3)} \lesssim \|P_{1 \leq \cdot \leq t^{-\frac{1}{2}}} e^{t\Delta}f\|_{L^\infty} + \sum_{N > t^{-\frac{1}{2}}} \|P_N e^{t\Delta}f\|_{L^\infty}.$$

The sum over large N can be controlled by $O(1)$ using (4.3). Then one can see the operator $P_{1 \leq \cdot \leq t^{-1/2}} e^{t\Delta}$ is uniformly bounded on L^∞ by appealing to Lemma 4.7.1.

Finally, (4.6) follows from (4.3) in a similar (more straightforward) way. Because $\alpha > 0$, the low modes are summable even when t is small. $\square$

Chapter 5

Miscellany

The contents of this chapter are the result of discussions with Tim Binz and Stan Palasek.

This chapter presents miscellaneous results on self-similar solutions of the incompressible Euler equations that complement our work in [9] on the stability of the inviscid power-law vortex. Our work in the first section shows that any homogeneous self-similar solution of the Euler equations cannot have an unstable eigenvalue. Our work in the second section proves that bounded self-similar solutions of the two-dimensional Euler equations are necessarily equal to zero (and a bit more).

5.1 Linearizing around Homogeneous Solutions

Consider the incompressible Euler equations in $\mathbb{R}^n$ (where $n = 2$ or $n = 3$) in vorticity form:

$$\begin{aligned}\partial_t \omega + (u \cdot \nabla) \omega &= (\omega \cdot \nabla) u \\ K_{BS} * \omega(\cdot, t) &= u(\cdot, t) \\ \omega(\cdot, t = 0) &= \omega_{in}(\cdot)\end{aligned}\tag{5.1}$$

We have denoted the Biot-Savart kernel by K_{BS} . When $n = 2$ the vortex stretching term $\omega \cdot \nabla u$ disappears, but this will have no influence on what follows. Let α be a nonzero real parameter. We

say that $\Omega(\xi)$ is a self-similar profile or a self-similar solution to the Euler equations if and only if

$$\omega(x, t) := t^{-1} \Omega(xt^{-1/\alpha})$$

is a solution to the Euler equations. We consider self-similar profiles Ω that are stationary in the physical coordinates. In particular, if ω is defined as above, then $\partial_t \omega = 0$. We call these self-similar profiles **Homogeneous Solutions** to the Euler equations. Evidently, a self-similar profile is a homogeneous solution to the Euler equations if and only if

$$\Omega + \frac{x}{\alpha} \cdot \nabla \Omega = 0.$$

It is not difficult to see that any homogeneous solution to the Euler equations is of the form

$$\Omega(x) = |x|^{-\alpha} f\left(\frac{x}{|x|}\right)$$

for some function f . Expressing coordinates in $\mathbb{R}^n$ as (r, θ) where $\theta \in S^{n-1}$, we may write

$$\Omega(r, \theta) = r^{-\alpha} f(\theta).$$

We also make the remark that any “del” operation in these polar coordinates will always be of the form

$$\nabla \simeq \partial_r + \frac{1}{r} D_\theta,$$

where D_θ is a linear differential operator involving only derivatives with respect to components of θ and multiplication by functions of θ . We now introduce the following operator, invertible on practically any space we could consider:

$$S_s \Omega(x) = \Omega(s \cdot x).$$

The inverse is evidently given by $S_s^{-1} = S_{s^{-1}}$. By our remark on “del” operations ∇ , it is clear that

$$\nabla S_s U = s S_s \nabla U, \quad (5.2)$$

where ∇ may denote the gradient, divergence, directional derivative, etc. and U may denote a function, vector field, tensor, etc.

Now let us suppose that we have a homogeneous self-similar solution Ω . Evidently, Ω is a stationary solution of the Euler equations; therefore, we may linearize the Euler equations around Ω and obtain the operator

$$\begin{aligned} L_\phi \Omega &= -(\bar{U} \cdot \nabla) \Omega - (U \cdot \nabla) \bar{\Omega} + (\bar{\Omega} \cdot \nabla) U + (\Omega \cdot \nabla) \bar{U} \\ K_{BS} * \Omega(\cdot) &= U(\cdot) \\ K_{BS} * \bar{\Omega}(\cdot) &= \bar{U}(\cdot) \end{aligned} \quad (5.3)$$

Before continuing, we prove a short lemma about commuting S_s with the Biot-Savart operator.

Lemma 5.1.1. *The Biot Savart kernel K_{BS} has the property that*

$$K_{BS} * (S_s \Omega) = s^{-1} S_s (K_{BS} * \Omega).$$

Proof. In $\mathbb{R}^n$, the Biot-Savart kernel, or the kernel of the operator $\nabla^\perp \Delta^{-1}$, is given, up to constants, by

$$K_{BS}(y) = \frac{y^\perp}{|y|^n}.$$

Therefore, up to constants, we have

$$K_{BS} * (S_s \Omega)(x) = \int_{\mathbb{R}^n} \frac{(x-y)^\perp}{|x-y|^n} \Omega(sy) dy.$$

Performing the change of variables $sy \rightarrow y$, we get

$$K_{BS} * (S_s \Omega)(x) = \int_{\mathbb{R}^n} \frac{s^{-1}(sx-y)^\perp}{|sx-y|^n} \Omega(y) dy = s^{-1} S_s (K_{BS} * \Omega).$$

□

We call a Banach space $\mathfrak{B}$ **Scaling-Adapted** if and only if S_s is a bounded invertible operator on $\mathfrak{B}$. We are now fully equipped to prove the following theorem.

Theorem 5.1.2. *Let $\mathfrak{B}$ be any scaling-adapted Banach space on which L_ϕ is a densely defined operator generating a continuous semigroup. Then*

$$\sigma(L_\phi, \mathfrak{B}) \subset i\mathbb{R}.$$

In fact, we have:

$$\sigma(L_\phi, \mathfrak{B}) \in \{\emptyset, i\mathbb{R}, i(0, \infty), i[0, \infty), i(-\infty, 0), i(-\infty, 0], i\mathbb{R} \setminus \{0\}, \{0\}\}.$$

Proof. Since $\mathfrak{B}$ is a scaling-adapted Banach space, S_s is a bounded invertible operator on $\mathfrak{B}$ with bounded inverse. In particular, for any $s > 0$

$$\sigma(L_\phi, \mathfrak{B}) = \sigma(S_s^{-1}L_\phi S_s, \mathfrak{B}).$$

Note indeed, that conjugation of L_ϕ with the bounded operator S_s does not change the domain or spectral property of the operator. On the other hand, we may use Equation (5.2) and Lemma 5.1.1 to conclude

$$S_s^{-1}L_\phi S_s = s^\alpha L_\phi.$$

We conclude that for any $s > 0$:

$$\sigma(L_\phi, \mathfrak{B}) = \sigma(S_s^{-1}L_\phi S_s, \mathfrak{B}) = \sigma(s^\alpha L_\phi, \mathfrak{B}). \quad (5.4)$$

Now suppose there exists an element of the spectrum $\lambda \in \sigma(L_\phi, \mathfrak{B})$ with $\operatorname{Re}(\lambda) \neq 0$. By time irreversibility of the Euler equations, we may assume without loss of generality that $\operatorname{Re}(\lambda) > 0$. By Equation (5.4), we conclude that $s^\alpha \lambda \in \sigma(L_\phi, \mathfrak{B})$ for any $s > 0$. This would contradict that L_ϕ

generates a continuous semigroup on $\mathfrak{B}$. The second claimed conclusion is obvious. $\square$

Interestingly, a similar result holds for stationary self-similar solutions of the Navier-Stokes equations. It turns out that self-similar stationary solutions of the Navier-Stokes equations have (-1) -homogeneous velocity, (-2) -homogeneous vorticity, and are in fact Landau solutions, see [92] and [86]. The linearization of the Navier-Stokes equations around such a (-2) -homogeneous vorticity $\bar{\Omega}$ is precisely

$$L_{NS} := L_\phi + \Delta\Omega.$$

It turns out that our simple rescaling argument can (more or less) rule out unstable eigenvalues of Landau solutions:

Theorem 5.1.3. *Let $\mathfrak{B}$ be any scaling-adapted Banach space on which L_{NS} is a densely defined operator generating a continuous semigroup. Then*

$$\sigma(L_{NS}, \mathfrak{B}) \subset \{z : \operatorname{Re}(z) \leq 0\}.$$

Proof. Since $\mathfrak{B}$ is a scaling-adapted Banach space, S_s is a bounded invertible operator on $\mathfrak{B}$ with bounded inverse. In particular, for any $s > 0$

$$\sigma(L_{NS}, \mathfrak{B}) = \sigma(S_s^{-1}L_{NS}S_s, \mathfrak{B}).$$

Note indeed, that conjugation of L_{NS} with the bounded operator S_s does not change the domain or spectral property of the operator. On the other hand, we may use Equation (5.2) and Lemma 5.1.1 to conclude

$$S_s^{-1}L_\phi S_s = s^2L_\phi$$

and

$$S_s^{-1}\Delta S_s = s^2\Delta.$$

We conclude that for any $s > 0$:

$$\sigma(L_{NS}, \mathfrak{B}) = \sigma(S_s^{-1} L_{NS} S_s, \mathfrak{B}) = \sigma(s^2 L_{NS} \mathfrak{B}). \quad (5.5)$$

Now suppose there exists an element of the spectrum $\lambda \in \sigma(L_{NS}, \mathfrak{B})$ with $\operatorname{Re}(\lambda) > 0$. By Equation (5.5), we conclude that $s^2 \lambda \in \sigma(L_{NS}, \mathfrak{B})$ for any $s > 0$. This would contradict that L_{NS} generates a continuous semigroup on $\mathfrak{B}$. $\square$

We encourage the reader to compare our result above with linear asymptotic stability theorems such as in [65] and the references therein.

5.2 Irregularity of Self-Similar Solutions

We now focus on self-similar profiles of the Euler equations in dimension two. It is not difficult to see that a self-similar profile $\Omega(\xi)$ satisfies

$$\begin{aligned} -\left(1 + \frac{\xi}{\alpha} \cdot \nabla\right) \Omega + (U \cdot \nabla) \Omega &= 0 \\ K_{BS} * \Omega(\cdot, \tau) &= U(\cdot, \tau). \end{aligned} \quad (5.6)$$

The first thing we do is prove the following interesting integral identity:

Proposition 5.2.1. *Let $0 < \alpha \leq 1$. Let Ω be a self similar profile with velocity U . Suppose $\Omega \in C^1(\mathbb{R}^2 \setminus \{0\})$. Then there exists a constant K such that on any circle $\partial B_R(0)$ we have*

$$\int_0^{2\pi} |\Omega(R, \theta)|^{2/\alpha} \left(1 - \frac{\alpha}{R} U_r(R, \theta)\right) d\theta = \frac{K}{R^2},$$

where U_r denotes the radial component of U .

Proof. We consider polar coordinates (r, θ) . Let $R_1 < R_2$ be arbitrary. Then Equation (5.6) is satisfied pointwise in the annulus $B_{R_2}(0) \setminus B_{R_1}(0)$ and we may test Equation (5.6) with $\Omega|\Omega|^{\frac{2}{\alpha}-2}$

and integrate to get

$$\iint_{B_{R_2}(0) \setminus B_{R_1}(0)} \left((U - \frac{\xi}{\alpha}) \cdot \nabla \Omega \right) \Omega |\Omega|^{-2+2/\alpha} d\xi = \iint_{B_{R_2}(0) \setminus B_{R_1}(0)} |\Omega|^{2/\alpha} d\xi.$$

We can rewrite the equality above as

$$\frac{\alpha}{2} \iint_{B_{R_2}(0) \setminus B_{R_1}(0)} (U - \frac{\xi}{\alpha}) \cdot \nabla (|\Omega|^{2/\alpha}) d\xi = \iint_{B_{R_2}(0) \setminus B_{R_1}(0)} |\Omega|^{2/\alpha} d\xi.$$

Integrating by parts yields

$$\iint_{B_{R_2}(0) \setminus B_{R_1}(0)} \operatorname{div} \left((U - \frac{\xi}{\alpha}) |\Omega|^{2/\alpha} \right) d\xi = 0.$$

Now the divergence theorem yields:

$$R_2^2 \int_0^{2\pi} |\Omega(R_2, \theta)|^{2/\alpha} (1 - \frac{\alpha}{R_2} U_r(R_2, \theta)) d\theta = R_1^2 \int_0^{2\pi} |\Omega(R_1, \theta)|^{2/\alpha} (1 - \frac{\alpha}{R_1} U_r(R_1, \theta)) d\theta.$$

Since $R_1 < R_2$ were arbitrary, the conclusion of the proposition is finished. □

Equipped with Proposition 5.2.1, we can prove:

Theorem 5.2.2. *Let $0 < \alpha \leq 1$. Let $\Omega \in C^1(\mathbb{R}^2 \setminus \{0\})$ be a self-similar profile. Assume that $\Omega \in L_{loc}^q(\mathbb{R}^2)$ for some $q \geq 1 + \frac{2}{\alpha}$ and that $\Omega \sim |\xi|^{-\alpha}$ as $\xi \rightarrow \infty$. Assume that the velocity $U := (K_{BS} * \Omega)(0) = 0$. With these assumptions, we have $\Omega \equiv 0$ in $\mathbb{R}^2$.*

Proof. Without loss of generality, we may assume the weakest hypothesis, namely that $\Omega \in L_{loc}^{1+\frac{2}{\alpha}}(\mathbb{R}^2)$. Since $U = K_{BS} * \Omega$, we have that $U \in W_{loc}^{1,1+2/\alpha}(\mathbb{R}^2)$ and by the Sobolev embedding, $U \in C^{\frac{2-\alpha}{2+\alpha}}(\mathbb{R}^2)$. By our assumption that $U(0) = 0$, we conclude there exists a constant C such that $\frac{|U(\xi)|}{|\xi|} \leq C|\xi|^{\frac{-2\alpha}{2+\alpha}}$ in $B_1(0)$. Now from the integral identity in Proposition 5.2.1, we get

$$\frac{K}{R} \leq C_\alpha R^{\frac{2-\alpha}{2+\alpha}} \int_0^{2\pi} |\Omega(R, \theta)|^{2/\alpha} d\theta \leq C'_\alpha R^{\frac{-\alpha}{2+\alpha}} \left(\int_0^{2\pi} |\Omega(R, \theta)|^{\frac{2}{\alpha}+1} R d\theta \right)^{\frac{2}{2+\alpha}},$$

where C_α and C'_α are constants depending only on α . Rearranging we have

$$KR^{\frac{-2}{2+\alpha}} \leq C''_\alpha \left(\int_0^{2\pi} |\Omega(R, \theta)|^{\frac{2}{\alpha}+1} R d\theta \right)^{\frac{2}{2+\alpha}}$$

or

$$KR^{-1} \leq C'''_\alpha \int_0^{2\pi} |\Omega(R, \theta)|^{\frac{2}{\alpha}+1} R d\theta.$$

The right-hand-side above is integrable in R on the interval $(0, 1)$, but the left-hand-side is evidently not. We necessarily conclude that the constant $K = 0$.

In this case, the integral identity from Proposition 5.2.1 gets us for all R :

$$\int_0^{2\pi} |\Omega(R, \theta)|^{2/\alpha} R d\theta = \alpha \int_0^{2\pi} |\Omega(R, \theta)|^{2/\alpha} U_r(R, \theta) d\theta. \quad (5.7)$$

However, assuming $\Omega \sim |\xi|^{-\alpha}$ as $\xi \rightarrow \infty$, Equation (5.7) yields

$$R^{-1} \sim \int_0^{2\pi} |\Omega(R, \theta)|^{2/\alpha} R d\theta = \alpha \int_0^{2\pi} |\Omega(R, \theta)|^{2/\alpha} U_r(R, \theta) d\theta \sim R^{-1-\alpha}$$

as $R \rightarrow \infty$, which is a contradiction. We conclude that $\Omega \equiv 0$ for R large. Likewise, for R small, we use the bound on $|U|$ to get from Equation (5.7):

$$\int |\Omega(R, \theta)|^{2/\alpha} R d\theta \leq R^{\frac{2-\alpha}{2+\alpha}} \alpha \int_0^{2\pi} |\Omega(R, \theta)|^{2/\alpha} d\theta.$$

From this inequality, we see that $\Omega \equiv 0$ when R is small. In particular, $\Omega \in C^1(\mathbb{R}^2)$ and is compactly supported. Now we may test Equation (5.6) by Ω itself and integrate by parts to get

$$\left(\frac{2}{\alpha} - 1\right) \int_{\mathbb{R}^2} |\Omega|^2 dx = 0,$$

which implies that $\Omega \equiv 0$ everywhere in $\mathbb{R}^2$. □

We remark that the assumption that we are in two-dimensions is essential for the proof above, since, otherwise, the vortex stretching term presents a major obstacle to a similar result.

Finally, we present some arguments explaining why the results above are certainly not optimal. In two dimensions, the vorticity is transported along the flowlines of the solution, in particular every L^p and $L^{p,\infty}$ norm should be conserved by the evolution. The idea is that a self-similar profile to the Euler equations is more or less a “re-arrangement of vorticity” of the inviscid power-law vortex $\omega = |\xi|^{-\alpha}$.

Suppose we have a self-similar profile $\Omega(\xi) \in L^{q,\infty}$. Given the self-similar profile $\Omega(\xi)$, the corresponding self-similar solution in-time is

$$\omega(x, t) = t^{-1}\Omega(xt^{-1/\alpha}),$$

and vice versa.

$$\|\Omega\|_{L^{q,\infty}}^q = \left(\sup_{s>0} s^q m(\{\xi : |\Omega(\xi)| > s\}) \right) = C,$$

where m is the usual Lebesgue measure on $\mathbb{R}^2$. We also have that the following norm *should* be conserved in time t :

$$\|\omega(\cdot, t)\|_{L^{q,\infty}}^q = \left(\sup_{s>0} s^q m(\{x : |\omega(x, t)| > s\}) \right) = \left(\sup_{s>0} s^q m(\{x : |t^{-1}\Omega(xt^{-1/\alpha})| > s\}) \right).$$

Let $S_s = \{\xi : |\Omega(\xi)| > s\}$. Thus, we know

$$\left(\sup_{s>0} s^q m(\{x : |t^{-1}\Omega(xt^{-1/\alpha})| > s\}) \right) = \left(\sup_{s>0} s^q m(t^{1/\alpha} S_{t \cdot s}) \right) = t^{2/\alpha - q} \left(\sup_{s>0} (t \cdot s)^q m(S_{t \cdot s}) \right) = C t^{\frac{2}{\alpha} - q}.$$

If $q \neq 2/\alpha$, then as $t \rightarrow 0$, this quantity goes to zero (or ∞), so it is not conserved. We conclude that necessarily $q = 2/\alpha$. We observe that the self-similar profile $\Omega(\xi) = |\xi|^{-\alpha}$ lies in $L^{2/\alpha,\infty}$, so this is of course not an empty class.

Also, suppose we have a self-similar profile $\Omega(\xi) \in L^q$ (or $L^1 \cap L^q$ to make sure the Biot-Savart law behaves nicely). The corresponding self-similar solution is $\omega(x, t) := t^{-1}\Omega(xt^{-1/\alpha})$ with homogeneous initial data that does not lie in any Lebesgue space. Nevertheless, at $t = 1$, $\omega(x, 1) = \Omega(x) \in L^q$. By standard existence theory, there exists a solution $\omega_1(x, t)$ solving backwards (and

forwards) in time from $t = 1$ that always lies in L^q . Now, it is straightforward to observe that

$$\|\omega_1\|_{L^q} = t^{\frac{2}{\alpha q}-1} \|\Omega\|_{L^q},$$

so it *should* be that $q = 2/\alpha$. Indeed, if uniqueness for this class of initial data held, then at $t = 0$, ω_1 would cease to be in L^q , a contradiction. Thus, either no self-similar profile $\Omega \in L^q$ with $q \neq 2/\alpha$ exists or the hypothesis that solutions from integrable vorticity data are unique fails. Our conclusion is that, given the scaling parameter α , the only natural class in which to search for self-similar solutions should be $L^{2/\alpha}$ or $L^{2/\alpha,\infty}$ (the latter class is known to contain solutions such as the homogeneous self-similar solutions).

However, we argue in [9] that any weighted L^2 space X is inadequate for the proof of non-uniqueness, following Vishik in [93] and [94], if the background vortex lies in the class $L^{2/\alpha,\infty}$. The inability to perform the linear analysis in a Hilbert space presents great difficulties for Vishik's approach towards the "Yudovich Problem" of uniqueness for integrable vorticity solutions of the incompressible Euler equations.

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