Annals of Mathematics, 122 (1985), 540–574

The metric entropy of diffeomorphisms Part II: Relations between entropy, exponents and dimension

By F. LEDRAPPIER AND L.-S. YOUNG

We continue to consider $f: (M, m) \to (M, m)$, a C^2-diffeomorphism f of a compact Riemannian manifold M preserving a Borel probability measure m. As before, let $h_m(f)$ denote the metric entropy of f, let $\lambda_1(x) > \cdots > \lambda_{r(x)}(x)$ be the distinct Lyapunov exponents at x and let $\oplus_{i \leqq r(x)} E_i(x)$ be the corresponding decomposition of $T_x M$. Both parts of this paper concern the relation between entropy and Lyapunov exponents. As we have indicated in the introduction to Part I, much of this work is motivated by the theorems of Margulis, Ruelle [Ru 1] and Pesin [Pe]. Part I is about a necessary and sufficient condition on the measure under which Pesin's entropy formula holds. The main goal here in Part II is to prove an analogous formula that works for all diffeomorphisms and all invariant measures.

Without any hypotheses of absolute continuity on m, any equation relating entropy and exponents must involve some notion of fractional dimension. This fact has been observed for certain specific examples for some time (see for instance [Bi]). General results relating these quantities have so far been confined to maps that are one-dimensional in nature (e.g. [L], [Y]). Our main theorem (Theorem C) extends these results. We show that for any invariant probability measure m,

$$(**) \qquad h_m(f) = \int \sum_i \lambda_i^+ \gamma_i \, dm$$

where γ_i denotes, roughly speaking, the dimension of m "in the direction of the subspace E_i". (Precise definitions are given in § 7, in particular in (7.3) and (7.4). Unfortunately it takes some work to see that these numbers as defined in Section 7 have the intuitive interpretation mentioned above.)

Research supported in part by NSF Grant MCS 8120790 and AFOSR Grant AFOSR-83-0265.

Since γ_i lies between 0 and $\dim E_i$, our formula implies the inequality of Margulis and Ruelle [Ru 1] (although this is a rather indirect way of arriving at their result). Pesin's formula corresponds to the case where $\gamma_i = \dim E_i$.

We give an indication of the proof of $(**)$. When two or more distinct positive exponents are present, it is difficult to estimate dimension as is done in [Y]. In these situations, however, there is a hierarchy of unstable manifolds corresponding to the largest k exponents, $k = 1, \ldots, u(x)$ (see (1.1)). These unstable manifolds give rise to a nested family of invariant foliations. Conditioning our estimates on the leaves of these foliations and working our way up successive layers, we are able to focus on one exponent at a time.

In our argument we use the notion of "entropy along an invariant foliation" or "partial entropies". This allows us to distinguish between randomness occurring in different directions in much the same way that one studies the growth of the derivative by decomposing T_xM into subspaces corresponding to distinct exponents. These ideas may have further ramifications.

We have expressed entropy in terms of exponents and dimension. One may wish also to express dimension in terms of the other invariants. We do not have a complete solution to this problem. We show however that the dimension of m is bounded above by $\sum_i \gamma_i$ (Theorem F). A corollary to this (Corollary I) partially confirms a conjecture of Yorke's on the dimension of attractors [FKYY]. Incidentally, Pesin's formula can also be deduced from theorem F and $(**)$ in the case when m is equivalent to Lebesgue.

This paper begins with Section 7 in which we give precise statements of results and related definitions. Section 8 contains local unstable manifolds estimates. The notion of entropy along an unstable foliation is introduced in Section 9. Theorem C is proved in Sections 10 and 11 and Theorem F is proved in Section 12.

Some of the proofs in Section 8 and especially in Section 11 are similar to those given in Part I. We try to make the statements of lemmas as self-contained as possible here but will refer the reader to Part I for certain proofs.

We remark also that while our theorems contain no ergodic assumptions, most of them can be proved by reduction to the ergodic case. For this reason–and for simplicity of exposition–we have chosen to present most of our ideas and proofs assuming that m is ergodic.

Finally we should mention that the theory of dimensions, as well as its relation to entropy and exponents, seems to be of some interest to mathematical physicists. In fact, while this manuscript was under preparation, two physicists, Grassberger and Procaccia (see [G]), derived heuristically (and independently of this work) a relation very similar to $(**)$. For a discussion of these invariants that is more oriented towards applications we refer the reader to the survey article [ER].

Standing Hypotheses

M is a C^∞ compact Riemannian manifold without boundary;

f: $M \hookleftarrow$ is a C^2 diffeomorphism;

m is an f-invariant Borel probability measure on M;

and except in (7.5), (7.6) and Section 12,

m is ergodic.

7. Definitions and statements of results

(7.1) *Unstable manifolds*. As in Section 1, let Γ' be the set of points that are regular in the sense of Oseledec [O]. For $x \in \Gamma'$, let

$$\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{r(x)}(x)$$

denote its distinct Lyapunov exponents and let

$$T_xM = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$$

be the corresponding decomposition of its tangent space. Since m is assumed to be ergodic, the functions $x \mapsto r(x)$, $\lambda_i(x)$ and $\dim E_i(x)$ are constant m-a.e.

Let $u = \max\{i: \lambda_i > 0\}$. For $1 \le i \le u$, define

$$W^i(x) = \left\{ y \in M: \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) \le -\lambda_i \right\}.$$

Then $W^i(x)$ is a C^2 $(\sum_{j \le i} \dim E_j)$-dimensional immersed submanifold of M tangent at x to $\oplus_{j \le i} E_j(x)$. (See for instance [Ru2].) It is called the i^{th} *unstable manifold* of f at x. We sometimes refer to $\{W^i(x): x \in \Gamma'\}$ as the W^i*-foliation* on M. Using this language then, we have a nested family of a.e. foliations

$$W^1 \subset W^2 \subset \cdots \subset W^u$$

corresponding to the distinct positive exponents of f.

As an immersed submanifold, each $W^i(x)$ inherits a Riemannian structure from M. This gives rise to a Riemannian metric on each leaf of W^i. We denote these metrics by d^i.

The measure m also defines conditional measures on the leaves of W^i. More precisely, a measurable partition ξ of M is said to be *subordinate* to the W^i-foliation if for a.e. x, $\xi(x) \subset W^i(x)$ and contains an open neighborhood of x in $W^i(x)$. Associated with each measurable partition is a system of conditional measures. (See (1.3) of Part I or [Ro].) In the next two subsections we shall define some invariants using the conditional measures for partitions subordinate to W^i.

(7.2) *Entropy along unstable manifolds*. Fix $1 \le i \le u$. We now define a notion of *entropy along the* W^i*-foliation*. In the ergodic case this notion is described by a number h_i which measures the amount of randomness along the leaves of W^i. There are several equivalent definitions. Following [BK] we take a pointwise approach:

Let $\varepsilon > 0$. For $x \in \Gamma'$ and $n \in \mathbf{Z}^+$, define

$$V^i(x, n, \varepsilon) = \left\{ y \in W^i(x) \colon d^i\left(f^k x, f^k y\right) < \varepsilon \text{ for } 0 \leq k < n \right\}.$$

Let ξ be a measurable partition subordinate to W^i and let $\{m_x\}$ be a system of conditional measures associated with ξ. Define

$$\underline{h}_i(x, \varepsilon, \xi) = \liminf_{n \to \infty} -\frac{1}{n} \log m_x V^i(x, n, \varepsilon)$$

and

$$\overline{h}_i(x, \varepsilon, \xi) = \limsup_{n \to \infty} -\frac{1}{n} \log m_x V^i(x, n, \varepsilon).$$

(The cautious reader may wish to verify that these functions are indeed measurable.)

PROPOSITION 7.2.1. *At m-a.e. x,*

$$\lim_{\varepsilon \to 0} \underline{h}_i(x, \varepsilon, \xi) = \lim_{\varepsilon \to 0} \overline{h}_i(x, \varepsilon, \xi).$$

These limits exist because $\underline{h}_i(x, \varepsilon, \xi)$ and $\overline{h}_i(x, \varepsilon, \xi)$ increase as $\varepsilon \downarrow 0$. The proof of this proposition occupies Section 9.

Let $h_i(x, \xi)$ denote the limit in Proposition 7.2.1. Using the essential uniqueness of conditional measures and the invariance of m it is easy to verify that $h_i(fx, \xi) = h_i(x, \xi)$ m-a.e. and hence $h_i(x, \xi)$ is constant a.e. The reader should check also that this constant is independent of the choice of ξ or $\{m_x\}$. This completes the definition of h_i.

The concept of entropy along a foliation has in fact been exploited in Part I though we did not explicitly introduce this terminology. The main proposition of Part I (see Section 5) combined with the discussion in Section 9 gives:

COROLLARY 7.2.2. $h_u = h_m(f)$.

(7.3) *Dimension of conditional measures on unstable manifolds.* Again fix $1 \leq i \leq u$. Let $B^i(x, \varepsilon)$ denote the d^i-ball in $W^i(x)$ centered at x of radius ε. Let ξ be a measurable partition subordinate to W^i with conditional measures $\{m_x\}$. For $x \in \Gamma'$ define

$$\underline{\delta}_i(x, \xi) = \liminf_{\varepsilon \to 0} \frac{\log m_x B^i(x, \varepsilon)}{\log \varepsilon}$$

and

$$\overline{\delta}_i(x, \xi) = \limsup_{\varepsilon \to 0} \frac{\log m_x B^i(x, \varepsilon)}{\log \varepsilon}.$$

As in the last section one verifies that $\underline{\delta}_i(x, \xi)$ and $\bar{\delta}_i(x, \xi)$ are constant along orbits and that the two numbers $\underline{\delta}_i$ and $\bar{\delta}_i$ are well defined independent of ξ.

Propositions 7.3.1 and 7.3.2 are consequences of our results in Sections 10 and 11.

PROPOSITION 7.3.1. $\underline{\delta}_i = \bar{\delta}_i$.

We denote this common value by δ_i and call it the *dimension of m on W^i-manifolds*. To justify this terminology let us recall the following known fact (see for instance [Y]):

Suppose μ is a finite Borel measure on M and $X \subset M$ is a Borel set with $\mu X > 0$. If for *every* $x \in X$,

$$\liminf_{\varepsilon \to 0} \frac{\log \mu B(x, \varepsilon)}{\log \varepsilon} \geq \underline{\delta}$$

and

$$\limsup_{\varepsilon \to 0} \frac{\log \mu B(x, \varepsilon)}{\log \varepsilon} \leq \bar{\delta},$$

then the *Hausdorff dimension* of X, written $\mathrm{HD}(X)$, satisfies

$$\underline{\delta} \leq \mathrm{HD}(X) \leq \bar{\delta}.$$

Suppose now that $2 \leq i \leq u$. Remember that W^i is a $(\Sigma_{j \leq i} \dim E_j)$-dimensional foliation on M each leaf of which is in turn foliated by the $(\Sigma_{j \leq i-1} \dim E_j)$-dimensional foliation W^{i-1}. We shall see in Section 11 that the number $\delta_i - \delta_{i-1}$ can be interpreted as the "transverse dimension" of m on W^i/W^{i-1}. In particular, we shall prove:

PROPOSITION 7.3.2. $0 \leq \delta_i - \delta_{i-1} \leq \dim E_i$ *for* $2 \leq i \leq u$.

(7.4) *The main result: ergodic case.* In (7.2) and (7.3) we described some natural invariants associated with the dynamical system $f\colon (M, m) \hookleftarrow$. Our main result establishes the connection between these numbers and the Lyapunov exponents of f.

THEOREM C′. *Let $f\colon M \hookleftarrow$ be a C^2 diffeomorphism of a compact Riemannian manifold and let m be an ergodic Borel probability measure on M. Let $\lambda_1 > \cdots > \lambda_u$ denote the distinct positive Lyapunov exponents of f, and let δ_i be the dimension of m on W^i-manifolds. Then for $1 \leq i \leq u$ there are numbers*

γ_i *with* $0 \leq \gamma_i \leq \dim E_i$ *such that*

$$\delta_i = \sum_{j \leq i} \gamma_j$$

for $i = 1, \ldots, u$ *and*

$$h_m(f) = \sum_{i \leq u} \lambda_i \gamma_i.$$

Theorem C′ is the amalgamation of the following three partial results:

(i) $h_1 = \lambda_1 \delta_1$,

(ii) $h_i - h_{i-1} = \lambda_i(\delta_i - \delta_{i-1})$ for $i = 2, \ldots, u$, and

(iii) $h_u = h_m(f)$.

Setting $\gamma_1 = \delta_1$ and $\gamma_i = \delta_i - \delta_{i-1}$ for $i = 2, \ldots, u$, one obtains the conclusion of Theorem C′ immediately by adding formulas (i), (ii) and (iii).

We explain these formulas before proceeding further. The dimension of a measure is directly related to dynamical invariants such as entropy and Lyapunov exponents if one can dynamically generate sets that are essentially *round* balls. This can be done when all the exponents are equal. In this case equation (i) is the basic principle relating entropy, exponents and dimension. In general, one attempts to decompose the dynamical structures associated with a map into directions corresponding to the different rates of expansion. Equation (ii) is then a restatement of (i) concerning the transverse dynamics and dimension of W^{i-1} inside W^i-leaves. This is the idea of part of the proof of (ii). (See Section 11.) To complete the proof of (ii) we need another argument, which is explained in Section 10. Finally equation (iii) says that what happens in the contracting and neutral directions does not contribute to the entropy of f. That was basically the concern of Part I. (See Section 9.)

The next two corollaries follow immediately from Theorem C′.

Corollary D′. *With the same hypotheses as in Theorem* C′, *and with* $h_0 = 0$,

$$\delta_i = \sum_{j \leq i} \frac{h_j - h_{j-1}}{\lambda_j} \qquad \textit{for } i = 1, \ldots, u.$$

In (7.3) we defined $\delta_i = \delta_i(f)$ and in Theorem C′ we defined $\gamma_i = \gamma_i(f)$ for $i = 1, \ldots, u$. Remember that these γ_i's carry geometric meaning as the transverse dimension of m between successive W^i-foliations. Now consider f^{-1}: $(M, m) \hookleftarrow$ and define analogously $\tilde{\gamma}_i = \gamma_i(f^{-1})$ corresponding to the positive exponents of f^{-1} (or equivalently the negative exponents of f). Let $\gamma_i = \tilde{\gamma}_{r-i+1}$ when $\lambda_i < 0$ and $\gamma_i = \dim E_i$ if $\lambda_i = 0$.

COROLLARY E′. *With the same hypotheses as in Theorem* C′ *and with* γ_i *as defined above,*

$$\sum_i \lambda_i \gamma_i = 0.$$

This follows from Theorem C′ and the fact that $h_m(f) = h_m(f^{-1})$. We state two more corollaries of Theorem C′ that are known results:

COROLLARY 7.4.1. $h_m(f) \leq \sum_{i \leq u} \lambda_i \dim E_i$.

COROLLARY 7.4.2. *If* m *has absolutely continuous conditional measures on unstable manifolds* (*see Definition* 1.4.2), *then*

$$h_m(f) = \sum_{i \leq u} \lambda_i \dim E_i.$$

Corollary 7.4.1 follows readily from Theorem C′ and Proposition 7.3.2, as does Corollary 7.4.2 since $\delta_u = \sum_{i \leq u} \dim E_i$ when m has absolutely continuous conditional measures on W^u.

The nonergodic version of these results is stated in the next section. We mention here that Corollary 7.4.1 (without the assumption of ergodicity) was proved independently by Margulis and Ruelle [Ru1]. Corollary 7.4.2 (again not necessarily ergodic) is the "if" part of Theorem A and was proved in [LS]. As indicated in Part I, this gives an alternate approach, though the proofs in [Ru1] and [LS] require weaker differentiability assumptions and are much more direct. Theorem C′ was first proved in the Axiom A dimension 2 case by Manning [Mg].

(7.5) *Definitions and the main results*: *nonergodic case*. Unlike the situation when m is ergodic, the functions $x \mapsto r(x)$, $\lambda_i(x)$ and $\dim E_i(x)$ (see (7.1)) are now no longer constant a.e. Let $u(x) = \max\{i\colon\ \lambda_i(x) > 0\}$. Then for $i = 1, \ldots, u(x)$, we can define $W^i(x)$ as in (7.1) except that λ_i should now be relaced by $\lambda_i(x)$.

Let $\Gamma_i = \{x \in \Gamma'\colon\ i \leq u(x)\}$. A measurable partition ξ of M is said to be subordinate to W^i on Γ_i if for m-a.e. $x \in \Gamma_i$, $\xi(x) \subset W^i(x)$ and contains an open neighborhood of x in $W^i(x)$. We extend the notion of entropy along W^i and dimension of m on W^i-manifolds to the nonergodic case using conditional measures for partitions of this type. These extensions are rather formal. To make geometric sense out of it the reader should consider sets of the form $\Gamma_{i,k} = \{x \in \Gamma_i\colon\ \sum_{j \leq i} \dim E_j(x) = k\}$ one at a time or simply disintegrate m into its ergodic components. Note that the entire leaf $W^i(x)$ is contained in the ergodic component of x (see e.g. (6.2)).

First we define the notion of entropy along W^i, which is now a class of measurable functions defined on Γ_i. Let ξ be a measurable partition subordinate to W^i with conditional measures $\{m_x\}$. Define h_i: $\Gamma_i \to \mathbf{R}$ by

$$\begin{aligned} h_i(x) &= h_i(x, \xi, \{m_x\}) \\ &= \lim_{\varepsilon \to 0} \liminf_{n \to 0} -\frac{1}{n} \log m_x V^i(x, n, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to 0} -\frac{1}{n} \log m_x V^i(x, n, \varepsilon). \end{aligned}$$

We know that the lim inf and lim sup definitions coincide m-a.e. on Γ_i because Proposition 7.2.1 says they do for the ergodic components of m. To verify that h_i is indeed a well defined class of measurable functions one must show that if ξ' is a measurable partition of the same type as ξ and $\{m'_x\}$ is a system of conditional measures for ξ' then $h_i(x, \xi, \{m_x\}) = h_i(x, \xi', \{m'_x\})$ m-a.e. on Γ_i.

Corollary 7.2.2 now reads

$$\int h_{u(x)}(x)\, dm(x) = h_m(f).$$

Similarly there are two measurable functions (or more precisely two classes of measurable functions) $\underline{\delta}_i, \overline{\delta}_i$: $\Gamma_i \to \mathbf{R}$ such that if ξ is a measurable partition subordinate to W^i on Γ_i and $\{m_x\}$ is a system of conditional measures for ξ, then at m-a.e. $x \in \Gamma_i$,

$$\liminf_{\varepsilon \to 0} \frac{\log m_x B^i(x, \varepsilon)}{\log \varepsilon} = \underline{\delta}_i(x)$$

and

$$\limsup_{\varepsilon \to 0} \frac{\log m_x B^i(x, \varepsilon)}{\log \varepsilon} = \overline{\delta}_i(x).$$

Propositions 7.3.1 and 7.3.2 are valid (so we can write $\delta_i = \underline{\delta}_i = \overline{\delta}_i$) again because they are valid for the ergodic components of m. The function δ_i is called the dimension of m on W^i-manifolds.

We are finally ready to state the main result of this paper:

THEOREM C. *Let f: $M \hookleftarrow$ be a C^2 diffeomorphism and let m be an f-invariant Borel probability measure on M. Let $\lambda_1(x) > \cdots > \lambda_{u(x)}(x)$ be the distinct positive exponents at x and let δ_i: $\Gamma_i \to \mathbf{R}$ be the dimension of m on W^i. Then there exist measurable functions γ_i: $\Gamma_i \to \mathbf{R}$ with $0 \le \gamma_i(x) \le \dim E_i(x)$ such that at m-a.e. x,*

$$\delta_i(x) = \sum_{j \le i} \gamma_j(x)$$

for $i = 1, \dots, u(x)$ and

$$h_m(f) = \int \sum_{i \le u(x)} \lambda_i(x) \gamma_i(x)\, dm(x).$$

Theorem C follows immediately from Theorem C$'$ by decomposition of m into ergodic components.

The nonergodic versions of Corollaries D$'$, E$'$, 7.4.1 and 7.4.2 are obvious.

(7.6) *A volume lemma and some consequences.* Theorem C and Corollary D give some information on the dimension of the conditional measures of m on unstable manifolds. We wish now to discuss the dimension of m itself. Since this discussion does not involve the other foliations, we shall refer exclusively to $W^{u(x)}$ when we speak of unstable manifolds—which we denote simply by W^u. Note also that no ergodicity is assumed in this section unless otherwise stated.

THEOREM F. *Let $f: M \hookleftarrow$ be a C^2 diffeomorphism of a compact Riemannian manifold and let m be an f-invariant Borel probability measure on M. Let δ^u be the dimension of m on W^u, δ^s be the dimension of m on W^s, and $\delta^c(x)$ be the multiplicity of* 0 *as an exponent at x. Then at m*-a.e. x,

$$\limsup_{\varepsilon \to 0} \frac{\log mB(x, \varepsilon)}{\log \varepsilon} \leq \delta^u(x) + \delta^s(x) + \delta^c(x).$$

Note that $\delta^u + \delta^s + \delta^c = \sum_i \gamma_i$. The influence of the neutral exponent on dimension is unpredictable and without further assumptions one cannot expect a sharper estimate. Theorem F is proved in Section 12.

Let us say that a measure on M has *full dimension* if at m-a.e. x,

$$\limsup_{\varepsilon \to 0} \frac{\log mB(x, \varepsilon)}{\log \varepsilon} = \dim M.$$

COROLLARY G. *Let $f: M \hookleftarrow$ be a C^2 diffeomorphism of a compact Riemannian manifold and let m be an f-invariant Borel probability measure with full dimension. Then m has absolutely continuous conditional measures on both stable and unstable manifolds and the following formulas hold*:

$$h_m(f) = \int \sum_i \lambda_i^+ \dim E_i \, dm,$$

$$h_m(f) = \int \sum_i \lambda_i^- \dim E_i \, dm,$$

and

$$\int \log|\mathrm{Det}\, Df| \, dm = 0.$$

Whether a measure m has full dimension is experimentally relatively easy to verify. Note also that full dimensionality is a weaker assumption than absolute continuity to Lebesgue. Hence Corollary G can be regarded as a slight extension of some of the results in [Pe].

COROLLARY H. *Let f be a C^2 diffeomorphism of a compact Riemannian manifold. An invariant Borel probability measure with full dimension is absolutely continuous with respect to Riemannian measure if any one of the following conditions hold:*

(1) *There is no zero exponent, i.e.* $\delta^c(x) = 0$ m-a.e.;

(2) $\delta^c(x) = 1$ m-a.e. *and f is the time-one map of a flow generated by a C^2 vector field;*

(3) *M is an n-torus and f is an ergodic algebraic automorphism.*

Assertions (1) and (2) follow from absolute continuity properties of the stable foliation. See [Be] for (3).

When the measure is not of full dimension, our results are related to a conjecture of Yorke's (see [FKYY]). Suppose m is ergodic. We define the Lyapunov dimension of m, written $\operatorname{Lyap\,dim}(m)$, to be $\dim M$ if $\Sigma_i \lambda_i \dim E_i \geq 0$. Otherwise define

$$\operatorname{Lyap\,dim}(m) = \sum_{j<i} \dim E_j + \frac{\Sigma_{j<i} \lambda_j \dim E_j}{|\lambda_i|}$$

where $i < r$ is the largest integer with $\Sigma_{j<i} \lambda_j \dim E_j \geq 0$.

COROLLARY I. *Let f: $M \hookleftarrow$ be a C^2 diffeomorphism of a compact Riemannian manifold and let m be an f-invariant ergodic Borel probability measure on M. Then*

$$\limsup_{\varepsilon \to 0} \frac{\log mB(x, \varepsilon)}{\log \varepsilon} \leq \operatorname{Lyap\,dim}(m)$$

at m-a.e. x. Equality holds almost everywhere only if m has absolutely continuous conditional measures on unstable manifolds. In fact, if equality holds, then there is some i with $\gamma_j = \dim E_j$ for $j < i$ and $\gamma_j = 0$ for $j > i$, γ_j being the numbers in Corollary E′.

A weaker inequality involving lim inf can be obtained by a simpler proof. (See [L].)

To prove Corollary I, observe that by 7.3.2 we have

$$(*) \qquad \sum_j (\lambda_j - \lambda_i)\gamma_j \leq \sum_{j<i} (\lambda_j - \lambda_i) \dim E_j.$$

Dividing by $-\lambda_i$ and applying Corollary E′, we get

$$\sum_j \gamma_j \leq \operatorname{Lyap\,dim}(m)$$

which together with Theorem F gives the inequality in Corollary I. Suppose now that equality is attained. This forces equality in (*), from which follows the last assertion in Corollary I. That m has absolutely continuous conditional measures in this case is a consequence of Theorem A and the fact that $i \geq u$.

8. More on local unstable manifolds

In the first half of Section 8 we record some estimates in Lyapunov charts. (8.1) and (8.2) parallel (2.1), (2.2), (2.3) and (4.2). For the convenience of the reader we reintroduce the notation here, referring to Part I (or [Ru2]) for proofs. The second half of Section 8 contains a description of some special coordinates on W^u-manifolds. These coordinates are used mainly in Section 11.

(8.1) *Lyapunov charts*. As usual, d denotes the Riemannian metric on M. We write $\mathbf{R}^{\dim M} = \mathbf{R}^{\dim E_1} \times \cdots \times \mathbf{R}^{\dim E_r}$ and for $x \in \mathbf{R}^{\dim M}$, let $(x_1, \ldots, x_r)$ be its coordinates with respect to this splitting. Define

$$|x| = \max_i |x_i|_i$$

where $|\cdot|_i$ is the Euclidean norm on $\mathbf{R}^{\dim E_i}$. Let

$$R^i(\rho) = \left\{x_i \in \mathbf{R}^{\dim E_i}\colon |x_i|_i \leq \rho\right\}$$

and $R(\rho) = \{x \in \mathbf{R}^{\dim M}\colon |x| \leq \rho\}$.

We now describe some changes of coordinates. Given $\varepsilon < (1/100) \times \min_{a \neq b \in S}|a - b|$ where $S = \{0\} \cup \{\lambda_1, \ldots, \lambda_r\}$, there exists a measurable function $l\colon \Gamma' \to (1, \infty)$ with $l(f^{\pm}x) \leq e^{\varepsilon} l(x)$ and an embedding $\Phi_x\colon R(l(x)^{-1}) \to M$ for each $x \in \Gamma'$ such that the following five conditions hold.

(i) $\Phi_x 0 = x$, and $D\Phi_x(0)$ takes $\mathbf{R}^{\dim E_i}$ to $E_i(x)$ for $i = 1, \ldots, r$.

(ii) $\exp_x^{-1} \circ \Phi_x$ coincides with $D\Phi_x(0)$ on $R(l(x)^{-1})$.

(iii) Let $\tilde{f}_x = \Phi_{fx}^{-1} \circ f \circ \Phi_x$ be the connecting map between the chart at x and the chart at fx, defined wherever it makes sense, and let $\tilde{f}_x^{-1} = \Phi_{f^{-1}x}^{-1} \circ f^{-1} \circ \Phi_x$ be defined analogously. Then for all $v \in \mathbf{R}^{\dim E_i}$,

$$e^{\lambda_i - \varepsilon}|v| \leq |D\tilde{f}_x(0)v| \leq e^{\lambda_i + \varepsilon}|v|.$$

(iv) If $L(g)$ denotes the Lipschitz constant of the function g, then

$$L\big(\tilde{f}_x - D\tilde{f}_x(0)\big) \leq \varepsilon,$$
$$L\big(\tilde{f}_x^{-1} - D\tilde{f}_x^{-1}(0)\big) \leq \varepsilon$$

and

$$L\big(D\tilde{f}_x\big), L\big(D\tilde{f}_x^{-1}\big) \leq l(x).$$

(v) For all $z, z' \in R(l(x)^{-1})$,

$$K_r^{-1} d(\Phi_x z, \Phi_x z') \leq |z - z'| \leq l(x) d(\Phi_x z, \Phi_x z')$$

for some universal constant K_r.

Throughout Part II we shall refer to any system of local charts $\{\Phi_x: x \in \Gamma'\}$ satisfying (i)–(v) above as (ε, l)-charts. This definition of (ε, l)-charts is more restrictive than the one given in Part I, where no attempt was made to control the behavior in individual Oseledec subspaces. The construction of Lyapunov charts given in the appendix to Part I extends easily to meet these additional requirements.

(8.2) *Local unstable manifold estimates.* Let $\{\Phi_x: x \in \Gamma'\}$ be a system of (ε, l)-charts. For $x \in \Gamma'$ and $1 \le i \le u$, let $W_x^i(x)$ be the subset of $R(l(x)^{-1})$ the Φ_x-image of which is the component of $W^i(x) \cap \Phi_x R(l(x)^{-1})$ containing x. We write $R^{(i)}(\rho) = R^1(\rho) \times \cdots \times R^i(\rho)$ and $R^{r-(i)}(\rho) = R^{i+1}(\rho) \times \cdots \times R^r(\rho)$.

Lemma 8.2.1. *There is a function g_x^i: $R^{(i)}(l(x)^{-1}) \to R^{r-(i)}(l(x)^{-1})$ such that*

(1) $g_x^i(0) = 0$, $Dg_x^i(0) = 0$,
(2) $\|Dg_x^i\| \le \frac{1}{3}$, *and*
(3) $\operatorname{graph}(g_x^i) = W_x^i(x)$.

Clearly, $\tilde{f}_x^{-1} W_x^i(x) \subset W_{f^{-1}x}^i(f^{-1}x)$.

Lemma 8.2.2. *Let $x \in \Gamma'$. Then there exists τ, $0 < \tau < \frac{1}{2}$, such that for all $y \in \Phi_x(W_x^u(x) \cap R(\tau l(x)^{-1}))$ and $1 \le i < u$, there is a function $g_{x,y}^i$: $R^{(i)}(l(x)^{-1}) \to R^{r-(i)}(l(x)^{-1})$ such that*

(1) $\Phi_x^{-1} y \in \operatorname{graph} g_{x,y}^i$,
(2) $\|Dg_{x,y}^i\| \le \frac{1}{3}$, *and*
(3)

$$\operatorname{graph} g_{x,y}^i = \left\{ z \in W_x^u(x) \colon \limsup_{n\to\infty} \frac{1}{n} \log |\tilde{f}_x^{-n} z - \tilde{f}_x^{-n} \Phi_x^{-1} y| \le -\lambda_i + 2\varepsilon \right\}.$$

We denote $W_x^u(x) \cap R(\tau l(x)^{-1})$ by $W_{x,\tau}^u(x)$. Note that $W_{x,\tau}^u(x) = \operatorname{graph}(g_x^u | R^{(u)}(\tau l(x)^{-1}))$. Also we denote the graph of $g_{x,y}^i$ by $W_x^i(y)$ and call its Φ_x-image a *local i^{th} unstable manifold* at y. The following corollary is an easy consequence of 8.2.2(3).

Corollary 8.2.3. *Let $x \in \Gamma'$ and let τ be as in 8.2.2. Then for all $y \in \Phi_x W_{x,\tau}^u(x)$,*

(1) $W_x^i(y) \subset W_x^{i+1}(y)$,
(2) *If $z \in \Phi_x W_{x,\tau}^u(x)$, then either $W_x^i(y) = W_x^i(z)$ or $W_x^i(y) \cap W_x^i(z) = \varnothing$, and*
(3) *If $y \in \Gamma'$, then $\Phi_x^{-1} W^i(y) \cap W_x^u(x) = W_x^i(y)$.*

Assertion (2) above says that $\{W_x^i(y): y \in \Phi_x W_{x,\tau}^u(x)\}$ is a foliation on some set containing $W_{x,\tau}^u(x)$. (1) says that for different i's these foliations are "nested", and (3) says that for each i the global and local W^i-foliations induce the same partition on $W_{x,\tau}^u(x)$, a fact we will need in Section 11.

LEMMA 8.2.4. *Let $x \in \Gamma'$.*
(1) *If $z, z' \in R(e^{-\lambda_1 - 3\varepsilon} l(x)^{-1})$ then $\tilde{f}_x z, \tilde{f}_x z' \in R(l(fx)^{-1})$ and*

$$|\tilde{f}_x z - \tilde{f}_x z'| \leq |z - z'| e^{\lambda_1 + 2\varepsilon}.$$

(2) *If $y \in \Phi_x W_{x,\tau}^u(x)$ and $z, z' \in W_x^i(y)$, then*

$$|\tilde{f}_x^{-1} z - \tilde{f}_x^{-1} z'| \leq e^{-\lambda_i + 2\varepsilon} |z - z'|,$$

Finally we state a lemma that tells us that the W^i-foliation inside local W^u-leaves is at least Lipschitz. Let $L(\mathbf{R}^n, \mathbf{R}^k)$ denote the space of linear maps from $\mathbf{R}^n$ to $\mathbf{R}^k$. For $x \in \Gamma'$ and $1 \leq i \leq u$, let G_x^i: $\bigcup_{y \in \Phi_x W_{x,\tau}^u(x)} W_x^i(y) \to L(\mathbf{R}^{\Sigma_{j \leq i} \dim E_j}, \mathbf{R}^{\Sigma_{i < j \leq r} \dim E_j})$ be defined by

$$G_x^i(z) = Dg_{x,y}^i(z_1, \ldots, z_i)$$

where $y \in \Phi_x W_{x,\tau}^u(x)$ and $z \in W_x^i(y)$. In particular, if $\Phi_x z \in \Gamma'$, then $D\Phi_x(z)(\text{graph } G_x^i(z)) = \oplus_{j \leq i} E_j(\Phi_x z)$.

LEMMA 8.2.5. *There is a constant $D > 0$ such that for all $x \in \Gamma'$ and for every $1 \leq i \leq u$, the map G_x^i is Lipschitz with* Lip constant $\leq Dl(x)$.

The proof is identical to that in (4.2).

(8.3) *Special coordinates on local unstable manifolds*. For each $x \in \Gamma'$ we now define a coordinate system on the local W^u-manifold at x. These coordinates "straighten out" local W^i-manifolds into parallel planes and at the same time preserve good dynamical estimates.

We first state a lemma about Lipschitz foliations that has nothing to do with dynamics. The proof is quite straightforward and will be omitted. Fix positive integers $n_1, \ldots, n_k$ and a number $0 < \rho < 1$. Let $B^i(\rho)$ be a closed disk centered at 0 of radius ρ in $\mathbf{R}^{n_i}$. Consider $B(\rho) = B^1(\rho) \times \cdots \times B^k(\rho)$ as a subset of $\mathbf{R}^{n_1 + \cdots + n_k}$.

LEMMA 8.3.1 (*straightening out lemma*). *For $i = 1, \ldots, k-1$, let F_i be a Lipschitz foliation with C^1 leaves on some subset of $\mathbf{R}^{n_1 + \cdots + n_k}$ containing $B(\rho)$. Assume that each leaf of F_i is the graph of a function*

$$g\colon B^1(2\rho) \times \cdots \times B^i(2\rho) \to \mathbf{R}^{n_{i+1} + \cdots + n_k}$$

with $\|Dg\| \leq \frac{1}{3}$ and that the function $x \mapsto T_x F_i$ has Lipschitz constant $\leq$ some number C. Assume also that the F_i's are nested; i.e., if $F_i(x)$ denotes the leaf of

F_i containing the point x, then $F_i(x) \subset F_{i+1}(x)$ for all $x \in B(\rho)$ and for all i. Define $\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_k)$: $B(\rho) \to \mathbf{R}^{n_1 + \cdots + n_k}$ as follows: For $x = (x_1, \dots, x_k) \in B(\rho)$, let $\mathcal{O}_1(x) = x_1$, and let $\mathcal{O}_i(x)$ be the i^{th} coordinate of the unique point of intersection of $F_{i-1}(x)$ and $\{0\} \times \cdots \times \{0\} \times \mathbf{R}^{n_i + \cdots + n_k}$ for $i \geq 2$. Then

(1) $\mathcal{O}$ is a homeomorphism between $B(\rho)$ and its image;

(2) For every $x, y \in B(\rho)$, $\mathcal{O}_j x = \mathcal{O}_j y$ for $j = i+1, \dots, u$ if and only if $y \in F_i(x)$ and

(3) Both $\mathcal{O}$ and $\mathcal{O}^{-1}$ are Lipschitz with Lip constant depending only on C (assuming k is fixed).

Consider now a system of Lyapunov charts $\{\Phi_x\}$. For $x \in \Gamma'$ let p_x: $R(l(x)^{-1}) \to R^{(u)}(l(x)^{-1})$ be the natural projection. Then $p_x | W_x^u(x)$ is a lipeomorphism and the p_x-images of $W_x^i(y)$, $y \in \Phi_x W_{x,\tau}^u(x)$, form the leaves of a Lipschitz foliation F_i. It is easy to check that these F_i's ($i = 1, \dots, u-1$) satisfy the hypotheses of 8.3.1 with $B(\rho) = R^{(u)}(\tau l(x)^{-1})$ and $C = Dl(x)$ (see 8.2.5). Let $\mathcal{O}_x$: $R^{(u)}(\tau l(x)^{-1}) \to \mathbf{R}^{\Sigma_{i \leq u} \dim E_i}$ be the map in 8.3.1.

We are now ready to define our "special coordinates" on $W_{x,\tau}^u(x)$. Let π_x: $W_{x,\tau}^u(x) \to \mathbf{R}^{\Sigma_{i \leq u} \dim E_i}$ be given by

$$\pi_x = \mathcal{O}_x \circ p_x.$$

From 8.3.1 we can conclude that π_x is a lipeomorphism between $W_{x,\tau}^u(x)$ and its image with $\operatorname{Lip}(\pi_x), \operatorname{Lip}(\pi_x^{-1}) \leq N_x$, where N_x depends only on $l(x)$ (and other constants determined by f). Moreover, $\pi_x W_x^i(y)$ lies on a $(\Sigma_{j \leq i} \dim E_j)$-dimensional plane parallel to $\mathbf{R}^{\Sigma_{j \leq i} \dim E_j} \times \{0\} \times \cdots \times \{0\}$ and if $W_x^i(y) \neq W_x^i(y')$ then $\pi_x W_x^i(y)$ and $\pi_x W_x^i(y')$ lie on distinct planes.

LEMMA 8.3.2. *Let $z \in W_{x,\tau}^u(x)$ and suppose $|z| \leq \tau e^{-\lambda_1 - 3\varepsilon}$. When $\pi_x = (\pi_x^1, \dots, \pi_x^u)$,*

$$|\pi_{fx}^i \tilde{f}_x z| \leq e^{\lambda_i + 3\varepsilon} |\pi_x^i z| \qquad \text{for } i = 1, \dots, u.$$

Proof. Make the estimate one i at a time. For $i \geq 2$ the proof is identical to that of Lemma 2.3.2.

(8.4) *Special coordinates on a set of positive measure.* In (8.3) we describe some coordinates on local unstable manifolds that have desirable geometric as well as dynamical properties. These coordinates, however, are not compatible in the sense that for $D \subset \Phi_x W_{x,\tau}^u(x) \cap \Phi_{x'} W_{x',\tau}^u(x')$, $(\pi_x \circ \Phi_x^{-1}) | D$ need not be equal to $(\pi_{x'} \circ \Phi_{x'}^{-1}) | D$. The aim of this subsection is to construct a map π: $S \to \mathbf{R}^{\Sigma_{i \leq u} \dim E_i}$, where S is a set of positive measure, such that π restricted to each local W^u-leaf in S straightens out W^i-leaves as in (8.3).

Let l_0 be chosen so that $\Lambda = \{l \leq l_0\}$ has positive m-measure. For $x \in \Gamma'$, let $E^i(x) = \oplus_{j \leq i} E_j(x)$. As mentioned in (8.2), $E^i(y)$, $i = 1, \dots, u$, is actually defined for every $y \in \tilde{\Lambda} = \bigcup_{x \in \Lambda} \Phi_x W^u_{x,\tau}(x)$. In fact, there exists a constant l'_0 such that for every $y \in \tilde{\Lambda}$, $\|Df_y^{-n} v\| \leq l'_0 e^{-(\lambda_i - 3\varepsilon)n} \|v\|$ for all $n \geq 0$ and for all $v \in E^i(y)$. This implies that $y \mapsto E^i(y)$ is *continuous* on $\tilde{\Lambda}$.

Let $w \in \Lambda$ be a density point of $\chi_\Lambda \cdot m$. Then using the continuity of $y \mapsto E^i(y)$ on $\tilde{\Lambda}$, we can choose τ_0, $0 < \tau_0 < \tau$, so that for all $x \in A_{\tau_0} = \Lambda \cap \Phi_w R(\tau_0 l_0^{-1})$, if $y \in \Phi_x W^u_{x,\tau}(x) \cap \Phi_w R(\tau_0 l_0^{-1})$, then for each $i = 1, \dots, u$, $\Phi_w^{-1} \Phi_x W^i_x(y) \cap R(2\tau_0 l_0^{-1})$ is the graph of a function from $R^{(i)}(2\tau_0 l_0^{-1}) \to R^{r-(i)}(2\tau_0 l_0^{-1})$, the norm of the derivative of which is $\leq \frac{1}{3}$. For each $x \in A_{\tau_0}$ then, Lemma 8.3.1 gives a map $\mathcal{O}_{w,x}$: $R^{(u)}(\tau_0 l_0^{-1}) \to \mathbf{R}^{\sum_{i \leq u} \dim E_i}$ that straightens out the foliation whose leaves are subsets of $p_w \Phi_w^{-1} \Phi_x W^i_x(y)$.

Let $S = \bigcup_{x \in A_{\tau_0}} (\Phi_x W^u_x(x) \cap \Phi_w R(\tau_0 l_0^{-1}))$. Define π: $S \to \mathbf{R}^{\sum_{i \leq u} \dim E_i}$ by

$$\pi(y) = \mathcal{O}_{w,x} \circ p_w \circ \Phi_w^{-1}(y)$$

where $y \in \Phi_x W^u_x(x)$, $x \in A_{\tau_0}$. The reader should verify that this map is well-defined; i.e. the definition of $\pi(y)$ is independent of the choice of x.

We note also that

(1) S is the disjoint union of sets D_α where each $D_\alpha \cap \{l \leq l_0\} \neq \varnothing$, and if $x \in D_\alpha \cap \{l \leq l_0\}$ then $D_\alpha = S \cap \Phi_x W^u_x(x)$; also, $m(S \cap \{l \leq l_0\}) > 0$;

(2) Because $W^i|S$, $i = 1, \dots, u$, is a continuous foliation, it is easy to see from our definition of $\mathcal{O}$ in Lemma 8.3.1 that π is in fact a continuous map; and

(3) With the notation in (1), for each α, $\pi|D_\alpha$ and $(\pi|D_\alpha)^{-1}$ are Lipschitz maps with Lip constant $\leq$ some constant N_0.

9. Entropy along unstable foliations

(9.1) *Increasing partitions subordinate to W^i.* Let η_1 and η_2 be measurable partitions on M. Recall the following definitions: η_1 *refines* $\eta_2 (\eta_1 > \eta_2)$ if $\eta_1(x) \subset \eta_2(x)$ for m-a.e. x, and η_1 is *increasing* if $\eta_1 > f\eta_1$. In (3.1) we described certain increasing partitions that are subordinate to W^u. In this subsection we construct similar partitions subordinate to W^i, $i = 1, \dots, u$.

LEMMA 9.1.1. *There exist measurable partitions $\xi_1, \dots, \xi_u$ on M such that for each i,*

(1) ξ_i is an increasing partition subordinate to W^i,

(2) $\xi_i > \xi_{i+1}$ and

(3) $f^{-n}\xi_i$ generates as $n \to \infty$.

Outline of proof. We take a system of (ε, l)-local charts $\{\Phi_x\}$ and choose l_0 and $S = \bigcup D_\alpha$ as in (8.4). For $i = 1, \dots, u$, let

$$\hat{\xi}_i(x) = \begin{cases} W^i(x) \cap D_\alpha & \text{if } x \in D_\alpha \\ M - S & \text{if } x \notin S. \end{cases}$$

It follows from our definition of S and 8.2.3(3) that if $x \in D_\alpha \cap \{l \le l_0\}$, then every $\hat{\xi}_i$-element contained in D_α is equal to $S \cap \Phi_x W_x^i(y)$ for some $y \in \Phi_x W_{x,\tau}^u(x)$. Let $\xi_i = \bigvee_{n \ge 0} f^n \hat{\xi}_i$. Since $\hat{\xi}_1 > \cdots > \hat{\xi}_u$, obviously we have $\xi_1 > \cdots > \xi_u$.

Some care has to be taken to ensure that $\xi_i(x)$ contains an open neighborhood of x in $W^i(x)$ for m-a.e. x. (In fact, Lebesgue-a.e. τ_0 small enough for purposes (8.4) will do.) We omit further details because this argument is identical to that in [LS].

Note that ξ_i has the following characterization. Let $x \in \Gamma'$. Then $y \in \xi_i(x)$ if and only if for all $n \ge 0$,

(1) $f^{-n}y \in S$ if and only if $f^{-n}x \in S$,

(2) $f^{-n}x, f^{-n}y$ lie in the same D_α whenever $f^{-n}x \in S$, and

(3) $y \in W^i(x)$.

In (9.2) and (9.3) we shall prove

$$\lim_{\varepsilon \to 0} \underline{h}_i(x, \varepsilon, \xi_i) = H(\xi_i | f\xi_i) = \lim_{\varepsilon \to 0} \overline{h}_i(x, \varepsilon, \xi_i).$$

Proposition 7.2.1 follows immediately from this. Remark also that if ξ_i' is another increasing partition subordinate to W^i constructed in the proof of 9.1.1, then $H(\xi_i' | f\xi_i') = h_i$ as well.

In (9.2) and (9.3) let ξ_i $(i = 1, \ldots, u)$ be as above and for each i let $\{m_x^i\}$ be a system of conditional measures associated with ξ_i.

(9.2) *Proof that* $\underline{h}_i \ge H(\xi_i | f\xi_i)$. Let $\varepsilon > 0$ be given. For $\delta > 0$, we let $A_\delta = \{x\colon\ B^i(x, \delta) \subset \xi_i(x)\}$. Then $mA_\delta \uparrow 1$ as $\delta \downarrow 0$. Let $g(x) = -\log m_x^i(f^{-1}\xi_i)(x)$ and choose δ' small enough that

$$\int_{A_{\delta'}} g \ge H(f^{-1}\xi_i | \xi_i) - \varepsilon = H(\xi_i | f\xi_i) - \varepsilon.$$

Define

$$U^i(x, n, \delta) = \bigcap_{\substack{\{j:\, 0 \le j \le n \\ \text{and } f^j x \in A_\delta\}}} (f^{-j}\xi_i)(x).$$

Then by definition $V^i(x, n, \delta) \subset U^i(x, n, \delta)$ and $-\log m_x^i U^i(x, n, \delta) \ge \sum_{j=0}^n (\chi_{A_\delta} \cdot g)(f^j x)$. Thus whenever $\delta \le \delta'$ we have for m-a.e. x,

$$\begin{aligned} \underline{h}_i(x, \delta, \xi_i) &= \liminf_{n \to \infty} \frac{1}{n} \log m_x^i V^i(x, n, \delta) \\ &\ge \int_{A_\delta} g \\ &\ge H(\xi_i | f\xi_i) - \varepsilon. \end{aligned}$$

(9.3) *Proof that* $\overline{h}_i \le H(\xi_i | f\xi_i)$.

LEMMA 9.3.1. *Let $\mathscr{P}$ be a partition with $H_m(\mathscr{P}) < \infty$. Then for m-a.e. x,*

$$\lim_{n\to\infty} -\frac{1}{n}\log m_x^i(\mathscr{P}\vee \xi_i)_0^n(x) = H(\xi_i|f\xi_i)$$

where $\alpha_0^n(x)$ denotes the element of the partition $\bigvee_{j=0}^{n-1} f^{-j}\alpha$ that contains x.

Proof. Define the information function of a partition $\mathscr{Q}$ given η by $I(\mathscr{Q}|\eta)(x) = -\log m_x\mathscr{Q}(x)$ where $\{m_x\}$ is a family of conditional measures associated with η. Then

$$\frac{1}{n}I\big((\mathscr{P}\vee\xi_i)_0^n|\xi_i\big)(x)$$
$$= \frac{1}{n}I(\mathscr{P}|\xi_i)(x) + \frac{1}{n}\sum_{k=1}^{n-1} I\big(\mathscr{P}\vee\xi_i|f\xi_i\vee\mathscr{P}_{-k}^0\big)(f^kx).$$

Since $\sup_n I(\mathscr{P}\vee\xi_i|f\xi_i\vee\mathscr{P}_{-n}^0)$ is an integrable function, the second term above converges m-a.e. and in L^1 (see [Pa]). By ergodicity the limit function is constant almost everywhere and is therefore equal to

$$\lim_{n\to\infty}\frac{1}{n}H\big((\mathscr{P}\vee\xi_i)_0^n|\xi_i\big)$$

which can be written as

$$\lim_{n\to\infty}\frac{1}{n}H\big((\xi_i)_0^n|\xi_i\big) + \lim_{n\to\infty}\frac{1}{n}H\big(\mathscr{P}_0^n|f^{-n+1}\xi_i\big).$$

The first term is equal to $H(\xi_i|f\xi_i)$. The second term goes to 0 since $f^{-n}\xi_i$ generates. $\square$

We now construct a partition $\mathscr{P}$ where $\mathscr{P}_0^n(x)$ can be compared to $V^i(x,n,\varepsilon)$. Let $\{\Phi_x\}$, S and l_0 be as in (9.1) and let $S' = S\cap\{l\le l_0\}$. Then $mS'>0$. First we define n_+ and $n_-: S'\to\mathbf{N}$ by

$$n_+(x) = \operatorname{Inf}\{n>0: f^nx\in S'\}$$

and

$$n_-(x) = \operatorname{Inf}\{n>0: f^{-n}x\in S'\}.$$

Then we let $\psi: M\to\mathbf{R}$ be given by

$$\psi(x) = \begin{cases} \varepsilon' K_r^{-1} l_0^{-2} e^{-\max\{\lambda_1+3\varepsilon,\, -\lambda_r+3\varepsilon\}\cdot\max\{n_+(x),\, n_-(x)\}} & \text{if } x\in S' \\ 1 & \text{otherwise}\end{cases}$$

where ε' is a preassigned small number. Since $\int -\log\psi\, dm<\infty$, there is a partition $\mathscr{P}$ with $H(\mathscr{P})<\infty$ such that $\mathscr{P}(x)\subset B(x,\psi(x))$ for almost every x. (See [Mé] or (2.4) for a similar construction.) Finally we define ψ_+ by replacing $\max(n_+,n_-)$ in the definition of ψ by n_+.

LEMMA 9.3.2. *Let* $x \in S'$.

(1) *If* $y \in \Phi_x W_x^i(x)$ *satisfies* $|\Phi_x^{-1} y| \le l_0 \psi_+(x)$, *then*

$$d^i(f^j x, f^j y) < \varepsilon'$$

for $0 \le j \le n_+(x)$ *and*

$$f^{n_+(x)} y \in \Phi_{f^{n_+(x)}x} W^i_{f^{n_+(x)}x}(f^{n_+(x)} x).$$

(2) *If* $y \in \mathscr{P}_0^n(x) \cap \Phi_x W_x^i(x)$ *for some* $n \ge 0$, *then*

$$d^i(f^j x, f^j y) < \varepsilon'$$

for $0 \le j \le n$.

Proof. It follows from our assumptions on y and 8.2.4(1) that $\tilde{f}_x^j(\Phi_x^{-1} y) \in W^i_{f^j x}(f^j x)$ and

$$|\tilde{f}_x^j(\Phi_x^{-1} y)| \le |\Phi_x^{-1} y| e^{(\lambda_1 + 2\varepsilon) j} \qquad \text{for } j > 0$$

provided that $|\Phi_x^{-1} y| e^{(\lambda_1 + 2\varepsilon)k} \le l(f_x^k)^{-1} e^{-\lambda_1 - 3\varepsilon}$ for all $0 \le k < j$. This is guaranteed for $j \le n_+(x)$. Since $W^i_{f^j x}(f^j x)$ is a graph over $R^{(i)}(l(f^j x)^{-1})$ with slope $\le \frac{1}{3}$ and $|\cdot|$ is the max norm,

$$d^i(f^j x, f^j y) \le K_r |\Phi^{-1}_{f^j x}(f^j y)| < \varepsilon' \quad \text{for } 0 \le j \le n_+(x).$$

To prove (2), first observe that if $y \in \mathscr{P}(x)$, then $|\Phi_x^{-1} y| \le l_0 \psi(x) \le l_0 \psi_+(x)$ so that we have the desired conclusion for $0 \le j \le n_+(x)$. Furthermore, if $n > n_+(x)$ and $y \in \mathscr{P}_0^n(x)$, then

$$f^{n_+(x)} y \in \mathscr{P}(f^{n_+(x)} x) \cap \Phi_{f^{n_+(x)}x} W^i_{f^{n_+(x)}x}(f^{n_+(x)} x)$$

and we can apply (1) with $f^{n_+(x)} x$ and $f^{n_+(x)} y$ in place of x and y. An inductive argument completes the proof of (2).

LEMMA 9.3.3. *For every* $\varepsilon' > 0$, *there exists a partition* $\mathscr{P}$ *with* $H(\mathscr{P}) < \infty$ *and a function* n_0: $M \to \mathbf{N}$ *such that for* m-a.e. x,

$$\mathscr{P}_0^n(x) \cap \xi_i(x) \subset V^i(x, n, \varepsilon') \quad \textit{for all } n \ge n_0(x).$$

Proof. Let $\mathscr{P}$ be as above and let

$$n_0(x) = \operatorname{Inf}\{n \ge 0 \colon f^n x \in S'\}.$$

Consider an arbitrary point x with the property that $f^n x \in S'$ infinitely often as $n \to \pm\infty$. Let $y \in \mathscr{P}_0^{n_0(x)}(x) \cap \xi_i(x)$. We claim that $d^i(f^j x, f^j y) < \varepsilon'$ for $0 \le j \le n_0(x)$ and that $f^{n_0(x)} y \in \Phi_{f^{n_0(x)}x} W^i_{f^{n_0(x)}x}(f^{n_0(x)} x)$.

It suffices to prove this claim, for if $y \in \mathscr{P}_0^n(x) \cap \xi_i(x)$, then the second assertion above allows us to apply 9.3.2(2) to $f^{n_0(x)} y$, which then tells us that $d^i(f^j x, f^j y) < \varepsilon'$ for $n_0(x) \le j \le n$.

To prove the claim, let us assume that $x \notin S'$ and for simplicity of notation write $r_0 = -n_-(f^{n_0(x)}x) + n_0(x)$. That is, r_0 is the largest integer < 0 such that $f^{r_0}x \in S'$. Since ξ_i is increasing, we have $f^{r_0}y \in \xi_i(f^{r_0}x)$ which by our choice of S lies in $\Phi_{f^{r_0}x} W^i_{f^{r_0}x}(f^{r_0}x)$. Also, ψ is chosen in such a way that $f^{-j}\mathscr{P}(f^{n_0(x)}x)$ lies well inside the charts at $f^{-j+n_0(x)}x$ for $j = 1, 2, \ldots, n_-(f^{n_0(x)}x)$. Thus

$$|\Phi^{-1}_{f^{r_0}x} f^{r_0}y| \le |\Phi^{-1}_{f^{r_0+1}x} f^{r_0+1}y| \le \cdots \le |\Phi^{-1}_{f^{n_0(x)}x} f^{n_0(x)}y| \le l_0\psi(f^{n_0(x)}x),$$

which is $\le l_0\psi_+(f^{r_0}x)$. Lemma 9.2.2(1) gives exactly what we need. $\square$

Finally, combining Lemmas 9.3.1 and 9.3.3, we have at m-a.e. x and for every small ε',

$$\begin{aligned}\bar{h}_i(x, \varepsilon', \xi_i) &= \limsup_{n\to\infty} -\frac{1}{n}\log m^i_x V^i(x, n, \varepsilon') \\ &\le \limsup_{n\to\infty} -\frac{1}{n}\log m^i_x \mathscr{P}^n_0(x) \\ &\le \lim_{n\to\infty} -\frac{1}{n}\log m^i_x(\xi_i \vee \mathscr{P})^n_0(x) \\ &= H(\xi_i | f\xi_i).\end{aligned}$$

Proof of Corollary 7.2.2. It follows from (9.2), (9.3) and Corollary 5.3 that

$$h_u = H(\xi_u | f\xi_u) = h_m(f).$$

10. Relating entropy, exponents and dimension: Proof of Theorem C′

We now begin to prove Theorem C′. The five sets of inequalities listed below are dealt with separately:

(1) a. $\bar{\delta}_1 \le \dfrac{h_1}{\lambda_1}$,

b. $\underline{\delta}_1 \ge \dfrac{h_1}{\lambda_1}$,

(2) $h_i - h_{i-1} \ge \lambda_i(\bar{\delta}_i - \bar{\delta}_{i-1})$, $\quad 2 \le i \le u$,

(3) $h_i - h_{i-1} \le \lambda_i(\underline{\delta}_i - \underline{\delta}_{i-1})$, $\quad 2 \le i \le u$,

(4) $h_i - h_{i-1} \le \lambda_i \dim E_i$, $\quad 2 \le i \le u$,

(5) $h_u = h_m(f)$.

Inequalities (1)a. and b. tell us that $\underline{\delta}_1 = \bar{\delta}_1$ and that $h_1 = \delta_1\lambda_1$. This together with (2) and (3) prove inductively that for $i = 2, \ldots, u$, $\underline{\delta}_i = \bar{\delta}_i$ (Proposition 7.3.1) and $h_i - h_{i-1} = \lambda_i(\delta_i - \delta_{i-1})$. Adding these equations and setting $\gamma_i = \delta_i - \delta_{i-1}$ ($\delta_0 = 0$), we obtain $h_u = \sum_{i \le u}\gamma_i\lambda_i$. That $\gamma_i \le \dim E_i$ (Proposition 7.3.2) is a consequence of (4).

The fifth equation was stated as Corollary 7.2.2 and was proved in the last section. In Section 10 we prove (1) and (2). The remaining two inequalities involve explicit estimates on "transverse dimension", a notion we shall elucidate in Section 11.

(10.1) *Proof that* $h_1 = \delta_1\lambda_1$. Since there is only one rate of expansion along W^1 this proof mimics the arguments in [Y]. Let $\varepsilon > 0$ be given. Choose a system of (ε, l)-charts $\{\Phi_x\}$, an increasing partition ξ_1 subordinate to W^1 and a system of conditional measures $\{m_x^1\}$ for ξ_1. From Section 9, (9.2) in particular, we know that there exists $\delta > 0$ such that for m-a.e. x,

$$\underline{h}_1(x, K_r\delta, \xi_1) \geq h_1 - \varepsilon.$$

Consider $y \in B^1(x, e^{-n(\lambda_1+3\varepsilon)}\delta l(x)^{-2})$ and let $z = \Phi_x^{-1}y$. Then $z \in W_x^1(x)$ and $|z| \leq e^{-n(\lambda_1+3\varepsilon)}\delta l(x)^{-1}$. By 8.2.4(3), $|\tilde{f}_x^k z| \leq |z| e^{(\lambda_1+2\varepsilon)k}$ for $k > 0$ provided all iterates up through k stay well inside charts. This is guaranteed for $k = 1, \dots, n$. Thus $|\tilde{f}_x^k z| \leq \delta$ for $0 \leq k \leq n$ and since $\tilde{f}_x^k z \in W_{f^k x}^1(f^k x)$, we have shown that $B^1(x, e^{-n(\lambda_1+3\varepsilon)}\delta l(x)^{-2}) \subset V^1(x, n, K_r\delta)$. This gives

$$\begin{aligned}
\underline{\delta}_1(x) &= \liminf_{\rho\to 0} \frac{\log m_x^1 B^1(x,\rho)}{\log \rho} \\
&= \liminf_{n\to\infty} \frac{\log m_x^1 B^1\left(x, e^{-n(\lambda_1+3\varepsilon)}\delta l(x)^{-2}\right)}{-n(\lambda_1+3\varepsilon) + \log \delta l(x)^{-2}} \\
&\geq \frac{1}{\lambda_1 + 3\varepsilon} \liminf_{n\to\infty} -\frac{1}{n}\log m_x^1 V^1(x, n, K_r\delta) \\
&\geq \frac{h_1 - \varepsilon}{\lambda_1 + 3\varepsilon}
\end{aligned}$$

for m-a.e. x. The choice of ε being arbitrary, we have proved $\underline{\delta}_1 \geq (h_1/\lambda_1)$.

To prove the other inequality, let $\mathscr{P}$ be the partition in (9.3) with $i = 1$. It was shown in the proof of Lemma 9.3.3 that for almost every x, if $f^n x \in S'$, $n \geq 0$, then $f^n(\xi_1(x) \cap \mathscr{P}_0^n(x)) \subset \Phi_{f^n x} W_{f^n x}^1(f^n x)$. Fix x and $n \geq n_0(x)$. Consider $y \in \xi_1(x) \cap \mathscr{P}_0^n(x)$. Let k be the largest integer $< n$ such that $f^k x \in S'$. Since $f^k y \in \mathscr{P}(f^k x)$, $|\Phi_{f^k x}^{-1}(f^k y)| \leq l_0\psi(f^k x) \leq \varepsilon' K_r^{-1} l_0^{-1} e^{-(\lambda_1+3\varepsilon)(n-k)}$. Also, since $k \geq n_0$, $f^k y \in \Phi_{f^k x} W_{f^k x}^1(f^k x)$. It then follows from Lemma 8.2.4(2) that $|\Phi_x^{-1}y| \leq |\Phi_{f^k x}^{-1}(f^k y)| e^{-(\lambda_1-2\varepsilon)k} \leq K_r^{-1} e^{-(\lambda_1-2\varepsilon)n}$. Thus

$$\xi_1(x) \cap \mathscr{P}_0^n(x) \subset B^1(x, e^{-n(\lambda_1-2\varepsilon)})$$

for all $n \geq n_0(x)$. Proceeding as before, we have

$$\begin{aligned}\bar{\delta}_1(x) &= \limsup_{n\to\infty} \frac{\log m_x^1 B(x, e^{-n(\lambda_1 - 2\varepsilon)})}{-n(\lambda_1 - 2\varepsilon)} \\ &\leq \frac{1}{\lambda_1 - 2\varepsilon} \cdot \limsup_{n\to\infty} -\frac{1}{n}\log m_x^1 \mathscr{P}_0^n(x) \\ &\leq \frac{h_1}{\lambda_1 - 2\varepsilon}.\end{aligned}$$

We know from Corollary 4.1.4 that $\bar{\delta}_1 \leq \dim E_1$.

(10.2) *Proof that* $h_i - h_{i-1} \geq \lambda_i(\bar{\delta}_i - \bar{\delta}_{i-1})$. Let $\varepsilon > 0$ be given. Choose (ε, l)-charts $\{\Phi_x\}$ and measurable partitions ξ_i, $1 \leq i \leq u$, as in (9.1). In particular then, each ξ_i is an increasing partition subordinate to W^i and $\xi_1 > \cdots > \xi_u$. For each i, choose a system of conditional measures associated with ξ_i. We now fix i, $2 \leq i \leq u$, and focus on the relationships among the invariants defined on W^i and W^{i-1}.

LEMMA 10.2. *There exists a partition $\mathscr{P}$ with $H(\mathscr{P}) < \infty$ and a measurable function n_0: $M \to \mathbf{N}$ such that for m-a.e. x, the following four properties are satisfied for all $n \geq n_0(x)$:*

$$\frac{\log m_x^{i-1} B^{i-1}(x, e^{-n(\lambda_i - 2\varepsilon)})}{-n(\lambda_i - 2\varepsilon)} \leq \bar{\delta}_{i-1} + \varepsilon, \tag{1}$$

$$-\frac{1}{n}\log m_x^{i-1}\mathscr{P}_0^n(x) \geq h_{i-1} - \varepsilon, \tag{2}$$

$$\xi_i(x) \cap \mathscr{P}_0^n(x) \subset B^i(x, e^{-n(\lambda_i - 2\varepsilon)}), \tag{3}$$

$$-\frac{1}{n}\log m_x^i \mathscr{P}_0^n(x) \leq h_i + \varepsilon. \tag{4}$$

Proof. By definition of $\bar{\delta}_{i-1}$, there exists a function n_1: $M \to \mathbf{N}$ such that property (1) is satisfied for m-a.e. x and $n \geq n_1(x)$. By (9.2), we can choose ε^1 and n_2: $M \to \mathbf{N}$ such that for m-a.e. x and

$$n \geq n_2(x), \; -1/n \log m_x^{i-1} V^{i-1}(x, n, \varepsilon^1) \geq h_{i-1} - \varepsilon.$$

By 9.3.2 there are a partition $\mathscr{Q}$ and a function n_2': $M \to \mathbf{N}$ such that for m-a.e. x and $n \geq n_2'(x)$,

$$\mathscr{Q}_0^n(x) \cap \xi_{i-1}(x) \subset V^{i-1}(x, n, \varepsilon^1).$$

By 10.1.1 there is a partition $\mathscr{R}$ and a function n_3: $M \to \mathbf{N}$ such that for m-a.e. x and $n \geq n_3(x)$, $\xi_i(x) \cap \mathscr{R}_0^n(x) \subset B^i(x, e^{-n(\lambda_i - 2\varepsilon)})$. (Our argument in 10.1 is given for $i = 1$ but this is clearly valid for all $i = 1, \ldots, u$.) Let $\mathscr{P} = \mathscr{Q} \vee \mathscr{R}$. Then properties (2) and (3) are satisfied for $n \geq n_2(x)$, $n_2'(x)$ and $n_3(x)$. Finally, since $\mathscr{P}_0^n(x) \supset (\xi_i \vee \mathscr{P})_0^n(x)$, by 9.3.1 there is an n_4: $M \to \mathbf{R}$ such that for m-a.e. x and $n \geq n_4(x)$ property (4) is satisfied. We put $n_0 = \max(n_1, n_2, n_2', n_3, n_4)$ and the lemma is proved. □

For the remainder of this section let $\mathscr{P}$ and n_0 be as in the last lemma. Fix N_0 large enough that if $\Gamma = \{n_0 \leq N_0\}$ then $m\Gamma > 0$. Consider the following four additional conditions:

(5) $$L \overset{\text{def}}{=} B^{i-1}(x, e^{-n(\lambda_i - 2\varepsilon)}) \subset \xi_{i-1}(x),$$

(6) $$\frac{m_x^{i-1}(L \cap \Gamma)}{m_x^{i-1}L} \geq \frac{1}{2},$$

(7) $$\frac{\log m_x^i B^i(x, 2e^{-(\lambda_i - 2\varepsilon)n})}{-n(\lambda_i - 2\varepsilon)} \geq \bar{\delta}_i - \varepsilon \text{ and}$$

(8) $$\frac{1}{n \log 2} < \varepsilon.$$

We claim that for a.e. $x \in \Gamma$, there exists $n(x) \geq N_0$ such that properties (1)–(8) are satisfied with $n = n(x)$. First, (1)–(4) are satisfied for all $x \in \Gamma$ and $n \geq N_0$. For a.e. $x \in \Gamma$, condition (5) holds for all n large enough (depending on x), as does condition (6). (See (4.1.2).) By definition of $\bar{\delta}_i$, for a.e. x, (7) holds for an infinite number of n's.

To complete the argument we now pick (and fix) x and $n = n(x) \geq N_0$ for which (1)–(8) hold. By (6) and (1), $m_x^{i-1}(L \cap \Gamma) \geq \frac{1}{2} m_x^{i-1}(L) \geq \frac{1}{2} e^{-n(\lambda_i - 2\varepsilon)(\bar{\delta}_{i-1} + \varepsilon)}$. But for $y \in L \cap \Gamma$, we have by (2), $m_x^{i-1}\mathscr{P}_0^n(y) \leq e^{-(h_{i-1} - \varepsilon)n}$. It follows that the number of distinct $\mathscr{P}_0^n$-atoms intersecting $L \cap \Gamma$ is bigger than $m_x^{i-1}(L \cap \Gamma) e^{(h_{i-1} - \varepsilon)n}$. By (5), $L \subset \xi_{i-1}(x) \subset \xi_i(x)$; so distinct $\mathscr{P}_0^n$-atoms intersecting $L \cap \Gamma$ define distinct $(\xi_i \vee \mathscr{P}_0^n)$-atoms intersecting $L \cap \Gamma$. Now $\xi_i(y) \cap \mathscr{P}_0^n(y) \subset B^i(y, e^{-n(\lambda_i - 2\varepsilon)})$ for $y \in \Gamma \cap L$, which guarantees that all the $(\xi_i \vee \mathscr{P}_0^n)$-atoms intersecting $L \cap \Gamma$ are contained in $B^i(x, 2e^{-n(\lambda_i - 2\varepsilon)})$. (Here we used the fact that $d^i(x, y) \leq d^{i-1}(x, y)$ for $y \in L$.) Putting these together we can conclude that

$$\begin{aligned}
&m_x^i B^i(x, 2e^{-n(\lambda_i - 2\varepsilon)}) \\
&\quad \geq \#\{\text{distinct } (\mathscr{P}_0^n \vee \xi_i)\text{-atoms intersecting } L \cap \Gamma\} \\
&\qquad \times \text{minimal measure of such an atom} \\
&\quad \geq \tfrac{1}{2} e^{-n(\lambda_i - 2\varepsilon)(\bar{\delta}_{i-1} + \varepsilon)} e^{n(h_{i-1} - \varepsilon)} \cdot e^{-n(h_i + \varepsilon)}.
\end{aligned}$$

Comparing this with property (7), we have

$$(\bar{\delta}_i - \varepsilon)(\lambda_i - 2\varepsilon) \leq (\bar{\delta}_{i-1} + \varepsilon)(\lambda_i - 2\varepsilon) + \frac{1}{n \log 2} + h_i - h_{i-1} + 2\varepsilon$$

and by (8),

$$\bar{\delta}_i - \bar{\delta}_{i-1} - 2\varepsilon \le \frac{h_i - h_{i-1} + 3\varepsilon}{\lambda_i - 2\varepsilon}.$$

The desired inequality follows from the arbitrariness of ε.

11. Transverse dimension: Proof of Theorem C′ (continued)

In this section we prove the two remaining relations leading to Theorem C′. We have to estimate from above an increase in entropy, and this is similar to what we did in Part I. In fact, the proof follows the same scheme as in Sections 3 and 5, and we shall not give details when they can be obtained by a straightforward adaptation of the corresponding statements in Part I.

(11.1) *Construction of partitions and related notions.* We consider again the family of increasing partitions ξ_i ($i = 1, \dots, u$) subordinate to W^i given by Lemma 9.1. Remember that these partitions are defined using a set S with the properties discussed in (8.4). Let $\mathscr{P}$ be a finite entropy partition adapted to $(\{\Phi_x\}, \tau e^{-\lambda_1 - 3\varepsilon})$ (see Definition 2.4.1). We require that $\mathscr{P}$ refine $\{S, M - S\}$ and another partition to be specified later. Define

$$\eta_i = \xi_i \vee \mathscr{P}^+.$$

LEMMA 11.1.1. *The family of measurable partitions η_i, $i = 1, \dots, u$, has the following properties*:

(1) *Each η_i is increasing*,

(2) $\eta_i < \eta_{i-1}$, $2 \le i \le u$,

(3) $\eta_i(x) \subset \Phi_x(W_x^i(x) \cap R(\tau e^{-\lambda_1 - 3\varepsilon} l(x)^{-1}))$ *for* $i = 1, \dots, u$ *and* m-a.e. x,

(4) $h_m(f, \eta_i) = H(\xi_i | f\xi_i) = h_i$, $i = 1, \dots, u$.

Property 3) is a consequence of Lemma 2.2.3 B ii) and our definition of $\mathscr{P}$. Property 4) is Lemma 3.2.1.

LEMMA 11.1.2. *For* m-a.e. x *and every* $y \in \Gamma' \cap \eta_i(x)$,

$$\Phi_x W_x^{i-1}(y) \cap \eta_i(x) = \eta_{i-1}(y).$$

That $\Phi_x W_x^{i-1}(y) \cap \eta_i(x) \subset \eta_{i-1}(y)$ follows from our characterization of ξ_i in (9.1).

LEMMA 11.1.3. *For* m-a.e. x *and every* $y \in \Gamma' \cap \eta_i(x)$,

$$f^{-1}(\eta_{i-1}(y)) = \eta_{i-1}(f^{-1}y) \cap f^{-1}(\eta_i(x)).$$

The last two lemmas are analogs of Lemmas 3.3.1 and 3.3.2.

For each $i \geq 2$, we now define a metric on $\eta_i(x)/\eta_{i-1}$ for m-a.e. x. As in (3.4), this metric will depend on the arbitrary choice of a positive measure set E. Let $E \subset S \cap \{l \leq l_0\}$ be a measurable set with $mE > 0$. We assume that the partition $\mathscr{P}$ refines $\{E, M - E\}$ and define $\tilde{\pi}$: $\bigcup_{n \geq 0} f^n E \to \mathbf{R}^{\Sigma_{i \leq u} \dim E_i}$ as follows: If $x \in E$, $\tilde{\pi}(x) = \pi(x)$ where π: $S \to \mathbf{R}^{\Sigma_{i \leq u} \dim E_i}$ is as in (8.4), and in general $\tilde{\pi}(x) = \pi(f^{-n(x)}x)$ where $n(x)$ is the smallest nonnegative integer such that $f^{-n}x \in E$. Since $\eta_i > \{E, M - E\}$, for each $x \in \bigcup_{n \geq 0} f^n E$, either $f^{-n}(\eta_i(x)) \subset E$ or $f^{-n}(\eta_i(x)) \cap E = \varnothing$. Thus for each x, $\tilde{\pi}|\eta_i(x)$ is a lipeomorphism between $\eta_i(x)$ and a subset of $\mathbf{R}^{\Sigma_{j \leq i} \dim E_j}$ carrying distinct η_{i-1}-elements to subsets of distinct $\mathbf{R}^{\Sigma_{j<i} \dim E_j}$-planes. (See (8.4).)

For $x \in \bigcup_{n \geq 0} f^n E$, $i \geq 2$ and $y, y' \in \eta_i(x)$, define

$$\tilde{d}_\tau^i(y, y') = |\tilde{\pi}^i y - \tilde{\pi}^i y'|$$

where $\tilde{\pi} = (\tilde{\pi}^1, \ldots, \tilde{\pi}^u)$. This induces a metric on $\eta_i(x)/\eta_{i-1}$ making it isometric to a subset of $(\mathbf{R}^{\dim E_i}, |\cdot|)$. For each i, $i \geq 2$, let $\tilde{m}_x^i$ be a system of conditional measures associated with η_i and let

$$\tilde{B}_\tau^i(x, \rho) = \{y \in \eta_i(x) \colon \tilde{d}_\tau^i(x, y) \leq \rho\}.$$

Define:

$$\tilde{\gamma}_i(x) = \liminf_{\rho \to 0} \frac{\log \tilde{m}_x^i \tilde{B}_\tau^i(x, \rho)}{\log \rho}.$$

We call the function $\tilde{\gamma}_i$ the transverse dimension of $\eta_i(x)/\eta_{i-1}$.

LEMMA 11.1.4. $\tilde{\gamma}_i \leq \dim E_i$, $i = 2, \ldots, u$.

This is a consequence of the above discussion and (4.1.4).

LEMMA 11.1.5. *There exists a number* $N > 0$ *such that for all* $i \geq 2$, *for all* $x \in E$ *and for all* $y, y' \in \eta_i(x)$,

$$\frac{1}{N}|\pi_x^i \circ \Phi_x^{-1}(y) - \pi_x^i \circ \Phi_x^{-1}(y')| \leq \tilde{d}_\tau^i(y, y') \leq N|\pi_x^i \circ \Phi_x^{-1}(y) - \pi_x^i \circ \Phi_x^{-1}(y')|,$$

where π_x^i *is as in* 8.3.2.

This follows from remark (3) at the end of (8.4) and the uniform Lipschitz property of $\pi_x \circ \Phi_x^{-1}$ on $S \cap \{l \leq l_0\}$. Note that N does not depend on E as long as $E \subset S \cap \{l \leq l_0\}$.

(11.2) *Proposition* 5.1 *revisited.*

PROPOSITION 11.2. *Suppose* f: $M \hookleftarrow$ *is a* C^2 *diffeomorphism of a compact Riemannian manifold and* m *is an ergodic Borel probability measure on* M. *Let* $\beta > 0$ *be given. Then there exists a family of increasing partitions* η_i, $i =$

$1, \ldots, u$, *satisfying Lemma* 11.1.1 *such that for* m-a.e. x *and for* $i = 2, \ldots, u$,

$$(\lambda_i + \beta)\tilde{\gamma}_i(x) \geq (1 - \beta)(h_i - h_{i-1} - \beta),$$

where $\tilde{\gamma}_i$ *is the transverse dimension of* η_i/η_{i-1}.

The proof of Proposition 11.2 is parallel to that of Proposition 5.1, except that the exponent in the transverse direction is now λ_i. More precisely, if $x \in E$, $f^n x \in E$ and $y \in f^{-n}(\eta_i(f^n x))$, then

$$\tilde{d}^i_\tau(f^n x, f^n y) \leq N^2 e^{(\lambda_i + 3\varepsilon)n} \tilde{d}^i_\tau(x, y)$$

(apply Lemmas 8.3.2 and 11.1.5).

It follows from this proposition and Lemma 11.1.4 that

$$\lambda_i \dim E_i \geq h_i - h_{i-1}.$$

which is the fourth inequality stated at the beginning of Section 10.

To complete the proof of inequality (3), we need to relate $\tilde{\gamma}_i$ to $\underline{\delta}_i - \underline{\delta}_{i-1}$. We do this via a notion of transverse dimension that does not depend on the η_i's.

(11.3) *General lemmas in dimension theory.* We state two lemmas relating the dimension of a measure and those of its decompositions.

LEMMA 11.3.1. *Let* m *be a probability measure on* $\mathbf{R}^p \times \mathbf{R}^q$, π *the projection onto* $\mathbf{R}^p$, m_t *a disintegration of* m *with respect to* π. *Define*

$$\gamma(t) = \liminf_{\rho \to 0} \frac{\log m \circ \pi^{-1} B^p(t, \rho)}{\log \rho}$$

and let $\delta \geq 0$ *be such that at* m-a.e. (s, t)

$$\delta \leq \liminf_{\rho \to 0} \frac{\log m_t B^q(s, \rho)}{\log \rho}.$$

Then, at m-a.e. (s, t):

$$\delta + \gamma(t) \leq \liminf_{\rho \to 0} \frac{\log m B^{p+q}((s, t), \rho)}{\log \rho}.$$

Proof. Fix $\sigma > 0$; we can find N_1 and a set A_1 with $mA_1 \geq 1 - \sigma$ such that for all $(s, t) \in A_1$ and $n \geq N_1$,

$$m_t B^q(s, 2e^{-n}) \leq e^{-n\delta} e^{n\sigma}.$$

By the Lebesgue density theorem (see 4.1.2) we can find N_2 and a set A_2 with $mA_2 \geq 1 - 2\sigma$ such that for all $(s, t) \in A_2$ and $n \geq N_2$,

$$m(A_1 \cap B^{p+q}((s, t), e^{-n})) \geq \tfrac{1}{2} m B^{p+q}((s, t), e^{-n}).$$

If $(s_0, t_0) \in A_2$ and $n \geq N_2$, we have

$$mB^{p+q}((s_0, t_0), e^{-n}) \leq 2\int_{B^p(t_0, e^{-n})} m_t(A_1 \cap B^q(s_0, e^{-n}))m \circ \pi^{-1}(dt)$$
$$\leq 2e^{-n\delta}e^{n\sigma}m \circ \pi^{-1}B^p(t_0, e^{-n})$$

because for each t, there exists some $u(t)$ with $(t, u(t)) \in A_1 \cap B^q(s_0, e^{-n})$ and thus $A_1 \cap B^q(s_0, e^{-n}) \cap \pi^{-1}\{t\} \subset B^q(u(t), 2e^{-n}) \cap \pi^{-1}\{t\}$. The lemma follows when $n \to \infty$ and $\sigma \to 0$.

If instead of $\mathbf{R}^p$, we have no structure at all, a similar proof gives the following lemma:

LEMMA 11.3.2. *Let (Ω, ν) be an abstract probability space and m a probability on $\Omega \times \mathbf{R}^q$ written $m(d\omega, ds) = \int m_\omega(ds)\nu(d\omega)$. Let $\tilde{\gamma} \geq 0$ be such that at m-a.e. (ω, s)*

$$\tilde{\gamma} \leq \liminf_{\rho \to 0} \frac{\log m_\omega(B^q(s, \rho))}{\log \rho}.$$

Then,

$$\tilde{\gamma} \leq \liminf_{\rho \to 0} \frac{\log m(B^q(s, \rho))}{\log \rho} \, m\text{-a.e.}$$

(11.4) *Transverse dimension*. Let $\beta > 0$ be given. We choose ε, l_0, S, π: $S \to \mathbf{R}^{\Sigma_{i \leq u} \dim E_i}$, E and η_i, $i = 1, \ldots, u$, as above and let $\tilde{\gamma}_i$ be the transverse dimension of η_i/η_{i-1}. For each i, let $\{\hat{m}_x^i\}$ be a system of conditional measures associated with $\hat{\xi}_i$ (see 9.1).

Fix $x \in S$ and i, $2 \leq i \leq u$. Let $\hat{\pi} = \pi|_{\xi_i(x)}$: $\xi_i(x) \to \mathbf{R}^{\Sigma_{j \leq i} \dim E_j} = \mathbf{R}^{\Sigma_{j < i} \dim E_j} \times \mathbf{R}^{\dim E_i}$ and write $m_x^i = \hat{m}_x^i \circ \hat{\pi}^{-1}$. Define:

$$\gamma_i(y) = \liminf_{\rho \to 0} \frac{\log m_x^i\{|z_i - y_i| < \rho\}}{\log \rho}.$$

Note that by the Lipschitz property of π and $\tilde{\pi}$,

$$\tilde{\gamma}_i(\hat{\pi}^{-1}y) = \liminf_{\rho \to 0} \frac{\log \tilde{m}^i_{\tilde{\pi}^{-1}y} \circ \tilde{\pi}^{-1}\{|z_i - y_i| < \rho\}}{\log \rho}$$

for m_x^i-a.e. y, if we assume $\hat{m}_x^i(\bigcup_{n \geq 0} f^n E) = 1$. Also we may assume that

$$\tilde{\gamma}_i(\hat{\pi}^{-1}y) \geq \frac{(1-\beta)[h_i - h_{i-1} - \beta]}{\lambda_i + \beta}$$

for m_x^i-a.e. y. Applying (11.3.2) with $\Omega = \hat{\xi}_i(x)/\eta_i$ and $\mathbf{R}^q = \mathbf{R}^{\dim E_i}$, we can conclude that

$$\gamma_i \geq \frac{(1-\beta)[h_i - h_{i-1} - \beta]}{\lambda_i + \beta} \quad m_x^i\text{-almost everywhere.}$$

Consider now the partition of $\hat{\pi}(\hat{\xi}_i(x))$ into planes of the form $\{z_i = \text{constant}\}$. We may assume that at m_x^i-a.e. y:

$$\liminf_{\rho \to 0} \frac{\log m_x^i\{|z - y| < \rho\}}{\log \rho} = \underline{\delta}_i \quad \text{and}$$

$$\liminf_{\rho \to 0} \frac{\log \hat{m}^{i-1}_{\hat{\pi}^{-1}y} \circ \hat{\pi}^{-1}\{z_i = y_i, |z - y| < \rho\}}{\log \rho} = \underline{\delta}_{i-1}.$$

Lemma 11.3.1 then tells us that for m_x^i-a.e. y:

$$(*) \qquad \underline{\delta}_i - \underline{\delta}_{i-1} \geq \gamma_i(y) \geq \frac{(1-\beta)[h_i - h_{i-1} - \beta]}{\lambda_i + \beta}$$

which completes the proof of inequality (3) and hence of Theorem C′.

Remark. Observe that γ_i as defined above has geometric meaning as the dimension of m on W^i/W^{i-1}. It is not hard to see that this quantity is in fact independent of our choice of S and π. It then follows from ($*$) and inequality (2) that

$$\gamma_i = \delta_i - \delta_{i-1}.$$

12. Volume lemma and dimension of invariant measures: Proof of Theorem F

(12.1) *Some reductions*. We do not assume in this section that the measure m is ergodic. Instead, we divide M into a countable number of invariant sets on each one of which all relevant functions are more or less constant. By (4.1.2) it suffices to consider these sets one at a time.

For $x \in \Gamma'$, set

$$E^u(x) = \bigoplus_{\lambda_i(x) > 0} E_i(x), \quad E^s(x) = \bigoplus_{\lambda_i(x) < 0} E_i(x), \quad E^c(x) = E_{i_0(x)}(x),$$

where $\lambda_{i_0(x)}(x) = 0$; W^u and W^s have the same meaning as in (7.6), and δ^u and δ^s are the dimension of m on W^u and W^s manifolds as defined in (7.5). In the proof of Theorem F we do not distinguish between exponents of the same sign. Thus the charts and estimates in Part I are more suitable for this section.

For $\varepsilon > 0$, integers u_0, s_0 and numbers λ^+, λ^-, δ^+, δ^- and h, with $\lambda^+, -\lambda^- > 100\varepsilon$, let $\Gamma(\varepsilon, u_0, s_0, \lambda^+, \lambda^-, \delta^+, \delta^-, h) =$

$$\begin{aligned}\{x \in \Gamma' \colon \ &\text{(i) } \dim E^u(x) = u_0, \dim E^s(x) = s_0,\\ &\text{(ii) } \min_{\lambda_i(x)>0} \lambda_i(x) \geq \lambda^+, \ \max_{\lambda_i(x)<0} \lambda_i(x) \leq \lambda^-,\\ &\text{(iii) } \delta^+ - \varepsilon \leq \delta^u(x) \leq \delta^+, \ \delta^- - \varepsilon \leq \delta^s(x) \leq \delta^-,\\ &\text{(iv) } h \leq h_{\overline{m}_x}(f) \leq h + \varepsilon\}.\end{aligned}$$

where $\{\overline{m}_x\}$ is a decomposition of m into ergodic components. Clearly, there is a countable number of invariant sets of this type, the union of which has measure 1.

First reduction. We may assume that for some ε, u_0, s_0, λ^+, λ^-, δ^+, δ^- and h,

$$m\Gamma(\varepsilon, u_0, s_0, \lambda^+, \lambda^-, \delta^+, \delta^-, h) = 1.$$

Let $\{\Phi_x\}$ be a system of (ε, l)-charts (see Remark (3) in the appendix at the end of Part I). We construct an increasing partition ξ^u subordinate to W^u and a decreasing partition ξ^s (i.e. $f\xi^s > \xi^s$) subordinate to W^s as in (3.1). We may assume that the same l_0 is used in both constructions, but since m is not ergodic, we can only ensure that Lemma 3.1.1 holds for both f and f^{-1} on a set B with $mB > 1 - \varepsilon'$. But again by (4.1.2) it suffices to prove the theorem for the induced measure on sets of measure arbitrarily close to 1.

Second reduction. We may assume that $mB = 1$.

More specifically we prove the following proposition:

PROPOSITION 12.1. *Suppose* (i), (ii), (iii), (iv) *are satisfied for* m-a.e.x, *and* ξ^u *and* ξ^s *are respectively increasing and decreasing partitions subordinate to* W^u *and* W^s *on* M. *Then, given* $\varepsilon' > 0$, *there exists a set* Λ *with* $m\Lambda > 1 - 5\varepsilon'$ *such that for* $x \in \Lambda$,

$$\limsup_{\rho \to 0} \frac{\log mB(x,\rho)}{\log \rho} \le \delta^+ + \delta^- + \delta^c + 5\varepsilon\left(1 + \frac{1}{|\lambda^-|} + \frac{1}{\lambda^+}\right)$$

where $\delta^c = \dim M - u_0 - s_0$ *is* $\dim E^c(x)$ *for* m-a.e.x.

We now pick once and for all a system of conditional measures m_x^u associated with ξ^u, a system of conditional measures m_x^s associated with ξ^s and a decomposition $\overline{m}_x$ of m into ergodic measures. Discarding a set of measure 0, we may assume further that each $\overline{m}_x$ is an ergodic invariant measure, for which $m^u_\cdot$ and $m^s_\cdot$ are conditional measures associated with ξ^u and ξ^s.

Let us review some of the facts at our disposal. First recall that at m-a.e.x, we have:

(1) $$H_{\overline{m}_x}(\xi^u | f\xi^u) = h_{\overline{m}_x}(f) \qquad \text{(Corollary 5.3)},$$

(2) $$\limsup_{\rho \to 0} \frac{\log m_x^u B^u(x,\rho)}{\log \rho} \le \delta^+, \quad \text{and}$$

(3) $$\limsup_{\rho \to 0} \frac{\log m_x^s B^s(x,\rho)}{\log \rho} \le \delta^- .$$

If $\mathscr{P}$ is a finite entropy partition and $a, b > 0$, then by the Shannon-Breiman-McMillan theorem,

$$\lim_n -\frac{1}{n}\log m(\mathscr{P}_{-nb}^{na}(x)) \leq (a+b)(h+\varepsilon) \tag{4}$$

where $\mathscr{P}_p^q(x)$ denotes the atom of the partition $\bigvee_{i=p}^{q-1} f^{-i}\mathscr{P}$ containing x. Moreover, if $\eta = \xi^u \vee \mathscr{P}_{-\infty}^0$ and $\{m_x\}$ is a system of conditional measures associated with η, then we claim that at m-a.e. x,

$$\lim_n -\frac{1}{n}\log m_x\mathscr{P}_0^n(x) = h_{\overline{m}_x}(f, \mathscr{P}) \quad \text{and} \tag{5}$$

$$\limsup_{\rho\to 0} \frac{\log m_x B^u(x,\rho)}{\log \rho} \leq \delta^+. \tag{6}$$

These two assertions are consequences of (1) and (2) and two other general facts, the proofs of which we postpone until (12.4).

From our results in Section 9 (it is not hard to adapt them to the non-ergodic case), it follows that there exist a finite entropy partition $\mathscr{P}_1$ and a set Λ_1 with $m\Lambda_1 > 1 - \varepsilon'$ such that at every x in Λ_1,

$$\begin{aligned} \limsup_n -\frac{1}{n}\log m_x^s(\mathscr{P}_1)_{-n}^0(x) &\geq H(\xi^s|f^{-1}\xi^s) - \varepsilon \\ &\geq h - \varepsilon. \end{aligned} \tag{7}$$

We may assume also that $h_{\overline{m}_x}(f, \mathscr{P}_1) \geq h - \varepsilon$ on Λ_1. Combining this with (5), we have

$$\lim_n -\frac{1}{n}\log m_x(\mathscr{P}_1)_0^n(x) \geq h - \varepsilon \quad \text{on } \Lambda_1. \tag{8}$$

In the case when there are no zero exponents, using arguments that are standard by now, we can choose a partition $\mathscr{P} > \mathscr{P}_1$ such that for some suitable choice of $a, b > 0$, the atoms of $\mathscr{P}_{-nb}^{na}$ have diameter $\leq e^{-n}$. Proposition 12.1 then follows from (3), (4), (6), (7), (8) and the counting argument we give in (12.3). In the presence of zero exponents, we have to further partition the atoms of $\mathscr{P}_{-nb}^{na}$—more or less arbitrarily—along the neutral direction. This accounts for the factor δ^c which appears in the statement of Proposition 12.1. This partition procedure is described in (12.2).

(12.2) *Local picture.*

LEMMA 12.2.1. *Let $a = (1/\lambda^+ - 2\varepsilon)$, $b = (1/-\lambda^- - 2\varepsilon)$. There exist a set Λ_2 with $m(\Lambda_2) > 1 - \varepsilon'$, an integer N_0, a constant C and a finite entropy partition $\mathscr{P}$ refining $\mathscr{P}_1$, with the following property: For every $n \geq N_0$, there exists a partition $\mathscr{Q}_n > \mathscr{P}_{-nb}^{na}$ such that if $x \in \Lambda_2$ and $q_n(x)$ is the atom of $\mathscr{Q}_n$*

containing x, *then*:

(i) $\operatorname{diam} q_n(x) \le 2e^{-n}$ *and*

(ii) $mq_n(x) \ge C^{-1}e^{-n\varepsilon}e^{-n\delta^c} m\mathscr{P}_{-nb}^{na}(x)$.

Proof of Lemma 12.2.1. Let (v_x^u, v_x^c, v_x^s) denote the coordinates in T_xM respecting the splitting $E^u(x) \oplus E^c(x) \oplus E^s(x)$. Via the $\exp_x$-map, this defines a coordinate system in a neighborhood of x. Our strategy is to divide the manifold into small sets, select a representative from each set and work with the coordinate systems associated with these points.

By Lusin's theorem, there exists a compact set Λ_3 with $m\Lambda_3 > 1 - (\varepsilon/3)$ such that on Λ_3, the four functions $E^u(x)$, $E^c(x)$, $E^s(x)$ and $l(x)$ are continuous. Let $L = \max\{l(x), x \in \Lambda_3\}$. Choose $\delta > 0$ small enough that if $d(x, x') < \delta$, then (i) $\exp_{x'}^{-1}\exp_x(v)$ is defined whenever $\|v\| \le 3\delta$, and (ii) $\exp_{x'}^{-1}\exp_x\{v_x^\alpha = \text{constant}\}$ has slope $\le (1/4KL)$ relative to $E^\alpha(x')$ for $\alpha = u$, c and s. Let $\mathscr{Q}_0$ be a partition of Λ_3 into sets of diameter smaller than δ and choose a point $z(q) \in q$ for every $q \in \mathscr{Q}_0$. For $x \in \Lambda_3$, write $\bar{x} = \bar{x}^u + \bar{x}^c + \bar{x}^s$ where $\bar{x}^\alpha = (\exp_{z(q(x))}^{-1}x)_{z(q(x))}^\alpha$, $\alpha = u, c, s$. We may assume that $\frac{1}{2}d(x, x') \le |\bar{x} - \bar{x}'| \le 2d(x, x')$ for $x' \in q(x)$.

SUBLEMMA 12.2.2. *Let δ be sufficiently small. For $x \in \Lambda_3$, define*

$$V(x, n) = \left\{ y \in q(x) \colon d(f^jx, f^jy) \le \frac{\delta}{2} l(f^jx)^{-2} \text{ for all } j, -nb \le j \le na \right\}.$$

If $y_1, y_2 \in V(x, n)$ satisfy $|\bar{y}_1^c - \bar{y}_2^c| \le (e^{-n}/12KL)$, then $|\bar{y}_1 - \bar{y}_2| \le e^{-n}$.

Proof of Sublemma 12.2.2. First, if $\max(|\bar{y}_1^u - \bar{y}_2^u|, |\bar{y}_1^s - \bar{y}_2^s|) \le 4KL|\bar{y}_1^c - \bar{y}_2^c|$, the conclusion is clearly true.

If not, then $|\bar{y}_1^c - \bar{y}_2^c| \le (1/4KL)\max(|\bar{y}_1^u - \bar{y}_2^u|, |\bar{y}_1^s - \bar{y}_2^s|)$. Writing $v_i^\alpha = (\exp_x^{-1}y_i)_x^\alpha$, $i = 1, 2$, we have $|v_1^c - v_2^c| \le (1/KL)\max(|v_1^u - v_2^u|, |v_1^s - v_2^s|)$ by our choice of δ. Set $w_i = \Phi_x^{-1}y_i$, where Φ_x is the (ε, l)-local chart at x. We have:

$$|w_1^c - w_2^c| \le \max(|w_1^u - w_2^u|, |w_1^s - w_2^s|),$$

and therefore either $|w_1 - w_2| = |w_1^u - w_2^u|$, or $|w_1 - w_2| = |w_1^s - w_2^s|$.

Suppose $|w_1 - w_2| = |w_1^u - w_2^u|$. This property is preserved by $\tilde{f}_x$ which expands in the u direction by at least $e^{\lambda^+ - 2\varepsilon}$ (see Lemma 2.3.1a). Since y_1, y_2 lie in $V(x, n)$, we can apply this expansion $[na]$ times and thus

$$|w_1^u - w_2^u| \le \delta e^{\lambda^+} e^{-n}.$$

In the other case, $|w_1 - w_2| = |w_1^s - w_2^s|$ and the same argument applied to $\tilde{f}_x^{-1}$ gives

$$|w_1 - w_2| \le \delta e^{-\lambda^-} e^{-n}.$$

Changing coordinates again we obtain in both cases $|\bar{y}_1 - \bar{y}_2| \leq 2L\delta \max(e^{\lambda^+}, e^{-\lambda^-})e^{-n}$. The choice of a sufficiently small δ gives Sublemma 12.2.2. □

We now choose the partition $\mathscr{P}$ in Lemma 12.2.1. Let $\mathscr{P}_1$ be the partition at the end of (12.1). Let $\mathscr{P}_2$ be the partition into $M \setminus \Lambda_3$ and $\mathscr{Q}_0$ on Λ_3. Using 2.4, we can find a finite entropy partition $\mathscr{P}_3$ and a function n_0 such that if $n \geq n_0(x)$, then $(\mathscr{P}_3)^{na}_{-nb}(x) \subset V(x, n)$. We take $\mathscr{P} = \mathscr{P}_1 \vee \mathscr{P}_2 \vee \mathscr{P}_3$ and choose N_0 such that $m\{n_0 \leq N_0\} \geq 1 - \varepsilon'/3$.

We now define the partitions $\mathscr{Q}_n$, $n \geq N_0$ on $\Lambda_3 \cap \{n_0 \leq N_0\}$. Since $\mathscr{P}$ refines $\mathscr{Q}_0$, the entire set $\mathscr{P}(x)$, and hence $\mathscr{P}^{na}_{-nb}(x)$, lies in $q(x)$. We shall subdivide the atoms of $\mathscr{P}^{na}_{-nb}$ that meet $\Lambda_3 \cap \{n_0 \leq N_0\}$ one at a time. Fix $x \in \Lambda_3 \cap \{n_0 \leq N_0\}$. Let $\mathscr{Q}_n|_{\mathscr{P}^{na}_{-nb}(x)}$ be a partition such that if y_1 and y_2 belong in the same element of $\mathscr{Q}_n$, then $|\bar{y}^c_1 - \bar{y}^c_2| \leq e^{-n}/12KL$. By Sublemma 12.2.2, these elements of $\mathscr{Q}_n$ have diameter less than $2e^{-n}$. Clearly, we can arrange to have the cardinality of $\mathscr{Q}_n|_{\mathscr{P}^{na}_{-nb}(x)}$ to be less than $C_1 e^{n\delta^c}$, where $C_1 = (24KL)^{\delta^c}$.

Let

$$A_n = \left\{x\colon m\left(q_n(x) \cap \mathscr{P}^{na}_{-nb}(x)\right) \leq \frac{e^{-n\delta^c}}{C_1}\frac{\varepsilon'}{3}e^{-n\varepsilon}(1 - e^{-\varepsilon})m\mathscr{P}^{na}_{-nb}(x)\right\}.$$

Then, $mA_n \leq (\varepsilon'/3)e^{-n\varepsilon}(1 - e^{-\varepsilon})$. Setting $\Lambda_2 = (\Lambda_3 \cap \{n_0 \leq N_0\}) \setminus \bigcup_n A_n$, we have $m\Lambda_2 \geq 1 - \varepsilon'$ and Lemma 12.2.1 is proved with $C = (3C_1/\varepsilon'(1 - e^{-\varepsilon}))$.

(12.3) *Proof of Proposition* 12.1. Set $a = (1/\lambda^+ - 2\varepsilon)$, $b = (1/-\lambda^- - 2\varepsilon)$. Consider a set Λ_2, $m\Lambda_2 > 1 - \varepsilon'$, an integer N_0 and a finite entropy partition $\mathscr{P}$ with the property in the statement of Lemma 12.2.1. By (4), (7) and (8) there exist a set $\Lambda_4 \subset \Lambda_1$ with $m\Lambda_4 > 1 - 2\varepsilon'$ and an integer N_1 such that if $x \in \Lambda_4$ and $n \geq N_1$, then

(a) $$m^s_x\left(\mathscr{P}^0_{-nb}(x)\right) \leq e^{-nb(h-2\varepsilon)},$$

(b) $$m_x\left(\mathscr{P}^{na}_0(x)\right) \leq e^{-na(h-2\varepsilon)},$$

(c) $$m\left(\mathscr{P}^{na}_{-nb}(x)\right) \geq e^{-n(a+b)(h+2\varepsilon)}.$$

Set $\Lambda_5 = \Lambda_2 \cap \Lambda_4$, $N_2 = \max(N_0, N_1)$. If $z \in \Lambda_5$ and $n \geq N_2$, then

(d) $$\operatorname{diam} q_n(z) \leq 2e^{-n} \quad \text{and}$$

(e) $$mq_n(z) \geq C^{-1}e^{-n\varepsilon}e^{-n\delta^c}e^{-n(a+b)(h+2\varepsilon)}.$$

By (6) and (4.1.2), we can choose $\Lambda_6 \subset \Lambda_5$ with $m\Lambda_6 > 1 - 4\varepsilon'$, and $N_3 \geq N_2$ such that if $y \in \Lambda_6$ and $n \geq N_3$, then:

(f) $$m_y\left(\Lambda_5 \cap B^u(y, e^{-n})\right) \geq \tfrac{1}{2}e^{-n(\delta^+ + \varepsilon)}.$$

By (3) and again (4.1.2), we can choose finally $\Lambda \subset \Lambda_6$ with $m\Lambda > 1 - 5\varepsilon'$ and $N_4 \geq N_3$ such that if $x \in \Lambda$ and $n \geq N_4$, then:

(g) $$m_x^s(\Lambda_6 \cap B^s(x, e^{-n})) \geq \tfrac{1}{2} e^{-n(\delta^- + \varepsilon)}.$$

The set of points for which the inequality in Proposition 12.1 holds will be this final set Λ. Fix x in Λ and set $\bar{\delta} = \limsup_n - 1/n \log mB(x, 4e^{-n})$. There exist infinitely many n such that

(h) $$mB(x, 4e^{-n}) \leq e^{-n(\bar{\delta} - \varepsilon)}.$$

Fix such an n, assuming that $n \geq N_4$ and $4C \leq e^{n\varepsilon}$. Consider the number

$$N = \sharp \{\text{atoms of } Q_n \text{ intersecting } \Lambda_5 \cap B(x, 2e^{-n})\}.$$

The minimum measure of these atoms imposes an immediate upper bound on N. More precisely, from (d), (e) and (h), we get

(*) $$N \leq Ce^{n\varepsilon} e^{n\delta^c} e^{n(a+b)(h+2\varepsilon)} e^{-n(\bar{\delta} - \varepsilon)}.$$

The estimates leading to a lower bound are slightly more involved. Basically, we use our upper estimate of the appropriate conditional measures of $\mathscr{P}_0^{na}$- and $\mathscr{P}_{-nb}^0$-atoms. To begin, since $x \in \Lambda$, we have (g). Then for every y in $\xi^s(x) \cap \Lambda_6 \cap B(x, e^{-n})$, we have by (a):

$$m_x^s(\mathscr{P}_{-nb}^0(y)) = m_y^s(\mathscr{P}_{-nb}^0(y)) \leq e^{-nb(h-2\varepsilon)}.$$

Thus:

$$\sharp \{\text{atoms of } \mathscr{P}_{-nb}^0 \text{ intersecting } \Lambda_6 \cap B(x, e^{-n})\} \geq \tfrac{1}{2} e^{-n(\delta^- + \varepsilon)} e^{nb(h-2\varepsilon)}.$$

Let us fix one of these atoms p_u and choose $y \in p_u \cap \Lambda_6 \cap B(x, e^{-n})$. We have (f), and for any z in $\eta(y) \cap \Lambda_5 \cap B(y, e^{-n})$, by (b):

$$m_y(\mathscr{P}_0^{na}(z)) = m_z(\mathscr{P}_0^{na}(z)) \leq e^{-na(h-2\varepsilon)}.$$

Thus if $n(X)$ denotes the number of atoms of $\mathscr{P}_0^{na}$ intersecting the set $X \cap \Lambda_5 \cap B(y, e^{-n})$, then:

$$n(\eta(y)) \geq \tfrac{1}{2} e^{-n(\delta^+ + \varepsilon)} e^{na(h-2\varepsilon)}.$$

Furthermore, since $\eta(y) \subset \mathscr{P}_{-\infty}^0(y) \subset p_u$, we have

$$n(p_u) \geq n(\eta(y)).$$

Let p_s be one of these atoms, $p_u \cap p_s$ is now an atom of $\mathscr{P}_{-nb}^{na}$ intersecting $\Lambda_5 \cap B(y, e^{-n})$ for some y with $d(y, x) \leq e^{-n}$. Therefore

(**) $$N \geq \sum_{\{p_u : p_u \cap \Lambda_6 \cap B(x, e^{-n}) \neq \varnothing\}} n(p_u)$$
$$\geq \tfrac{1}{4} e^{-n(\delta^- + \delta^+ + 2\varepsilon)} e^{n(a+b)(h-2\varepsilon)}.$$

Comparing (*) with (**), we complete the proof of Proposition 12.1 and hence that of Theorem F, provided we justify (5) and (6) in Section 12.1.

(12.4) *Two lemmas from measure theory*. In this section, we prove two general lemmas on conditional measures, leading to (5) and (6) in Section 12.1.

Suppose ξ is a measurable partition in a Lebesgue space (X, μ), $\mathscr{P}$ is a finite entropy partition and μ_x is a system of conditional measures associated with ξ. We denote

$$I(\mathscr{P}/\xi) = -\log \mu_x \mathscr{P}(x).$$

The following lemma tells us that (1) implies (5):

LEMMA 12.4.1. *Let f: $(X, \mu) \hookleftarrow$ be an ergodic automorphism of the Lebesgue space (X, μ), with finite entropy $h_\mu(f)$. Let ξ be an increasing measurable partition such that $H(\xi/f^{-1}\xi) = h_\mu(f)$ and let $\mathscr{P}$ be a finite entropy partition. Then, at* μ-a.e. x:

$$\lim_n \frac{1}{n} I\big(\mathscr{P}_0^n/\big(\xi \vee \mathscr{P}_{-\infty}^0\big)\big) = h_\mu(\mathscr{P}, f).$$

Proof. (Compare with 9.3.1). First:

$$\frac{1}{n} I\big(\mathscr{P}_0^n/\big(\xi \vee \mathscr{P}_{-\infty}^0\big)\big)(x) = \frac{1}{n} \sum_{i=0}^{n-1} I\big(\mathscr{P}/\big(\mathscr{P}_{-\infty}^0 \vee f^{-i}\xi\big)\big)(f^i x),$$

and the second term converges m-a.e. and in L^1 toward $H(\mathscr{P}/\Lambda_i(\mathscr{P}_{-\infty}^0 \vee f^{-i}\xi))$, which is clearly $\le h_\mu(\mathscr{P}, f)$. The limit, being constant, is also the limit of

$$\frac{1}{n} H\big(\mathscr{P}_0^n/\big(\xi \vee \mathscr{P}_{-\infty}^0\big)\big) = \frac{1}{n} H\big(\xi_1^n \vee \mathscr{P}_0^n/\big(\xi \vee \mathscr{P}_{-\infty}^0\big)\big) - \frac{1}{n} H\big(\xi_1^n/\big(\xi \vee \mathscr{P}_{-\infty}^0\big)\big).$$

We have

$$\lim_n \frac{1}{n} H\big(\xi_1^n \vee \mathscr{P}_0^n\big)/\big(\xi \vee \mathscr{P}_{-\infty}^0\big) = \lim_n \frac{1}{n} H\big(\xi_1^n \vee \mathscr{P}_0^n/\xi\big) \ge H\big(\xi/f^{-1}\xi\big) \quad \text{and}$$

$$\lim_n \frac{1}{n} H\big(\xi_1^n/\big(\xi \vee \mathscr{P}_{-\infty}^n\big)\big) = H\big(\xi/\big(f^{-1}\xi \vee \mathscr{P}_{-\infty}^{+\infty}\big)\big).$$

Let $\mathscr{Q}$ be a finite entropy partition such that $\xi = f^{-1}\xi \vee \mathscr{Q}$. Then:

$$\begin{aligned} H\big(\xi/\big(f^{-1}\xi \vee \mathscr{P}_{-\infty}^{+\infty}\big)\big) &\le H\big(\mathscr{Q}/\mathscr{Q}_{-\infty}^0 \vee \mathscr{P}_{-\infty}^{+\infty}\big) \\ &= h_\mu(\mathscr{Q} \vee P, f) - h_\mu(\mathscr{P}, f) \\ &\le h_\mu(f) - h_\mu(\mathscr{P}, f). \end{aligned}$$

Therefore the limit in question has to be larger than $H(\xi/f^{-1}\xi) - h_\mu(f) + h_\mu(\mathscr{P}, f)$, and this completes the proof.

The following lemma tells us that (2) implies (6):

LEMMA 12.1.2. *Let μ be a probability measure on $\mathscr{R}^q(1)$, ξ a measurable partition and μ_s a system of conditional measures associated with the partition ξ. Suppose Δ is such that*

$$\limsup_{\varepsilon \to 0} \frac{\log \mu B^q(s, \varepsilon)}{\log \varepsilon} \leq \Delta \quad \textit{for } m\text{-a.e.}\, s.$$

Then

$$\limsup_{\varepsilon \to 0} \frac{\log \mu_s B^q(s, \varepsilon)}{\log \varepsilon} \leq \Delta \; m\text{-a.e.}$$

Proof. Fix $\sigma > 0$ and call

$$B_n(\sigma) = \{t\colon \mu B^q(t, e^{-n}) \geq e^{-n(\Delta+\sigma)}\}.$$

Our hypothesis is $\liminf_n \mu B_n(\sigma) = 1$ for all σ. Call

$$A_n(\sigma) = \{s\colon \mu_s B^q(s, 2e^{-n}) \leq e^{-n(\Delta+2\sigma)}\}.$$

We have:

$$\mu(A_n(\sigma) \cap B^q(t, e^{-n})) = \int \mu_s(A_n(\sigma) \cap B^q(t, e^{-n}))\mu(ds).$$

If $\mu_s(A_n(\sigma) \cap B^q(t, e^{-n})) > 0$, then there exists $s' \in \xi(s) \cap A_n(\sigma) \cap B^q(t, e^{-n})$ and thus:

$$\mu_s(A_n(\sigma) \cap B^q(t, e^{-n})) \leq \mu_{s'}(B^q(s, 2e^{-n})) \leq e^{-n(\Delta+2\sigma)}.$$

Therefore, if $t \in B_n(\sigma)$, then

$$\mu(A_n(\sigma) \cap B^q(t, e^{-n})) \leq e^{-n\sigma}\mu B^q(t, e^{-n}).$$

Using the Besicovitch covering lemma (see 4.1) we conclude that $\mu(A_n(\sigma) \cap B_n(\sigma)) \leq e^{-n\sigma}c(q)$. Therefore, μ-a.e. point s belongs to a finite numbers of sets A_n and this completes the proof. □

UNIVERSITE DE PARIS VI, PARIS, FRANCE
MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN

REFERENCES

[Be] K. R. BERG, Convolution of invariant measures, maximal entropy, Math. Systems Theory 3 (1969), 146–150.

[Bi] P. BILLINGSLEY, *Ergodic Theory and Information*, Wiley, New York (1965).

[BK] M. Brin and A. Katok, On local entropy, in *Geometric Dynamics*, Springer Lect. Notes **1007** (1983), 30–38.

[FKYY] P. Frederickson, J. L. Kaplan, E. D. Yorke and J. A. Yorke, The Lyapunov dimension of strange attractors, Journal of Diff. Eq. 49 (1983), 185–207.

[L] F. Ledrappier, Some relations between dimension and Lyapunov exponents, Comm. Math. Phys. **81** (1981), 229–238.

[LS] F. Ledrappier and J.-M. Strelcyn, A proof of the estimation from below in Pesin entropy formula, Ergod. Th. and Dynam. Sys. **2** (1982), 203–219.

[Mé] R. Mañé, A proof of Pesin's formula, Ergod. Th. and Dynam. Sys. **1** (1981), 95–102.

[Mg] A. Manning, A relation between Lyapunov exponents, Hausdorff dimension and entropy, Ergod. Th. and Dynam. Sys. **1** (1981), 451–459.

[O] V. I. Oseledeč, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, Trans. Moscow Math. Soc. **19** (1968), 197–221.

[Pa] W. Parry, *Entropy and Generators in Ergodic theory*, Benjamin, New York (1969).

[Pe] Ia. B. Pesin, Lyapunov characteristic exponents and smooth ergodic theory, Russian Math. Surveys **32** (1977), 55–114.

[Ro] V. A. Rohlin, On the fundamental ideas in measure theory, A. M. S. Trans. 1, **10** (1962), 1–54.

[Ru1] D. Ruelle, An inequality for the entropy of differentiable maps, Bol. Soc. Bras. Mat. **9** (1978), 83–87.

[Ru2] ———, Ergodic theory of differentiable dynamical systems, Publ. Math. I.H.E.S. **50** (1979), 27–58.

[Y] L.-S. Young, Dimension, entropy and Lyapunov exponents, Ergod. Th. and Dynam. Sys. **2** (1982), 109–129.

[G] P. Grassberger, Information aspects of strange attractors, preprint Wüppertal (FRG) (1984).

[ER] J.-P. Eckman and D. Ruelle, Ergodic theory of chaos and strange attractors, preprint I.H.E.S. (1985).

(Received August 6, 1984)