Recall: 2D autonomous ODEs

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

X-nullcline

$$\{(x,y) \mid f(x,y) = 0\}$$

Y-nullcline

$$\{(x,y) \mid g(x,y) = 0\}$$

Intersections are fixed points

Example

$$\begin{cases} \frac{dx}{dt} = (x-1)(y+1) \\ \frac{dy}{dt} = (y+x)(x+2) \end{cases}$$

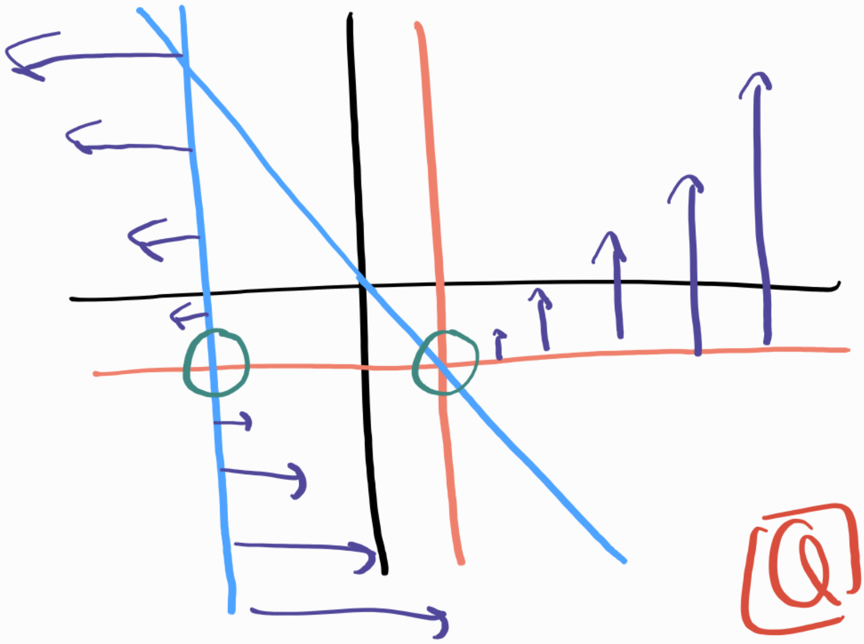
$\Rightarrow$ plot nullclines in the phase plane

X-nullcline ☆

$x=1$, y: any
or
$y=-1$, x: any

Y-nullcline ☆

$y=-x$
or
$x=-2$, y: any



2 fixed points:

① $x^*=-2$, $y^*=-1$

② $x^*=1$, $y^*=-1$

$\boxed{Q}$ What direction is flow on nullclines?

Linearization

Let

$$u = x - x^*$$
$$v = y - y^*$$

u, v small

$$\Rightarrow \frac{du}{dt} = \frac{dx}{dt} = f(x,y) = f(u+x^*, v+y^*)$$

Taylor series

$$\approx \cancel{f(x^*,y^*)}^{\,0} + u\frac{\partial f}{\partial x}(x^*,y^*) + v\frac{\partial f}{\partial y}(x^*,y^*)$$
$$+ \cancel{\frac{u^2}{2}\frac{\partial^2 f}{\partial x^2}(x^*,y^*)} + \cancel{uv\frac{\partial^2 f}{\partial x \partial y}(x^*,y^*)}$$
$$+ \cancel{\frac{v^2}{2}\frac{\partial^2 f}{\partial y^2}(x^*,y^*)} + O(u^3, uv^2, u^2v, v^3)$$

small!

$$\Rightarrow \begin{cases} \dfrac{du}{dt} \approx u \underline{\dfrac{\partial f}{\partial x}(x^*,y^*)} + v \underline{\dfrac{\partial f}{\partial y}(x^*,y^*)} \\[2ex] \dfrac{dv}{dt} \approx u \underline{\dfrac{\partial g}{\partial x}(x^*,y^*)} + v \underline{\dfrac{\partial g}{\partial y}(x^*,y^*)} \end{cases}$$

linear system

constants!

Jacobian

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = J\Big|_{x^*,y^*} \begin{bmatrix} u \\ v \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$\Longrightarrow$ find eigenvalues, classify them

Classification of fixed points

Suppose matrix $\underline{\underline{A}}^{2\times 2}$ has eigenvalues λ_1, λ_2

⇝ we know from linear algebra that:

$$\begin{cases} tr(\underline{\underline{A}}) = \tau = \lambda_1 + \lambda_2 \\ det(\underline{\underline{A}}) = \Delta = \lambda_1 \lambda_2 \end{cases}$$

Note: λ_1 and λ_2 are either both real or complex conjugates (because $\underline{\underline{A}}$ is real)

If they are complex conjugates, $\lambda_{1,2} = \alpha \pm \beta i$

$$\Rightarrow \tau = \alpha + \beta i + \alpha - \beta i = 2\alpha \in \mathbb{R}$$

$$\Rightarrow \Delta = (\alpha + \beta i)(\alpha - \beta i) = \alpha^2 + \beta^2 \in \mathbb{R}$$

So as expected, $tr(\underline{\underline{A}})$ & $det(\underline{\underline{A}})$ are real

We also know that

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

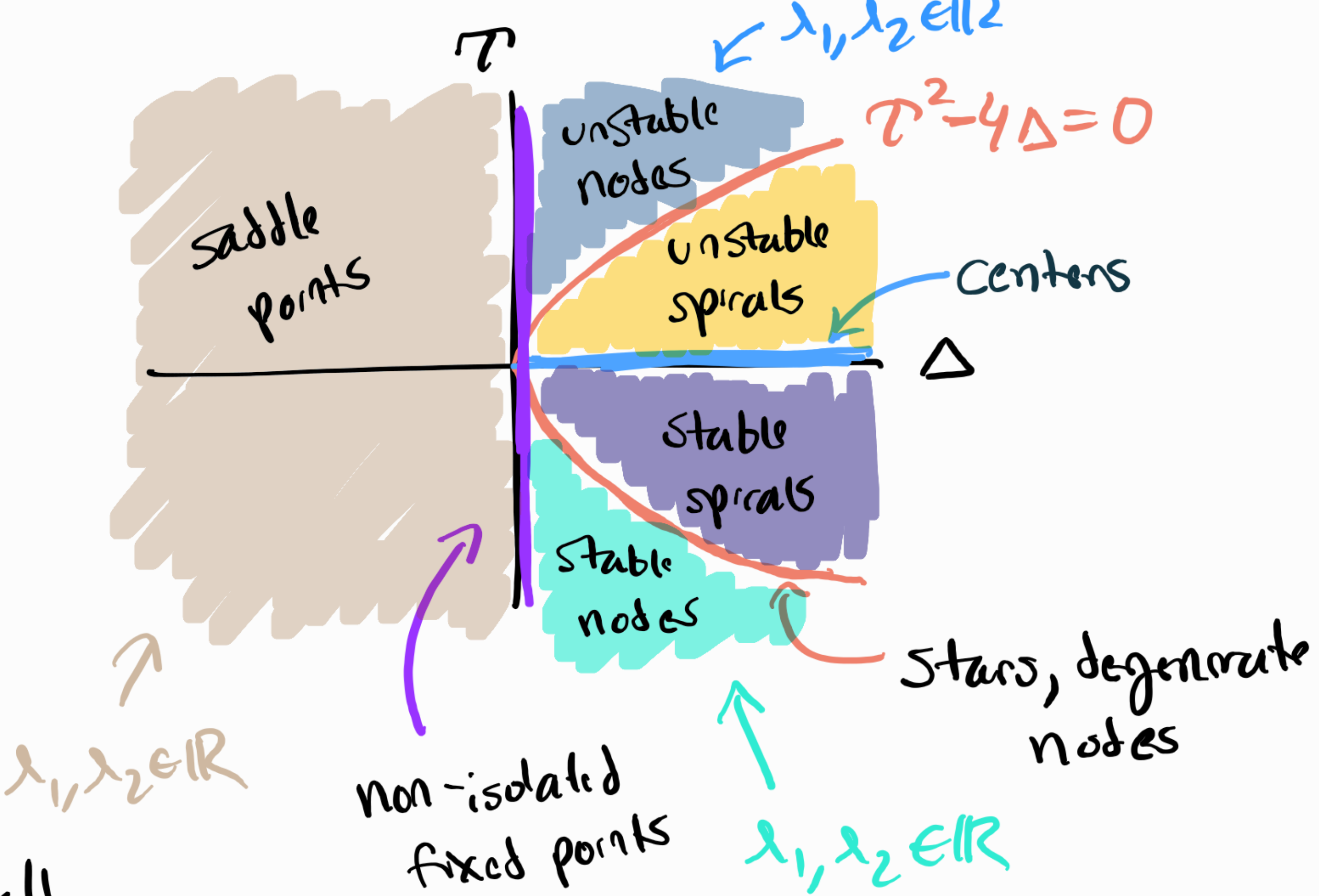
⇝ notice, if $\tau^2 - 4\Delta < 0$

$\Rightarrow$ eigenvalues complex

⇝ if $\tau = 0$, $\Delta > 0$

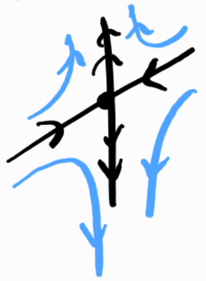
$\Rightarrow$ eigenvalues purely imaginary

⇒ can we classify eigenvalues $\lambda_{1,2}$ by looking at the (Δ, τ) space?



Recall

saddle



$\lambda_1 < 0$
$\lambda_2 > 0$

$\Delta < 0$

stable node

$\lambda_1 < 0$
$\lambda_2 < 0$

$$\begin{cases} \Delta > 0 \\ \tau^2 - 4\Delta > 0 \\ \tau < 0 \end{cases}$$

unstable node

$\lambda_1 > 0$
$\lambda_2 > 0$

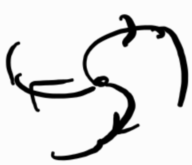
$$\begin{cases} \Delta > 0 \\ \tau^2 - 4\Delta > 0 \\ \tau > 0 \end{cases}$$

stable spiral

$\mathrm{Re}(\lambda) < 0$

$$\begin{cases} \Delta > 0 \\ \tau^2 - 4\Delta < 0 \\ \tau < 0 \end{cases}$$

unstable spiral

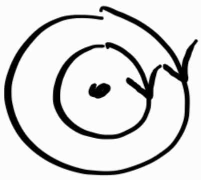


$\mathrm{Re}(\lambda) > 0$

$$\begin{cases} \Delta > 0 \\ \tau^2 - 4\Delta < 0 \\ \tau > 0 \end{cases}$$

What about borderline cases?

Centers



$$\begin{cases} \Delta = 0 \\ \tau^2 - 4\Delta < 0 \end{cases}$$

$$\lambda_{1,2} = \pm \beta i$$

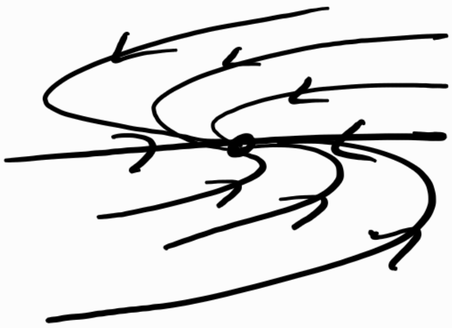
$$= \pm \frac{\sqrt{\tau^2 - 4\Delta}}{2}$$

degenerate node

$$\begin{cases} \tau^2 - 4\Delta = 0 \\ \Delta > 0 \end{cases}$$

$$\lambda_1, \lambda_2 = \tau/2 \neq 0$$

⊕ one eigenvector



as $t \to +\infty$, trajectories are parallel to the eigenvector

(like a combination of spiral & node)

Stars

$$\begin{cases} \tau^2 - 4\Delta = 0 \\ \Delta > 0 \end{cases}$$

$$\lambda_1, \lambda_2 = \tau/2 \neq 0$$

⊕ two distinct eigenvectors

all trajectories are straight lines

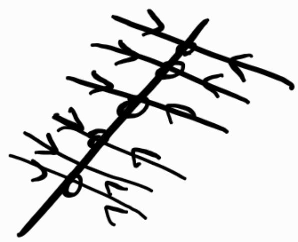
note: if $\lambda_1, \lambda_2 = 0$, whole plane is fixed pts

non isolated f.p.

$$\Delta = 0$$

$\Rightarrow$ at least one eigenvalue $= 0$

if $\lambda_1 = 0$, $\lambda_2 \neq 0$



Example

$$\begin{cases} \dfrac{dx}{dt} = (x-1)(y+1) \\ \dfrac{dy}{dt} = (y+x)(x+2) \end{cases} \Longrightarrow J = \begin{bmatrix} y+1 & x-1 \\ y+2x+2 & x+2 \end{bmatrix}$$

fixed pts: $(-2,-1)$ & $(1,-1)$

$$J(-2,-1) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \begin{matrix} \nearrow \tau = 0 \\ \searrow \Delta = -9 \end{matrix} \Rightarrow \text{saddle pt}$$

$$\lambda_1, \lambda_2 = \pm 3 \Rightarrow \text{saddle pt}$$

$$J(1,-1) = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} \begin{matrix} \nearrow \tau = 3 \\ \searrow \Delta = 0 \end{matrix} \Rightarrow \text{non isolated S.P.}$$

$$\lambda_1 = 3, \lambda_2 = 0 \Longrightarrow$$

(or is it? linearization can be inconclusive)

☆ note: this classification is only local, because we linearized for small u, v

Example: model for dynamics of love affairs (Strogatz 1988)

Consider Romeo and Juliet.

Suppose R represents Romeo's love/hate for Juliet, s.t. $R>0$ if he loves her
$R<0$ if he hates her

Similarly, let J represent Juliet's love ($J>0$) or hate ($J<0$) for Romeo

Consider the system

$$\begin{cases} R = aR + bJ \\ J = bR + aJ \end{cases}$$

$\boxed{Q}$ How can we interpret a and b?

e.g. $b>0$: if Juliet loves him, Romeo is more in love, but if she hates him, he will lose interest.
(responsiveness)

e.g. $a<0$: Romeo is cautious of his own feelings, in either direction
(cautiousness)

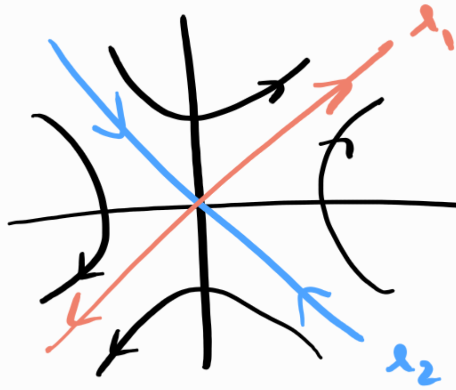
$\rightsquigarrow$ Juliet is similarly responsive and cautious

Let $a<0$, $b>0$. Then

$$\begin{cases} \tau = 2a < 0 \\ \Delta = a^2 - b^2 \end{cases} \quad \& \quad \begin{cases} \lambda_1 = a+b, \; v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 = a-b, \; v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$$

Case 1 $a^2 < b^2$

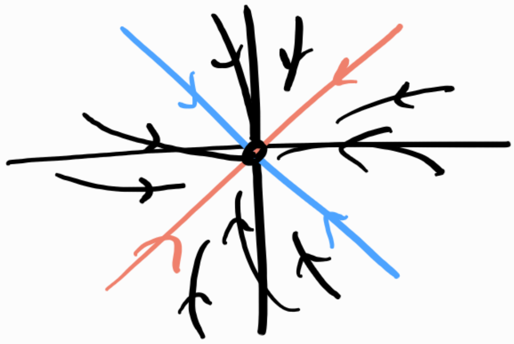
$\Rightarrow \Delta < 0 \Rightarrow$ saddle pt



Generally, either they become madly in love or hate each others' guts

Case 2 $a^2 > b^2$

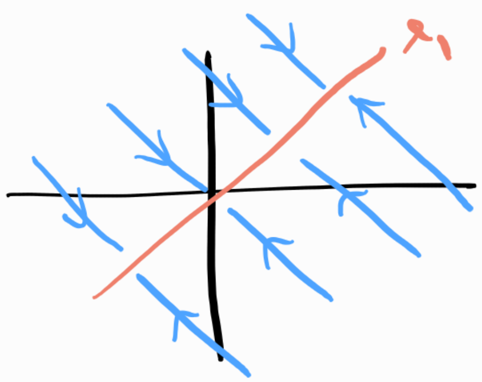
$\Rightarrow \Delta > 0 \Rightarrow$ stable node



The relationship will sizzle out

Case 3 $a^2 = b^2$

$\Rightarrow \Delta = 0 \Rightarrow$ nonisolated fp



$\lambda_1 = 0$

$\lambda_2 = 2a = -2b$

Their relationship will stable with mutual feelings of love or hate

Quiz

Part 1 Consider the system

$$\begin{cases} \frac{dx}{dt} = y^3 - 4x \\ \frac{dy}{dt} = y^3 - y - 3x \end{cases}$$

(a) Find the fixed points

(b) Classify them using linearization

Part 2 Nothing could ever change the way Romeo feels ($\frac{dR}{dt} = 0$), but Juliet ($\frac{dJ}{dt} = aR + bJ$) feels cautious of her own feelings ($b<0$) and is similarly responsive to Romeo's ($a>0$). Does Juliet end up loving or hating him? Sketch the phase plane flow.

Solution

$$\underline{\underline{J}} = \begin{bmatrix} -4 & 3y^2 \\ -3 & 3y^2-1 \end{bmatrix}$$

Part 1

$$(x^*, y^*) = (-2,-2),\ (0,0),\ (2,2)$$

$$\underline{J}(-2,-2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \rightarrow \Delta = -8$$

saddle pt

$$\underline{J}(0,0) = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \rightarrow \Delta = 4,\ \tau = -5$$

stable node

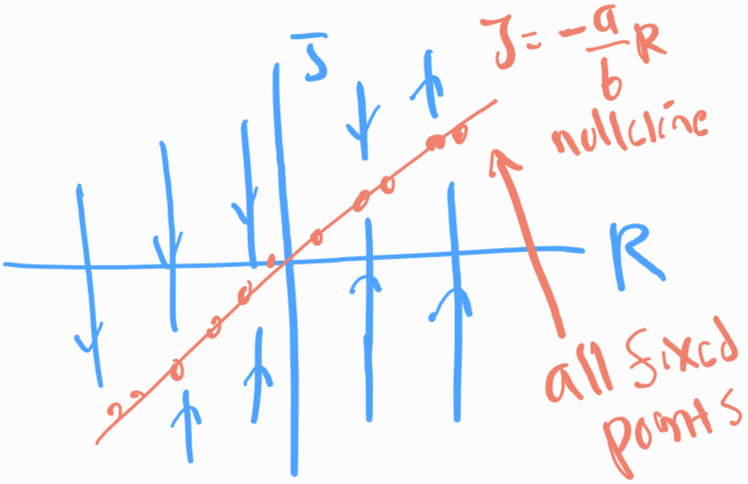
$$\underline{J}(2,2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \rightarrow \Delta = -8$$

saddle pt

Part 2

$$\begin{cases} \dfrac{dR}{dt} = 0R + 0J \\ \dfrac{dJ}{dt} = aR + bJ \end{cases} \Rightarrow \underline{\underline{A}} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

$$\begin{cases} \tau = b < 0 \\ \Delta = 0 \end{cases} \quad \& \quad \begin{aligned} \lambda_1 &= 0, & \bar{v}_1 &= \left(-\tfrac{b}{a}, 1\right) \\ \lambda_2 &= b < 0, & \bar{v}_2 &= (0,1) \uparrow \end{aligned}$$



If $R>0$ (he loves her), she will end up loving him. If $R<0$, she will end up hating him.