Derrida–Retaux model: from discrete to continuous time

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Discrete-time Derrida–Retaux model
Definition

- Re-introduced by Derrida–Retaux (2014) for studying the depinning transition.

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**Definition:** Start with a nonnegative random variable $X_0$ and, for any $n \geq 0$,

$$X_{n+1} = (X_n + \tilde{X}_n - 1)_+$$

where $\tilde{X}_n$ is an independent copy of $X_n$. 
Definition on a tree

Construction of $X_n$ on a binary tree:

If $X_0 \in \mathbb{N} := \{0, 1, 2, \ldots\}$, it can be seen as a parking procedure on the tree.
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(a + b - 1)_+ \quad \text{i.i.d. copies of } X_0
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- $X_n$ is constructed by i.i.d. copies of $X_0$.
- The process involves a series of decisions at each level of the tree, with choices depending on the values of $a$ and $b$.
- At each node, the process chooses a value from $\{0, 1, 2, \ldots\}$, which is then added to the cumulative sum of the parent nodes.

The diagram illustrates the path taken through the tree, with nodes representing the choices made at each step.
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Phase transition

Free energy: \( F_\infty \) := \( \lim_{n \to \infty} \frac{\mathbb{E}[X_n]}{2^n} \in [0, \infty] \).
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- \( F_\infty > 0: \) supercritical phase.
- \( F_\infty = 0: \) subcritical phase.

Assume that \( X_0 \in \mathbb{N} \) a.s. and that \( P(X_0 = 1) < 1. \)

- (supercritical) If \( \mathbb{E}[X_0^2X_0] > \mathbb{E}[2X_0^2] \) or \( \mathbb{E}[2X_0^2] = \infty, \) then \( F_\infty > 0 \) and \( X_n^2 n \) a.s. \( \to \) as \( n \to \infty. \)

- (subcritical) If \( \mathbb{E}[X_0^2X_0] \leq \mathbb{E}[2X_0^2] < \infty, \) then \( F_\infty = 0 \) and \( X_n \) probability \( \to 0 \) as \( n \to \infty. \)

Open question: Try to say something about the case where \( X_0 \) is not integer-valued.
Free energy: \( F_\infty := \lim_{n \to \infty} \frac{\mathbb{E}[X_n]}{2^n} \in [0, \infty]. \)

- \( F_\infty > 0 \): supercritical phase.
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Theorem (Collet–Eckmann–Glaser–Martin 1984): Assume that \( X_0 \in \mathbb{N} \) a.s. and that \( \mathbb{P}(X_0 = 1) < 1. \)
Phase transition

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  \[ F_\infty > 0 \quad \text{and} \quad \frac{X_n}{2^n} \xrightarrow{n \to \infty} F_\infty. \]
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**Open question:** Try to say something about the case where $X_0$ is not integer-valued.
Let ν be a probability measure on $(0, \infty)$, in the supercritical phase.
Let $\nu$ be a probability measure on $(0, \infty)$, in the supercritical phase. Consider $X_0 \overset{(d)}{=} (1 - p)\delta_0 + p\nu$ for each $p \in [0, 1]$. 
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Consider $X_0 \overset{(\text{d})}{=} (1 - p) \delta_0 + p \nu$ for each $p \in [0, 1]$.
Let $F_\infty(p)$ denote the free energy and $p_c := \inf\{p \in [0, 1] : F_\infty(p) > 0\}$. 

![Graph showing $F_\infty(p)$ with $\nu = \delta_2$, $p_c = \frac{1}{5}$]
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Let $F_\infty(p)$ denote the free energy and $p_c := \inf\{p \in [0, 1] : F_\infty(p) > 0\}$.

If $X_0 \in \mathbb{N}$ a.s., then $p_c$ is explicit by CEGM 1984.
Conjecture (Derrida–Retaux 2014):
If $p_c > 0$, then as $p \downarrow p_c$

$$F_\infty(p) = \exp\left( -\frac{K + o(1)}{(p - p_c)^{1/2}} \right).$$
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Theorem (Chen–Dagard–Derrida–Hu–Lifshits–Shi 2019+): If $\nu$ is supported by $\mathbb{N}^*$ and $\int_0^\infty x^3 2^x \nu(dx) < \infty$, then as $p \downarrow p_c$

$$F_\infty(p) = \exp \left( - \frac{1}{(p - p_c)^{1/2 + o(1)}} \right).$$
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▷ CDDFLS deal also with the case where $p_c > 0$ and $\int_0^\infty x^3 2^x \nu(dx) = \infty$. 

$\nu = \delta_2$  
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Theorem (Chen–Dagard–Derrida–Hu–Lifshits–Shi 2019+): If $\nu$ is supported by $\mathbb{N}^*$ and $\int_0^\infty x^{3/2} x^2 \nu(dx) < \infty$, then as $p \downarrow p_c$

$$F_\infty(p) = \exp\left(-\frac{1}{(p - p_c)^{1/2} + o(1)}\right).$$

▷ CDDFLS deal also with the case where $p_c > 0$ and $\int_0^\infty x^{3/2} x^2 \nu(dx) = \infty$.
▷ Hu–Shi 2018: case $p_c = 0$. 

$\nu = \delta_2$

$p_c = \frac{1}{5}$
Critical case for \( X_0 \in \mathbb{N} \): \( E[X_0 2^{X_0}] = E[2^{X_0}] < \infty \).
Behavior at criticality

- Critical case for $X_0 \in \mathbb{N}$: $\mathbb{E}[X_0 2^{X_0}] = \mathbb{E}[2^{X_0}] < \infty$.
- Recall that $X_n \to 0$ in probability.
Behavior at criticality

▷ **Critical case** for $X_0 \in \mathbb{N}$: $\mathbb{E}[X_0 2^{X_0}] = \mathbb{E}[2^{X_0}] < \infty$.

▷ Recall that $X_n \to 0$ in probability.

▷ **Theorem (Chen–Derrida–Hu–Lifshits–Shi 2017):** If $\mathbb{E}[X_0^3 2^{X_0}] < \infty$, then

$$\frac{c_1}{n} \leq \mathbb{E}[2^{X_n}] - 1 \leq \frac{c_2}{n}.$$ 

*In particular, $\mathbb{P}(X_n > 0) \leq \frac{c_2}{n}$.*
Critical case for $X_0 \in \mathbb{N}$: $\mathbb{E}[X_0 2^{X_0}] = \mathbb{E}[2^{X_0}] < \infty$.

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*In particular,* $\mathbb{P}(X_n > 0) \leq \frac{c_2}{n}$.

**Conjecture (Chen–Derrida–Hu–Lifshits–Shi 2017):** If $\mathbb{E}[X_0^3 2^{X_0}] < \infty$, then

$$\mathbb{P}(X_n > 0) \sim \frac{4}{n^2}.$$  

Moreover, given $X_n > 0$, $X_n$ converges in law to a geometric distribution with parameter $\frac{1}{2}$.  


Given that $X_n > 0$, we color in red the paths from a leaf to the root, where the operation “positive part” was not needed.
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The red vertices form a subtree, called the red tree.
Questions concerning the red tree

Question: Given $X_n > 0$, what does the red tree look like for large $n$?

$n = 200$

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Questions concerning the red tree

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▷ Scaling limit?

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Question: Given $X_n > 0$, what does the red tree look like for large $n$?

- Scaling limit?
- Number of red leaves?

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Continuous-time Derrida–Retaux model
Initial condition: a nonnegative random variable $X_0$. For $t > 0$, $X_t$ is defined using a painting procedure:
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For $t > 0$, $X_t$ is defined using a **painting procedure**:

1. Consider a **Yule tree** of height $t$ (binary tree with i.i.d. exponentially distributed lifetimes).
**Definition**

**Initial condition**: a nonnegative random variable $X_0$.

For $t > 0$, $X_t$ is defined using a **painting procedure**:

- Consider a **Yule tree** of height $t$ (binary tree with i.i.d. exponentially distributed lifetimes).
- **Initially**: painters start on the leaves with i.i.d. amount of paint chosen according to the law of $X_0$. 
  - Then, painters climb down the tree, painting the branches with a quantity 1 of paint per unit of branch length.
  - When two painters meet, they put their remaining paint in common.
  - $X_t$ is the remaining paint at the root.
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General properties

▷ Free energy: \( F_\infty := \lim_{t \to \infty} e^{-t} \mathbb{E}[X_t] \).
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Theorem: If \( F_\infty > 0 \), then \( e^{-t}X_t \xrightarrow{\text{law}} \text{Exp}(F_\infty^{-1}) \).
General properties

- Free energy: $F_\infty := \lim_{t \to \infty} e^{-t} \mathbb{E}[X_t]$.

- Theorem: If $F_\infty > 0$, then $e^{-t}X_t \xrightarrow{\text{law}} \text{Exp}(F_\infty^{-1})$.

- Open question: If $F_\infty = 0$, then prove that $X_t \xrightarrow{\text{probability}} 0$. 
General properties

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▷ **Open question:** If $F_\infty = 0$, then prove that $X_t \xrightarrow{\text{probability}} 0$.

▷ **Proposition:** Let $\mu_t$ denote the distribution of $X_t$ for each $t \geq 0$. Then, $(\mu_t)_{t \geq 0}$ is the unique family of positive measures on $\mathbb{R}$ solution (in the weak sense) of the PDE

$$
\partial_t \mu_t = \partial_x (1_{\{x>0\}} \mu_t) + \mu_t * \mu_t - \mu_t,
$$

with initial condition $\mu_0$. 
An exactly solvable family of solutions

\[ \partial_t \mu_t = \partial_x \left( \mathbb{1}_{x>0} \mu_t \right) + \mu_t * \mu_t - \mu_t \]

▷ From now on, consider \( \mu_0 = p_0 \delta_0(dx) + (1 - p_0) \lambda_0 e^{-\lambda_0 x} \ dx. \)
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▷ **Proposition:** For any \( t \geq 0 \), \( \mu_t = p(t) \delta_0(dx) + (1 - p(t)) \lambda(t) e^{-\lambda(t) x} dx \), where \( p: \mathbb{R}_+ \rightarrow [0, 1] \) and \( \lambda: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) are the unique solutions of the ODE

\[
\begin{align*}
p' &= (1 - p)(\lambda - p) \\
\lambda' &= -\lambda(1 - p)
\end{align*}
\]

with \( \begin{align*} p(0) &= p_0 \\
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  with \( \begin{cases} p(0) = p_0 \\ \lambda(0) = \lambda_0. \end{cases} \)
- \( H := \frac{p(t)}{\lambda(t)} + \log \lambda(t) \) is an invariant of the dynamics.
The phase transition

\[ X_t^{(d)} = \mu_t = p(t)\delta_0(dx) + (1 - p(t))\lambda(t)e^{-\lambda(t)x} \, dx \]

We have \( p(t) = H\lambda(t) - \lambda(t)\log\lambda(t) \) with \( H = \frac{p_0}{\lambda_0} + \log\lambda_0 \).
The phase transition

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Infinite order transition for the free energy with exponent \( 1/2 \).

Precise asymptotic behavior of \( p(t) \) and \( \lambda(t) \) in each phase.

F_{\infty} = 0 and \( X_t \to 0 \)
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One can make explicit computations:
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One can make explicit computations:

- Infinite order transition for the free energy with exponent \( \frac{1}{2} \).
- Precise asymptotic behavior of \( p(t) \) and \( \lambda(t) \) in each phase.
Theorem: With a critical initial condition ($\lambda_0 > 1$ and $p_0 = \lambda_0 - \lambda_0 \log \lambda_0$),

$$ \mathbb{P}(X_t > 0) = 1 - p(t) = \frac{2}{t^2} + \frac{16 \log t}{3t^3} + o\left(\frac{\log t}{t^3}\right).$$

Moreover, given $X_t > 0$, $X_t$ converges in law to $\text{Exp}(1)$.
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Moreover, given $X_t > 0$, $X_t$ converges in law to Exp(1).

Our goal: Given $X_t > 0$, what does the subtree bringing paint to the root look like?
Given that $X(t) = x$, the red tree of height $t$ is a time-inhomogeneous branching Markov process defined on $[0, t]$ such that:

- It starts at time 0 with a single particle with mass $x$.
- The mass of each particle grows linearly at speed 1.
- A particle of mass $m$ at time $s$ splits at rate $p(t - s)(1 - \lambda(t - s))m$ into two children, the mass $m$ being split uniformly.
- Particles behave independently after their splitting time.
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The scaling limit of the red tree

Let \((x_t)_{t \geq 0}\) be positive numbers such that \(\frac{x_t}{t} \to x \geq 0\).
The scaling limit of the red tree

Let \((x_t)_{t \geq 0}\) be positive numbers such that \(\frac{x_t}{t} \to x \geq 0\).

**Theorem:** Given that \(X_t = x_t\), the red tree of height \(t\), with time and masses rescaled by \(t\), converges locally in distribution to a time-inhomogeneous branching Markov process defined on \([0, 1)\) such that:

- It starts at time 0 with a single particle with mass \(x\).
- The mass of each particle grows linearly at speed 1.
- A particle of mass \(m\) at time \(s\) splits at rate \(\frac{2m}{(1 - s)^2}\) into two children, the mass \(m\) being split uniformly.
- Particles behave independently after their splitting time.

Simulations: the limit should be the same for the discrete-time model.

Wide open question: universality among other hierarchical renormalization models?
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**Theorem:** There exist $\gamma_1, \gamma_2 > 0$ such that, for any positive numbers $(x_t)_{t \geq 0}$ such that $x_t/t \to x \geq 0$, we have

$$\left( \frac{N_t}{t^2}, \frac{M_t}{t^2} \right) \text{ given } X_t = x_t \xrightarrow{\text{d}}_{t \to \infty} (\gamma_1 \eta_x, \gamma_2 \eta_x),$$

with $\eta_x := \int_0^1 r^2(s) \, ds$ and $r$ a 4-dimensional Bessel bridge from 0 to $2\sqrt{x}$. 

Idea of proof: The Laplace transform of $(N_t, M_t)$ given $X_t = x$ is solution of the following PDE, as a function of $t$ and $x$:

$$\partial_t \phi = \partial_x \phi + p(t)(1 - \lambda(t))(\phi^* \phi - x \phi).$$

It takes the particular form

$$\phi(t, x) = e^{-\left(\theta_1(t) + x \theta_2(t)\right)},$$

with $\theta_1' = \theta_2$ and $\theta_2' = p(1 - \lambda)\left(1 - e^{-\theta_1}\right)$. 

Last open question: What is the law of the mass of a typical red leaf?
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Thanks for your attention!