Absolute vs relative error

y: computed

y^*: true

absolute error: $= |y - y^*|$

relative error: $= \dfrac{|y - y^*|}{|y^*|}$

Example

$y = 15.934 \times 10^5$

$y^* = 15.136 \times 10^5$

2 digits

$|y - y^*| \sim 0.798 \times 10^5 \leftarrow$ "large"

approx

$\dfrac{|y-y^*|}{|y^*|} \sim 0.05272 = 5.272 \times 10^{-2}$

Example

$y = 0.1934$, $y^* = 0.19321$

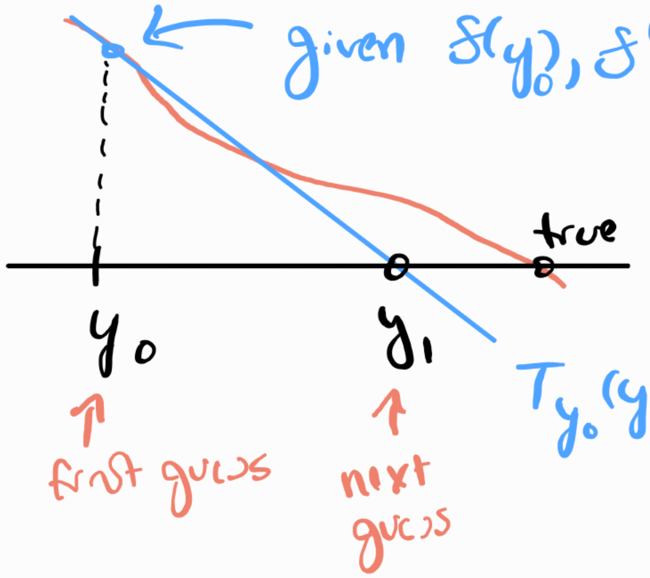
3 digits

$\dfrac{|y-y^*|}{|y^*|} \sim 9.834 \times 10^{-4}$

approx

Want $\bar{f}(\bar{y}) = \bar{0}$ $\rightsquigarrow$ in $\mathbb{R}$, $f(y) = 0$

Newton's method



given $f(y_0), f'(y_0)$

$$T_{y_0}(y) = f(y_0) + f'(y_0)\cdot(y - y_0)$$

$$\Downarrow$$

$$y_1 \text{ s.t. } T_{y_0}(y_1) = 0 = f(y_0) + f'(y_0)\cdot(y_1 - y_0)$$

$$\Rightarrow \quad y_1 = y_0 - \frac{f(y_0)}{f'(y_0)}$$

In general, given y_n step, define y_{n+1} s.t.

$$f'(y_n)\cdot(y_{n+1} - y_n) = -f(y_n)$$

equivalent

now: $\mathbb{R}^n$, $\bar{f}(\bar{y})$, $\bar{y} \in \mathbb{R}$

$$(*) \quad \underbrace{D\bar{f}(\bar{y}_n)}_{\text{Jacobian}}\cdot[\bar{y}_{n+1} - \bar{y}_n] = -\bar{f}(\bar{y}_n)$$

∇

$\Rightarrow$ this is a linear system we need to solve

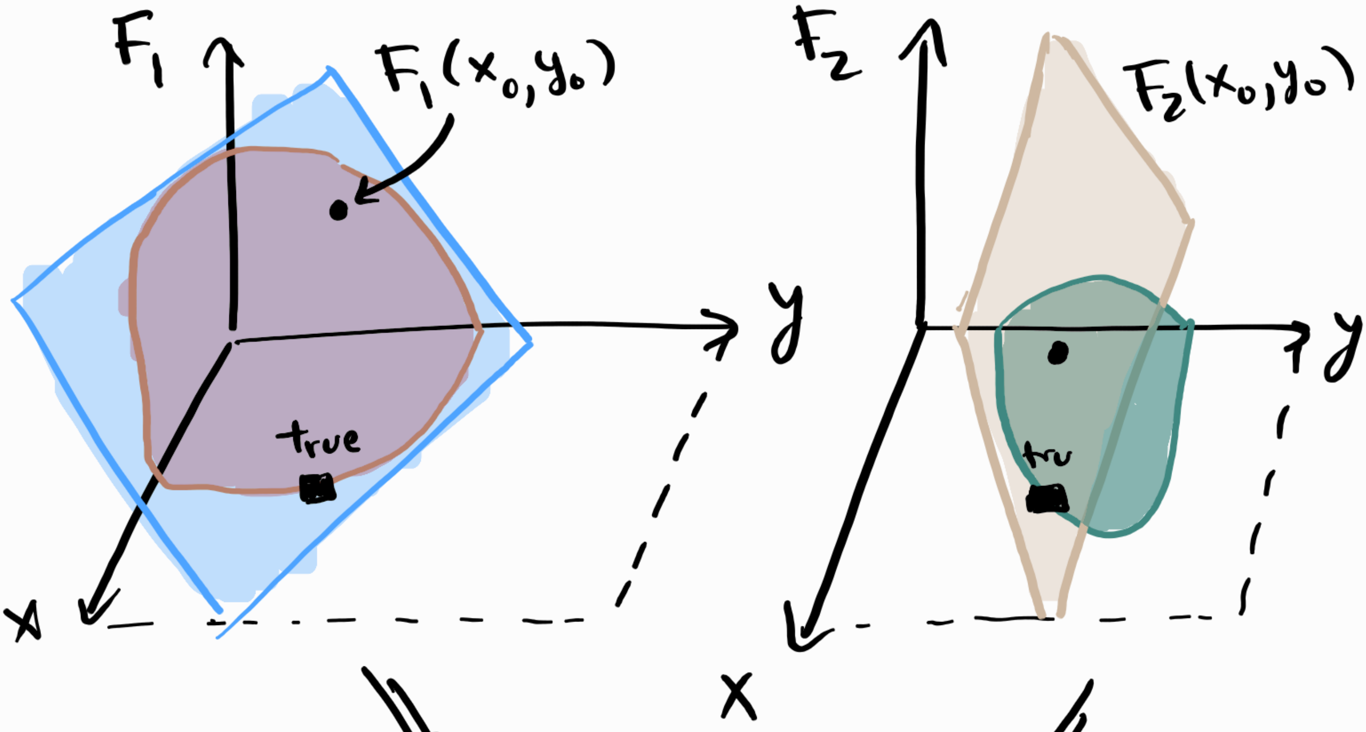
Consider $\bar{r} \in \mathbb{R}^2$, $\bar{f}(\bar{r}) \in \mathbb{R}^2$ ⇒ let's see how to get (*). Let $\bar{r}_i = \langle x_i, y_i \rangle$

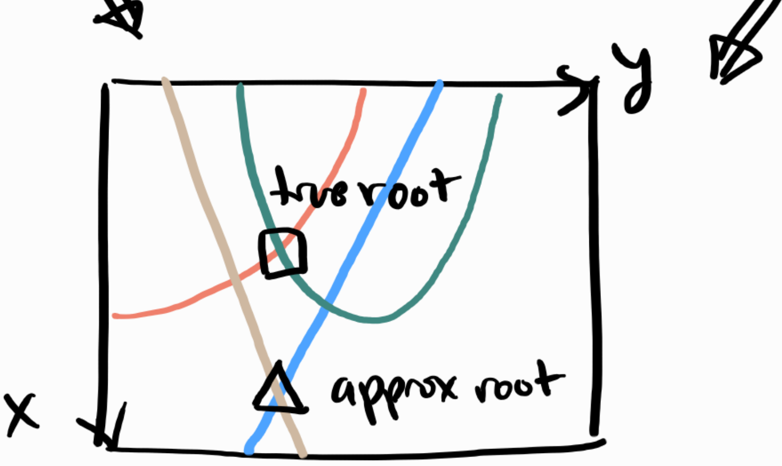
$$\bar{f}(\bar{r}) = \langle \underbrace{F_1(\bar{r})}, \underbrace{F_2(\bar{r})} \rangle$$

⇒ need to find tangent planes of these scalar functions

⇒ find when those tangent planes intersect with the $\bar{r}$ plane. (coastlines' approx)

⇒ then find their intersection to approx root





Tangent plane to $F_1(\bar{r}_0)$:

$$T_{1,\bar{r}_0}(\bar{r}) = F_1(\bar{r}_0) + \partial_x F_1(\bar{r}_0)(x_1 - x_0) + \partial_y F_1(\bar{r}_0)(y_1 - y_0)$$

$$= F_1(\bar{r}_0) + \underbrace{[\partial_x F_1, \partial_y F_1]\Big|_{\bar{r}_0}}_{\nabla F_1(\bar{r}_0)} \cdot \underbrace{[x_1 - x_0, y_1 - y_0]}_{\bar{r}_1 - \bar{r}_0}$$

$$T_{1,\bar{r}_0}(\bar{r}) = F_1(\bar{r}_0) + \nabla F_1(\bar{r}_0) \cdot (\bar{r}_1 - \bar{r}_0)$$

Tangent plane to $F_2(\bar{r}_0)$:

$$T_{2,\bar{r}_0}(\bar{r}) = F_2(\bar{r}_0) + \nabla F_2(\bar{r}_0) \cdot (\bar{r}_1 - \bar{r}_0)$$

need both to $=0$, i.e. intersection with x,y plane

$$\begin{bmatrix} T_{1,\bar{r}_0}(\bar{r}) \\ T_{2,\bar{r}_0}(\bar{r}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1(\bar{r}_0) \\ F_2(\bar{r}_0) \end{bmatrix} + \begin{bmatrix} \nabla F_1(\bar{r}_0) \cdot (\bar{r}_1 - \bar{r}_0) \\ \nabla F_2(\bar{r}_0) \cdot (\bar{r}_1 - \bar{r}_0) \end{bmatrix}$$

$$= \begin{bmatrix} F_1(\bar{r}_0) \\ F_2(\bar{r}_0) \end{bmatrix} + \overbrace{\begin{bmatrix} \partial_x F_1 & \partial_y F_1 \\ \partial_x F_2 & \partial_y F_2 \end{bmatrix}}^{\text{Jacobian}}\Bigg|_{\bar{r}_0} (\bar{r}_1 - \bar{r}_0)$$

$$\Rightarrow \quad \bar{0} = \bar{f}(\bar{r}_0) + D\bar{f}(\bar{r}_0)\cdot(\bar{r}_1 - \bar{r}_0)$$

So we need to solve linear system:

$$\underbrace{D\bar{f}(\bar{r}_0)}_{\text{matrix}} \cdot \underbrace{(\bar{r}_1 - \bar{r}_0)}_{\text{vector}} = - \underbrace{\bar{f}(\bar{r}_0)}_{\text{vector}}$$

we cannot do this (uniquely) if $D\bar{f}(\bar{r}_0)$ is singular, i.e. $\det(D\bar{f}(\bar{r}_0)) = 0$

$\rightarrow$ close to singular also bad

1D examples (MATLAB)

$$f(x) = x^2 + x + \frac{1}{4}\cos(4\pi x) - 2$$

$$f'(x) = 2x + 1 - \pi \sin(4\pi x)$$

$$f(x) = x^2 + x + \frac{1}{4}\cos(4\pi x) - 1$$

$$f'(x) = 2x + 1 - \pi \sin(4\pi x)$$

$$f(x) = 3.2\sin(\exp(-x)) - 0.5\cos(x-3) - 1$$

$$f'(x) = \frac{1}{2}\left(\sin(x-3) - 128.547\, e^{-x}\cos(e^{3-x})\right)$$

2D example (MATLAB)

$$\bar{f}(x,y) = \begin{bmatrix} -x^3 + y \\ x^2 + y^2 - 1 \end{bmatrix}$$

$$\Downarrow$$

$$D\bar{f}(\bar{x}) = \begin{bmatrix} -3x^2 & 1 \\ 2x & 2y \end{bmatrix}$$

Quiz

Consider $f(x) = \tan(x) - x$, and look at the interval $[4.0, 4.7]$. The true root is approximately $x^* \approx 4.4934095791$.

(a) Implement Newton's method for:

$$x_0 = 2.0, 4.4, \text{ and } 4.7$$

For each, determine how many steps it takes until the relative error is with x^* is $< 10^{-5}$.

(b) Explain the result for 2.0

$$\frac{|x_{computed} - x^*|}{|x^*|}$$

$x_0 = 4.4 \rightarrow$ 4 steps

$x_0 = 4.7 \rightarrow$ 8 steps

$x_0 = 2.0 \rightarrow$ never, because $\tan x - x$ has slope ~ 0 at $x \sim 3.3$