Autonomous ODEs in 1D

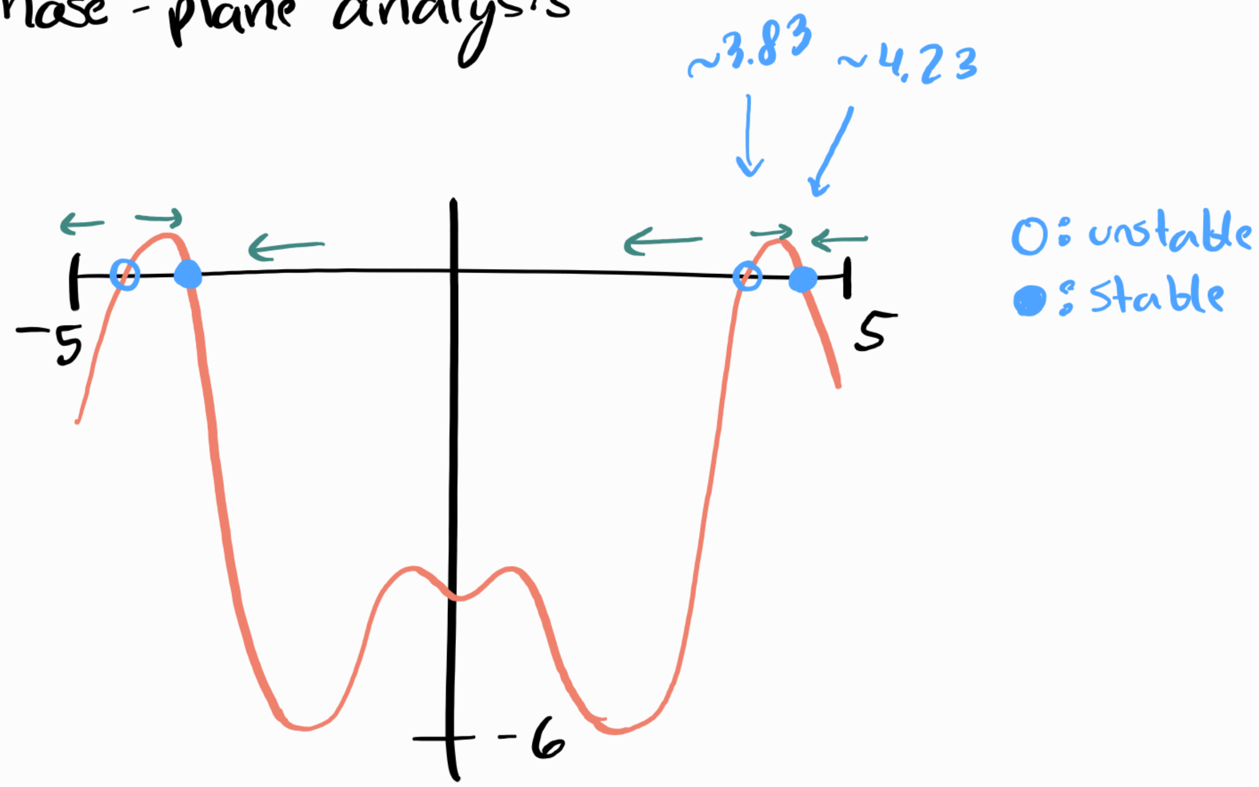
General form: $\frac{dy}{dt} = f(y)$

Example:

$$\frac{dy}{dt} = y \cdot \sin(2y) + \cos(y) - 3$$

$\boxed{Q}$ What are the fixed points in $(-5,5)$?
Are they stable?

$\Rightarrow$ phase-plane analysis

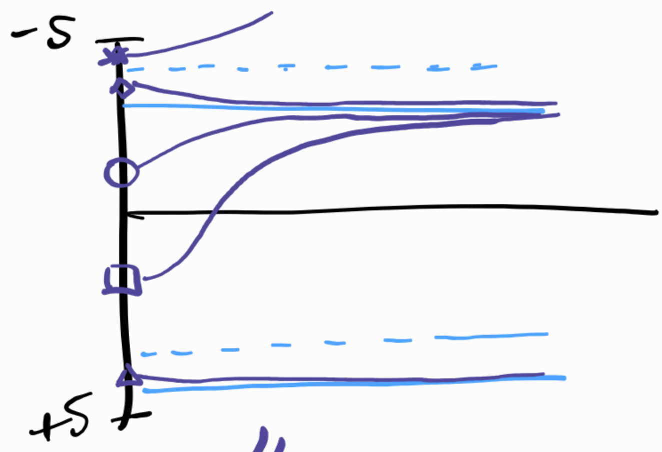


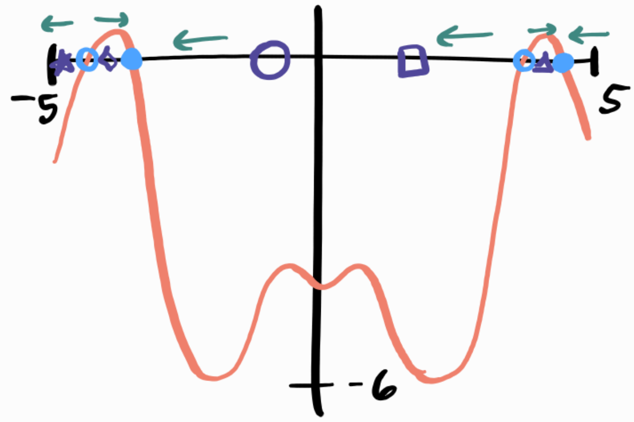
think about y: position of ball on a hill

$\frac{dy}{dt}$: velocity of ball

$\longrightarrow$ vs $\longleftarrow$

⇒ Sketch of solution:



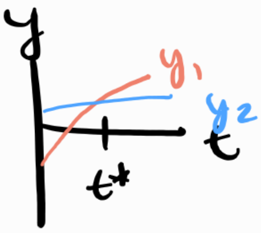


⇓

Trajectories should not cross! Why?

⇒ We assume the IVP has a unique soln

⇒ If you have solns $y_1(t)$ and $y_2(t)$ s.t.



$$y_1(t^*) = y_2(t^*) = y^*$$

notice ⟹
$$\left.\frac{dy_1}{dt}\right|_{t^*} = f(y^*) = \left.\frac{dy_2}{dt}\right|_{t^*}$$

↳ soln is identical after t^*

⇒ soln also identical before t^* b/c to have uniqueness, we need continuity of $f(y)$

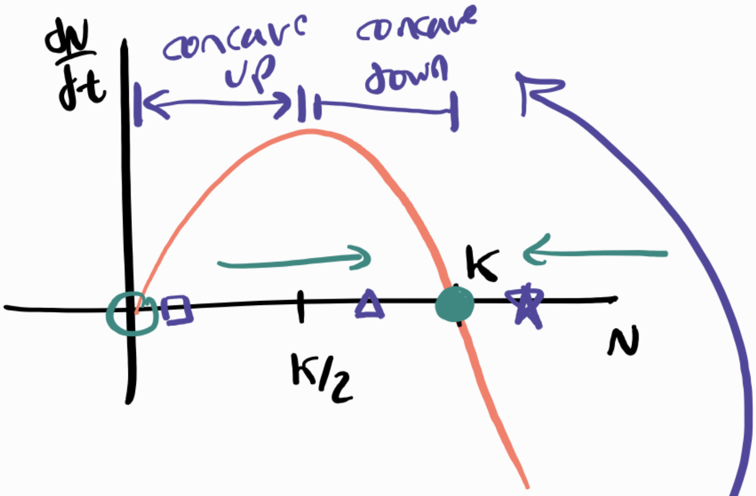
Example: Logistic equation for population growth

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

r: growth rate

K: carrying capacity

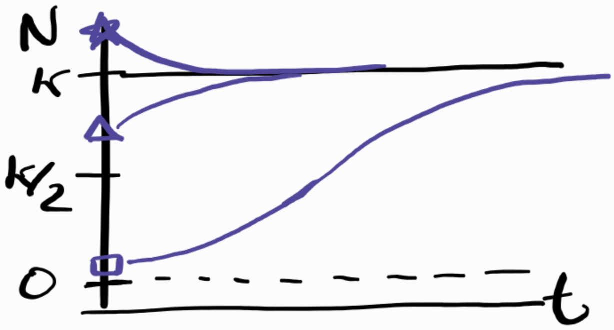
$\Rightarrow$ phase plane analysis:



$N^* = K$ stable

$N^* = 0$ unstable

$\Rightarrow$ sketch of soln:



think about qualitative soln features

Linear Stability Analysis

$\frac{dx}{dt} = f(x)$

Can we determine fixed point stability through analytical means?

$\Rightarrow$ let x^* be a fixed point

Let $\varepsilon(t) = x(t) - x^*$ be a small perturbation

$\rightsquigarrow$ how does $\varepsilon(t)$ change?

$$\frac{d}{dt}\varepsilon(t) = \frac{dx}{dt} = f(x(t)) = f(\varepsilon(t) + x^*)$$

Taylor expand

$$\frac{d\varepsilon}{dt} = \underbrace{f(x^*)}_{=0!} + \varepsilon f'(x^*) + O(\varepsilon^2)$$

$\Downarrow$

$$\frac{d\varepsilon}{dt} \approx \varepsilon f'(x^*) \xrightarrow{\text{sol'n}} \varepsilon(t) = \varepsilon_0 \exp\left(f'(x^*)\, t\right)$$

$\Rightarrow$ ① if $f'(x^*) > 0$, perturbation grows

② if $f'(x^*) < 0$, perturbation shrinks

Also, we now see perturbations evolve with timescale $\left|\frac{1}{f'(x^*)}\right|$

Going back to the example:

$$\frac{dN}{dt} = \underset{\substack{\parallel \\ f(N)}}{rN\left(1-\frac{N}{K}\right)}, \quad N^* = 0, K$$

$$f'(N) = r\left(1-\frac{N}{K}\right) - \frac{rN}{K}$$

$\Rightarrow f'(0) = r > 0 \rightsquigarrow$ unstable!

$\Rightarrow f'(K) = -r < 0 \rightsquigarrow$ stable!

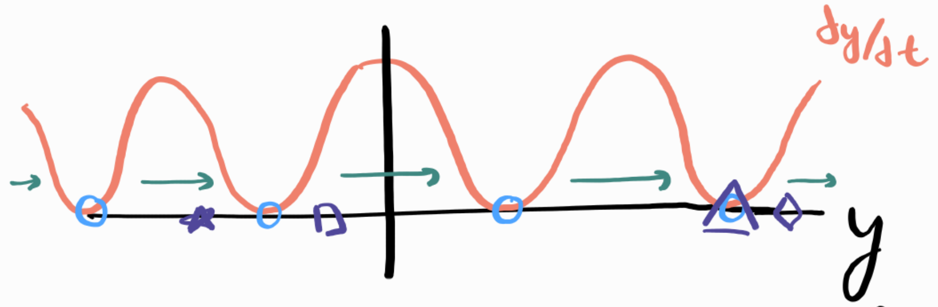
Example: "half-stable" fixed points

$$\frac{dy}{dt} = f(y) = \cos(y) + 1$$

$\rightsquigarrow$ $\frac{dy}{dt} = \cos(y) + 1 = 0 \Rightarrow y_j^* = (2j+1)\pi$

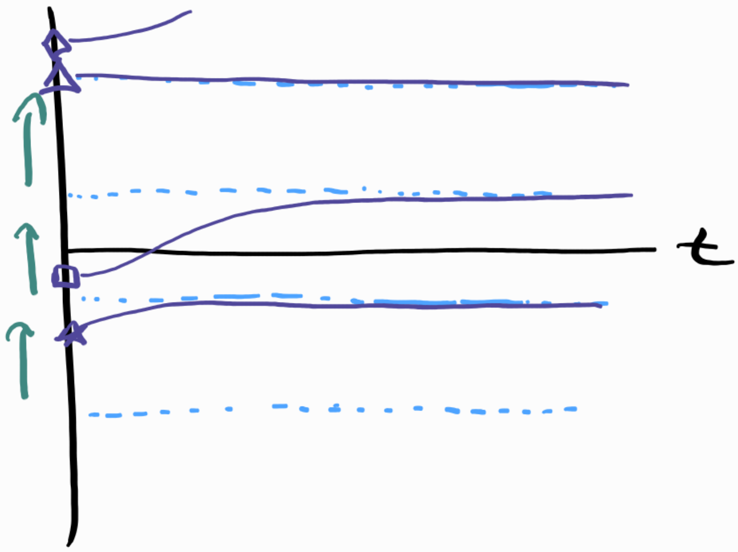
$j = 0, \pm 1, \pm 2, \dots$

$\Rightarrow$ phase plane



↪ attracting from left, repelling from right ("half-stable" – Strogatz)

$\Rightarrow$ sketch of solns



$\Rightarrow$ Notice

$f'(y) = \sin(y)$

$\downarrow$

$\sin(\pi) = 0$

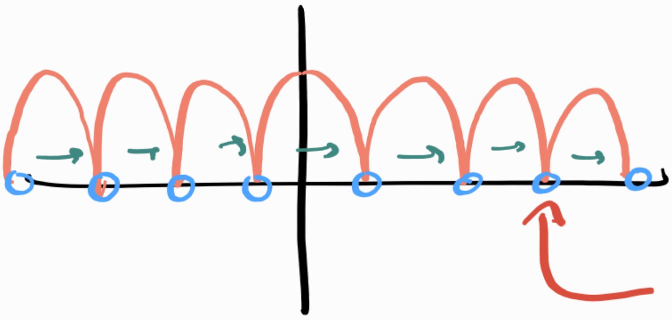
$\sin(\pm 3\pi) = 0$

$\sin(\pm 5\pi) = 0$

so linear stability analysis is inconclusive here! ← :)

Example:

What about $\frac{dx}{dt} = |\cos(x)|$?

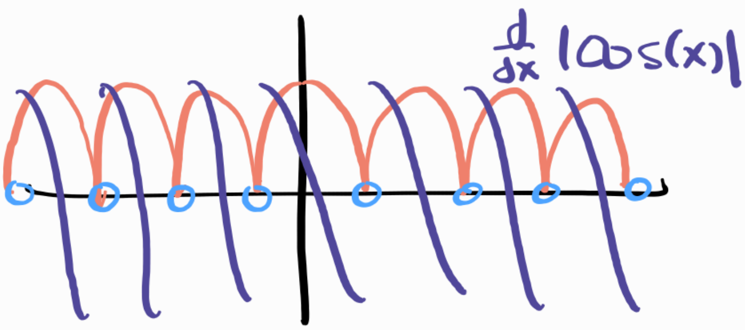


$\boxed{Q}$ Is $|\cos(x)|$ differentiable here?

no?

$$\frac{d}{dx}\left(|\cos(x)|\right) = -\frac{\sin(x)\cos(x)}{|\cos(x)|}$$

$\Rightarrow \frac{d}{dx}\left(|\cos(x)|\right) = \text{DNE}$ at fixed pts!



→ Convergence of numerical methods breaks down for non smooth functions

⇝ be careful if you try to solve $\frac{dx}{dt} = |\cos(x)|$ numerically

Autonomous ODEs in 2D

General form:
$$\begin{cases} \dfrac{dx_1}{dt} = F_1(x_1, x_2, \dots, x_N) \\ \dfrac{dx_2}{dt} = F_2(x_1, x_2, \dots, x_N) \\ \vdots \\ \dfrac{dx_N}{dt} = F_N(x_1, x_2, \dots, x_N) \end{cases}$$

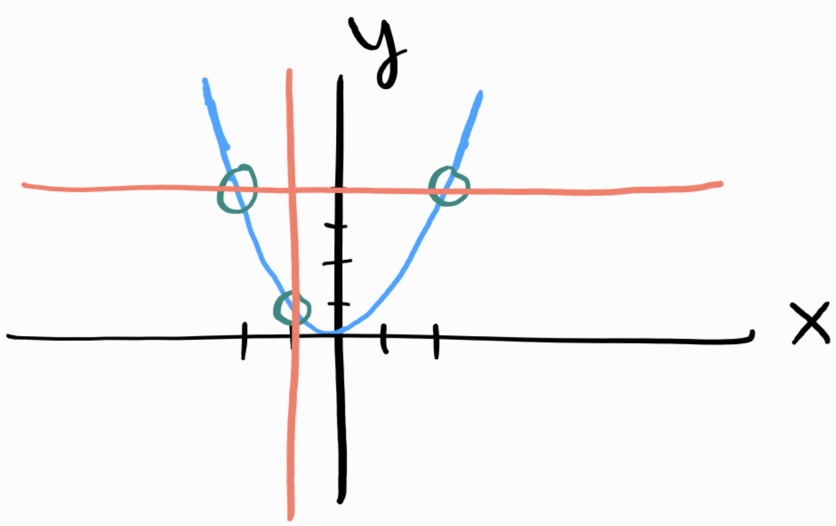
Example:
$$\begin{cases} \dfrac{dx}{dt} = (x+1)(y-4) \\ \dfrac{dy}{dt} = y - x^2 \end{cases}$$

$\Rightarrow$ X-nullclines: (x,y) s.t. $\dfrac{dx}{dt} = 0$

Y-nullclines: (x,y) s.t. $\dfrac{dy}{dt} = 0$

if both true, fixed point

i.e. fixed points are where nullclines intersect



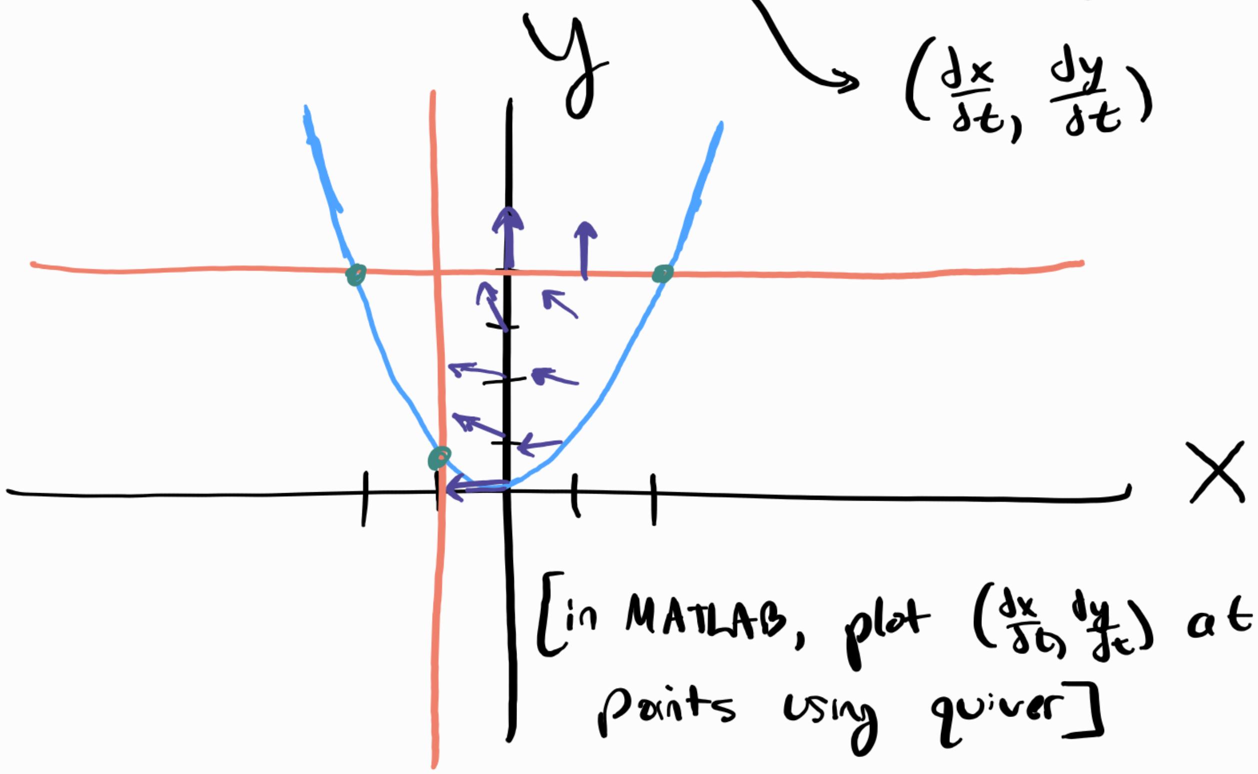
$(x^*, y^*) = (2, 4)$

$(x^*, y^*) = (-2, -4)$

$(x^*, y^*) = (1, 1)$

$\Rightarrow$ phase plane

$\hookrightarrow$ at (x,y), flow is $\frac{dy}{dx} \sim \frac{dy/dt}{dx/dt}$ $\rightarrow$ $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$



[in MATLAB, plot $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ at (x,y) points using quiver]

$(1,1) \rightarrow \frac{dy}{dt} = 0 \quad \frac{dx}{dt} = -6$

$(1,3) \rightarrow \frac{dy}{dt} = \frac{2}{-2}$

$(1,2) \rightarrow \frac{dy}{dx} = \frac{1}{-9}$

$(1,4) \rightarrow \frac{dx}{dt} = 0 \quad \frac{dy}{dt} = 3$

$(0,0) \rightarrow \frac{dy}{dt} = 0 \quad \frac{dx}{dt} = -4$

$(0,1) \rightarrow \frac{dy}{dx} = \frac{1}{-3}$

$(0,2) \rightarrow \frac{dy}{dt} = \frac{2}{-8}$

$(0,3) \rightarrow \frac{dy}{dx} = \frac{3}{-1}$

$(0,4) \rightarrow \frac{dx}{dt} = 0 \quad \frac{dy}{dt} = 4$

... and so on

$\rightsquigarrow$ how to do this more analytically?

$\Rightarrow$ let's first think about <u>linear</u> systems of ODEs

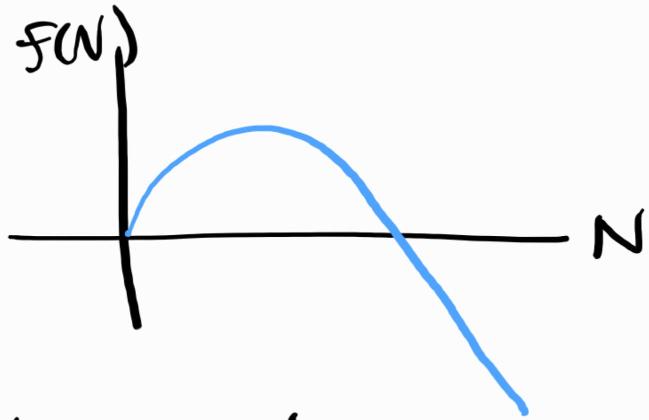
Quiz

[Note: no python/MATLAB/wolfram/etc. for this quiz → you do not need it]

Tumor growth can be modeled by the Gompertz model:

$$\dot{N} = -aN\ln(bN) = f(N)$$

tumor cells



a) Find the fixed points

b) Plot the phase plane (line) and determine the stability of the fixed points

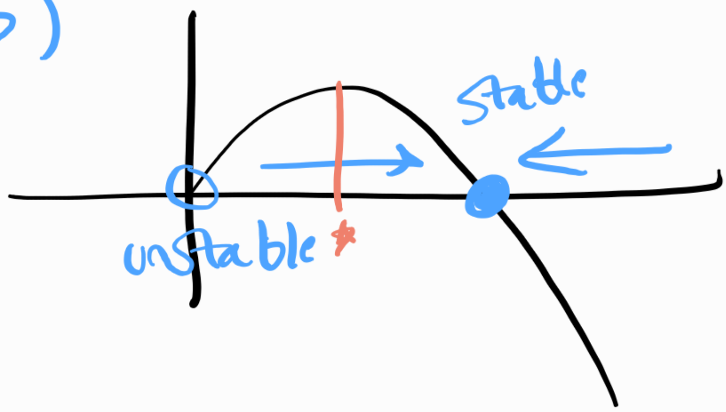
c) Sketch the solution N(t) for a couple initial values

d) Interpret a and b biologically

a) $N^* = 0$, $N^* = 1/b$

$f(N)$ is not technically defined here, but in practice from the perspective of modelling a physical system, we consider it

b)



d)

a: growth rate

b: 1/carrying capacity

c)