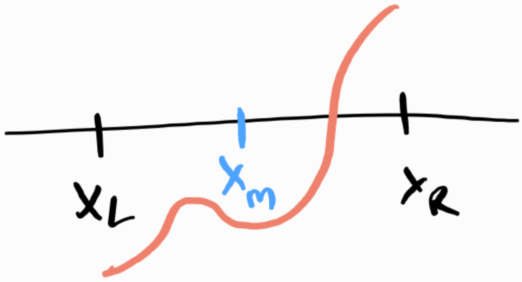
Finding equilibria

e.g. for $\frac{dx}{dt} = f(x)$, we want to find x^* s.t.

$$f(x^*) = 0$$ ← root-finding problem

Bisection method



Algorithm:

① Find x_L, x_R s.t. $f(x_L)\,f(x_R) < 0$

② $x_m = (x_L + x_R)/2$

↳ (3.1) if $f(x_L)\,f(x_M) < 0$

$\Rightarrow x_R = x_M$

↳ (3.2) if $f(x_M)\,f(x_R) < 0$

$\Rightarrow x_L = x_M$

Things to consider:

→ tolerance / accuracy

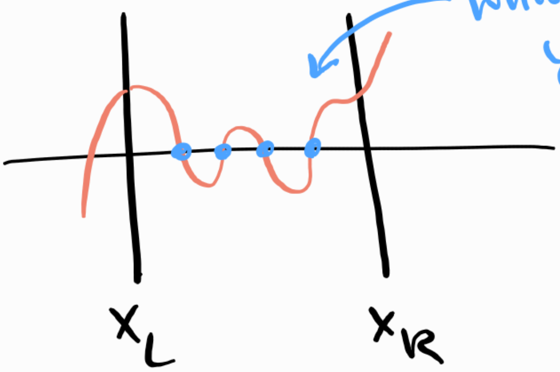
→ precomputing and saving computations

sometimes $f(x)$ is expensive!

Possible problems:

→ missing roots

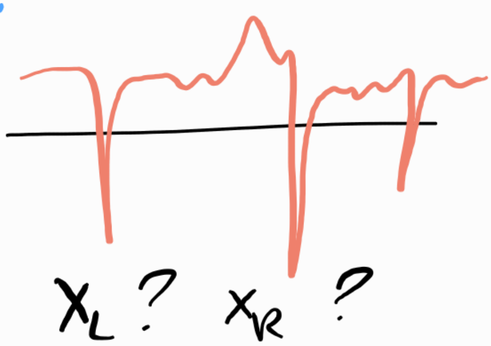
→ narrow regions of opposite sign



which will you find?

how many?

this is expensive!



Examples

① $f(x) = x^2 + x + 0.25\cos(4\pi x) - 2$

② $f(x) = x^2 + x + 0.25\cos(4\pi x) - 1$

③ $f(x) = 3.2\sin(\exp(-x)) - 0.5\cos(x) - 1$

④ $f(x) = 3.2\sin(\exp(-x+3)) - 0.5\cos(x-3) - 1$

⑤ $f(x) = 3.2\sin(\exp(-x+5)) - 0.5\cos(x-5) - 1$

Behavior of error:

$\Rightarrow$ At each step,

$$\text{error} \sim |x_M - x^*| < |x_R - x_L|$$

$\Rightarrow$ suppose initially you have $[x_L^0, x_R^0]$

$\hookrightarrow$ $x_M^0 = \frac{x_L^0 + x_R^0}{2}$, $\varepsilon_0 = |x_M^0 - x^*| < |x_R^0 - x_L^0|$

$\hookrightarrow$ Suppose $f(x_M) f(x_L^0) < 0$. Then

$$x_L^1 = x_L^0 \quad \& \quad x_R^1 = x_M^0$$

$\Rightarrow$ At next step,

$$x_M^1 = \frac{x_L^1 + x_R^1}{2}$$

$$|x_M^1 - x^*| < |x_R^1 - x_L^1| = |x_M^0 - x_L^0|$$

$$= \left|\frac{x_L^0 + x_R^0}{2} - x_L^0\right| = \left|\frac{x_R^0 - x_L^0}{2}\right|$$

$$\Longrightarrow |x_M^1 - x^*| < \frac{1}{2} |x_R^0 - x_L^0|$$

$\boxed{Q}$ What about $|x_M^2 - x^*|$?

$$|x_M^2 - x^*| < \frac{1}{4} |x_R^0 - x_L^0|$$

$\boxed{Q}$ What might the general formula be?

$$\varepsilon_N = |x_M^N - x^*| < \frac{1}{2^N} |x_R^0 - x_L^0|$$

Fixed point iteration

We want $\bar{x}^*$ s.t. $\bar{f}(\bar{x}^*) = \bar{0}$

$\Rightarrow$ equivalently, $\bar{g}(\bar{x}^*) = \bar{x}^*$

$\bar{g}(\bar{x}) = \bar{f}(\bar{x}) + \bar{x}$

Algorithm

① Consider $\bar{x}_0$

② Let $\bar{x}_{n+1} = g(\bar{x}_n)$ → hope this converges!

(until $|\bar{x}_{n+1} - \bar{x}_n| <$ tolerance)

Things to consider:

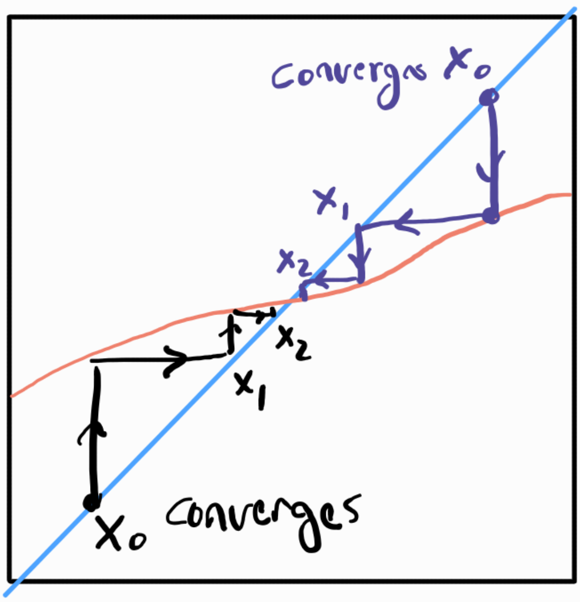
→ tolerance / accuracy

Possible problems:

→ function is nonsmooth

Example (1D)

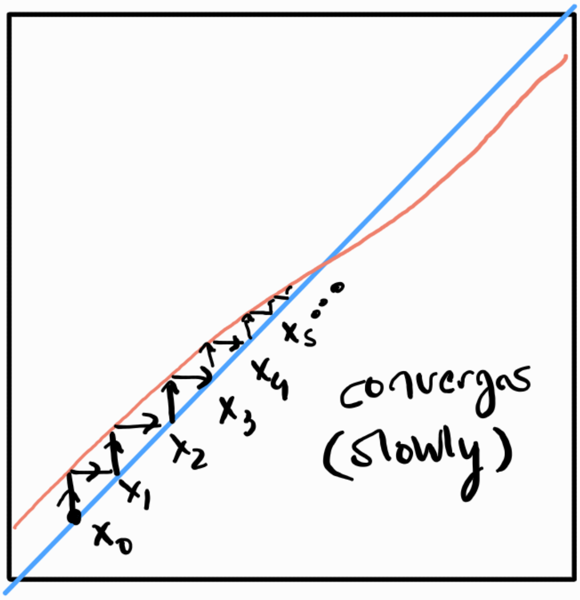
$y(x) = x \leftarrow y'(x) = 1$



$g(x) \leftarrow |g'(x)| < 1$

Example

$y = x$

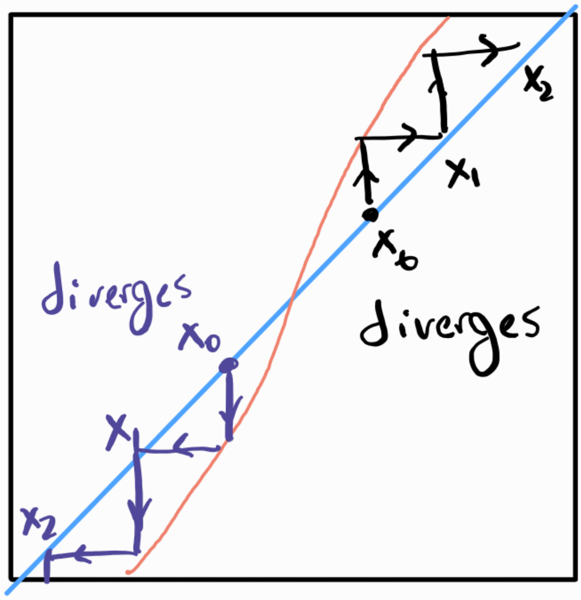


$g(x) \leftarrow |g'(x)| < 1$, but close to 1

Slow convergence to x^*

Example

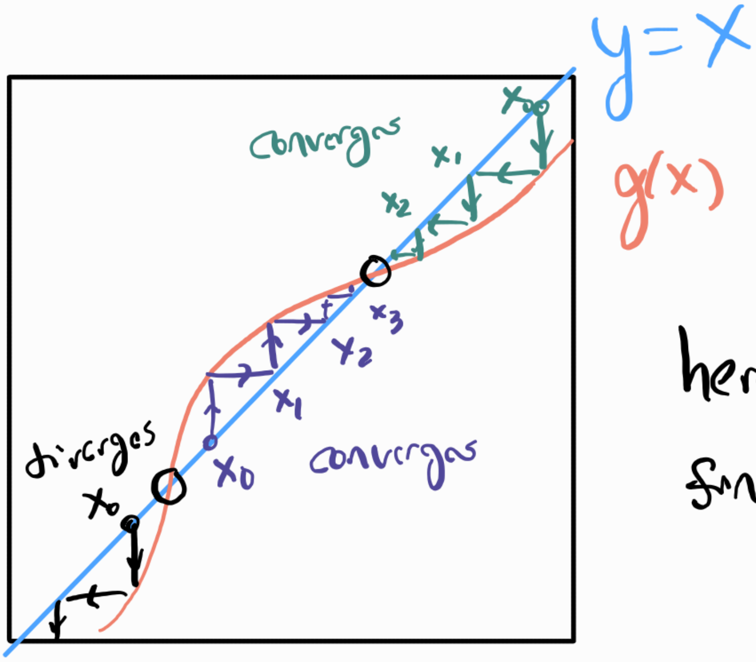
$y = x$



$g(x) \leftarrow |g'(x)| > 1$

No convergence to x^*

Example



here you can only find one of two roots

assuming $g \in C([a,b])$ & $g' \in C((a,b))$

<u>convergence and error</u>

suppose $\exists$ $a \le x^* \le b$ s.t. $g(x^*) = x^*$.
Consider the interval $[a,b]$. Mean-value thm tells us that $\exists$ c s.t.

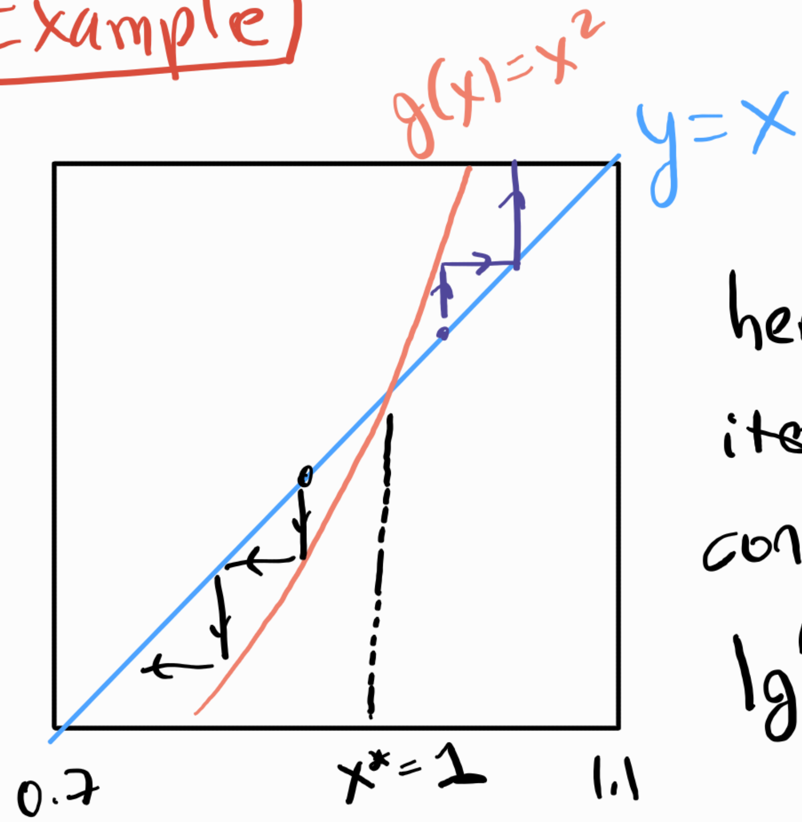
$$g(a) - g(b) = g'(c) \cdot (a-b)$$

Consider x_n interation step. Suppose $|g'(x)| \le \lambda$.

$$\Rightarrow |x_n - x^*| = |g(x_{n-1}) - g(x^*)|$$
$$= |g'(c)|\,|x_{n-1} - x^*| \le \lambda\,|x_{n-1} - x^*|$$
$$= \lambda\,|g(x_{n-2}) - g(x^*) \le \lambda^2\,|x_{n-2} - x^*|$$
$$\cdots \le \lambda^n |x_0 - x^*|$$

If $\lambda < 1$, then $\lim_{n\to\infty} |x_n - x^*| = 0 \Rightarrow$ convergence

Example

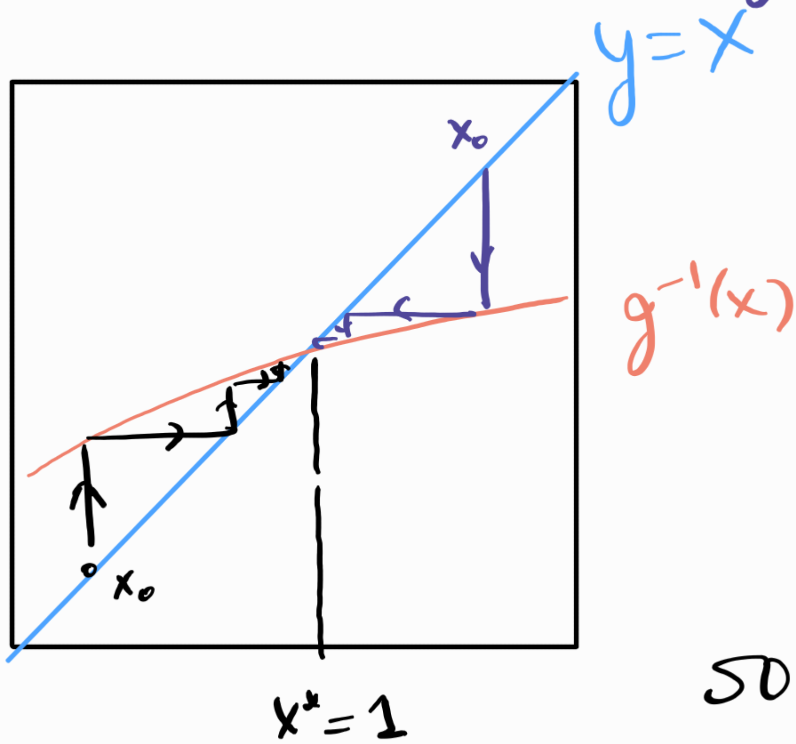


here, fixed point iteration will not converge, because

$$|g'(x)| = |2x| > 1$$

$\rightsquigarrow$ What about the inverse $g^{-1}(x)$?

$$y = x^2 \Longrightarrow x = \sqrt{y} \Longrightarrow g^{-1}(x) = \sqrt{x}$$



We found x^* s.t.

$$g^{-1}(x^*) = x^*$$

$$\Downarrow$$

$$x^* = g(x^*)$$

so we found a fixed point of $g(x)$! $\square$

Why does this work? Recall that for $g(x)=y$:

$$[g^{-1}(y)]' = \frac{1}{g'(x)} = \frac{1}{g'(g^{-1}(y))}$$

So if $|g'(x)| > 1$

$$\Rightarrow \frac{1}{|g'(x)|} < 1$$

Finding x^* s.t. $f(x)=h(x)$

Notice: $f(x) = h(x)$

$x = f^{-1}(h(x))$ assuming f is invertible

$\Rightarrow$ Define $g(x) = f^{-1}(h(x))$

$\Rightarrow$ Finding x^* s.t. $g(x^*) = x^*$ is equivalent to $f(x^*) = h(x^*)$

What about for vector valued functions?

$$\bar{g}(\bar{x}) = \bar{x}$$

$\Rightarrow$ this time, $\bar{x}^{(k+1)} = \bar{g}(\bar{x}^{(k)})$, $k = 0, 1, \ldots$

Convergence condition becomes that

$$\|\underline{\underline{J}}(\bar{x})\| < 1$$

matrix norm, e.g. $\|\underline{\underline{A}}\|_2 = \sqrt{\rho(A^* A)}$

$\rho(A) = \max|\lambda_i|$

Jacobian matrix for $\bar{g}$: $J_{ij} = \dfrac{\partial g_i}{\partial x_j}$

$\boxed{\text{Example}}$

Let $\bar{x} = \langle x_1, x_2 \rangle$, $\bar{g}(\bar{x}) = \langle x_2^2, \sqrt{1 - x_1^2} \rangle$

$\rightarrow$ this has f.p. $\bar{x} = \left\langle \frac{\sqrt{5}}{2} - \frac{1}{2}, \sqrt{\frac{\sqrt{5}}{2} - \frac{1}{2}} \right\rangle$

Will fixed point iteration work on $[0,1]^2$?

$$\Rightarrow \underline{\underline{J}}(\bar{x}) = \begin{bmatrix} 0 & 2x_2 \\ \dfrac{-2x_1}{\sqrt{1 - x_1^2}} & 0 \end{bmatrix}$$

will prevent $\|J\| < 1$

Quiz

Consider $f(x) = \tan(x) - x$, and look at the interval $[4.0, 4.7]$

(a) How many iterations of bisection method do you theoretically need to approximate a root in this interval with absolute error < 0.005?

(b) Implement bisection method for this problem with the number of steps you found in part (a). What is your approximate x^*? Show all steps.

(c) If the true $x^* \approx 4.4934095791$, at which step do you observe an absolute error < 0.005?