ERGODIC THEORY OF ATTRACTORS

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We begin with an overview of this article. Consider a dynamical system generated by a diffeomorphism f with an attractor Λ. We assume $f|\Lambda$ is sufficiently complex that it is impossible to have exact knowledge of every orbit. The ergodic theory approach, which we will take, attempts to describe the system in terms of the average or statistical properties of its "typical" orbits.

If Λ is an Axiom A attractor, then it follows from the work of Sinai, Ruelle and Bowen ([S],[R1],[BR]) in the 1970's that orbits starting from almost all initial conditions have a common asymptotic distribution. "Almost all" here refers to a full Lebesgue measure set in the basin of attraction of Λ. We will call this invariant measure the *SRB measure* of (f, Λ).

In the late 70's and early 80's the idea of a nonuniformly hyperbolic system was developed and the notion of an SRB measure was extended to this more general context. In Section 1 of this article I will define SRB measures and describe some of their ergodic and geometric properties, including their entropy and dimension.

While one could formally define SRB measures and study them abstractly, the question of how prevalent they are outside of the Axiom A category is not well understood. The first nonuniform (dissipative) examples for which SRB measures were constructed are the Hénon attractors. In Section 2, I will discuss briefly the analysis by Benedicks and Carleson [BC] of certain parameter values of the Hénon maps, and the subsequent work of Benedicks and myself [BY1] on the construction of SRB measures for these parameters.

In Section 3, I would like to present a recent work, also joint with Benedicks [BY2], in which we study stochastic processes of the form $\{\varphi \circ f^i\}_{i=0,1,2,\cdots}$ where f is a "good" Hénon map, the underlying measure is SRB, and φ is a Hölder continuous observable. We prove for these random variables the exponential decay of correlations and a central limit theorem.

While the results in Sections 2 and 3 are stated only for the Hénon family, our methods of proof are not particularly model-specific. I will conclude with some remarks on the types of situations to which these methods may apply.

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§1. Some ergodic and geometric properties of SRB measures

Let f be a C^2 diffeomorphism of a finite dimensional manifold M and let $\Lambda \subset M$ be a compact f-invariant set. We call Λ an attractor if there is a set $U \subset M$ with positive Riemannian measure such that for all $x \in U$, $f^n x \to \Lambda$ as $n \to \infty$.

Given an f-invariant Borel probability measure μ, let $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ denote the distinct Lyapunov exponents of (f, μ) and let E_i be the corresponding subspaces in the tangent space of each point. Stable and unstable manifolds are defined *a.e.* on sets with negative and positive Lyapunov exponents. They are denoted by W^s and W^u respectively.

Let (f, μ) be such that $\lambda_1 > 0$ *a.e.*, and let η be a measurable partition on M. Let $W^u(x)$ and $\eta(x)$ denote respectively the unstable manifold and element of η containing x. We say that η is *subordinate to* W^u if for *a.e.* $x, \eta(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$. For a given η, let $\{\mu_x^\eta\}$ denote a canonical family of conditional probabilities of μ with respect to η. We will use m_x^η to denote the Riemannian measure induced on $\eta(x)$ as a subset of the immersed submanifold $W^u(x)$.

Definition 1. *Let (f, μ) be as above. We say that μ has absolutely continuous conditional measures on W^u if for every measurable partition η subordinate to W^u, μ_x^η is absolutely continuous with respect to m_x^η for a.e. x.*

This definition has its origins in [S] and [R1]; in its present form it first appeared in [LS].

For Axiom A attractors the invariant measure we called *SRB* in the introduction has several equivalent definitions, one of which is that it has absolutely continuous conditional measures on W^u. My feelings are that as a working definition, this property is the most useful and the most straightforward to generalize. I therefore take the liberty to introduce the following definition:

Definition 2. *Let f and Λ be as in the beginning of this section. An f-invariant Borel probability measure μ on Λ is called an SRB measure* if $\lambda_1 > 0$ *a.e.* and μ has absolutely continuous conditional measures on W^u.

The physical significance of this property is that the set of points whose future trajectories are generic with respect to an *SRB* measure form a positive Lebesgue measure set. This is because we can "integrate out" from the attractor along W^s using the absolute continuity of the stable foliation. More precisely:

Theorem 1 ([P] [PS]). *Let μ be an ergodic SRB measure of f and assume that $\lambda_i \neq 0\ \forall i$. Then there is a set $\tilde{U} \subset M$ with positive Lebesgue measure such that if φ is a continuous function defined on a neighborhood of Λ then*

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \to \int \varphi d\mu$$

for every $x \in \tilde{U}$.

In general, entropy and Lyapunov exponents are different invariants, although both measure the complexity of a dynamical system. With respect to its SRB measure, however, the entropy of a map is equal to the sum of its positive Lyapunov exponents. Indeed, SRB measures are precisely the extreme points in the following variational principle:

Theorem 2 ([P], [R2], [LS], [L1], [LY1]). *Let μ be an f-invariant Borel probability measure. Then*

$$h_\mu(f) \le \int \sum_{\lambda_i > 0} \lambda_i \cdot \dim E_i \; d\mu \quad ;$$

and equality holds if and only if μ is SRB.

For arbitrary invariant measures, the difference between entropy and the sum of positive Lyapunov exponents can be understood in terms of the dimension of the measure. It is shown in [LY2] that if μ is ergodic, then corresponding to each $\lambda_i \neq 0$ there is a number δ_i with

$$0 \le \delta_i \le dim E_i$$

such that

$$h_\mu(f) = \sum_{\lambda_i > 0} \lambda_i \cdot \delta_i \;\; = - \sum_{\lambda_i < 0} \lambda_i \cdot \delta_i \; .$$

The number δ_i has the geometric interpretation of being the dimension of μ "in the direction of E_i"; it is equal to h_i/λ_i where h_i is the entropy "in the direction of E_i". (See [LY2] for precise definitions.)

These ideas have led to the following result on the dimension of SRB measures. For a finite measure μ, we write $dim(\mu) = \alpha$ if for $\mu - a.e.\ x$,

$$\lim_{r \to 0} \frac{\log \mu B(x,r)}{\log r} = \alpha$$

where $B(x,r)$ is the ball of radius r about x.

Theorem 3 ([L2], [LY2]). *Let μ be an SRB measure. We assume that (f,μ) is ergodic, and that $\lambda_i \neq 0 \; \forall i$. Then*

$$\dim(\mu) = \sum_{\text{all } i} \delta_i$$

where the δ_i's are as above. In particular, $\delta_i = \dim E_i$ for all i with $\lambda_i > 0$.

It is not known at this time whether this notion of dimension is well defined for arbitrary invariant measures. For a special case, see e.g. [Y1].

§2. SRB measures for Hénon maps

As we mentioned in the introduction, it follows from the work of Sinai, Ruelle and Bowen that every Axiom A attractor admits an SRB measure. It is natural to wonder to what extent this is true without the hypothesis of Axiom A. Mathematically very little has been proved, although the existence of SRB measures in general situations is often taken for granted in numerical experiments and by the physical scientist.

To the best of my knowledge, the first dissipative, genuinely nonuniformly hyperbolic attractor for which SRB measures were constructed are the Hénon attractors. (By "dissipative" I mean not volume preserving: if a volume preserving diffeomorphism has a positive Lyapunov exponent $a.e.$ then its volume measure satisfies the condition in Definition 2.) The Hénon maps are a 2-parameter family of maps $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}.$$

It is not hard to see that there is an open region in parameter space for which $T_{a,b}$ has an attractor; and that for (a,b) in the region, there is a continuous family of invariant cones on $\{|x| > \delta\}$ (δ depending on parameters) but that the attractor is not Axiom A.

In [BC2], Benedicks and Carleson proved that for b sufficiently small, there is a positive measure set of a's for which $T = T_{a,b}$ has a positive Lyapunov exponent on a dense subset of Λ. In addition to proving this result they devised a machinery for analyzing DT^n, the derivatives of the iterates of T, for certain orbits with controlled behavior. Without getting into the specifics of their machinery, let me try to explain the essence of what is going on:

Some of the ideas go back to 1-dimension, so let me first explain how expanding properties are proved for the quadratic family $f_a : x \to 1 - ax^2, x \in [-1,1], a \in [0,2]$. Yakobson [J] proved in 1981 that for a positive measure set of parameters a, f_a admits an invariant measure absolutely continuous with respect to Lebesgue and has a positive Lyapunov exponent $a.e.$ Roughly speaking, the "good" parameters are those for which the derivatives along the critical orbit have exponential growth. Away from the critical point 0, we could think of the map as essentially expanding, and for x near 0, the orbit of x stays near that of 0 for some period of time, giving $(f^n)'x \sim 2ax \cdot (f^{n-1})'(f0) \sim 2ax \cdot \lambda^{n-1}$ for some $\lambda > 1$. These ideas have been used by various authors studying 1-dimensional maps (see e.g. [CE] and [BC1] as well as [J]).

An obstacle to proving hyperbolicity in dimensions greater than 1 is the switching of expanding and contracting directions. For a 2×2 matrix A that is not an isometry, let $s(A)$ denote the direction that will be contracted the most by A. Suppose that for $m, n > 0$ we have proved hyperbolicity for the stretches from $T^{-m}x$ to x and from x to $T^n x$. In order to extend this hyperbolic behavior all the way from $T^{-m}x$ to $T^n x$ we must control $\angle(s(DT_x^{-m}), s(DT_x^n))$, the angle between $s(DT_x^{-m})$ and $s(DT_x^n)$.

In some sense then, the set of points x where $\angle(s(DT_x^{-m}), s(DT_x^n)) \to 0$ as $m, n \to \infty$ plays the role of the critical point in 1-dimension. An essential difference, however, is that the exact location of this "critical set" cannot be known ahead of time. To identify points with the property above, one must prove the hyperbolicity of DT_x^{-m} and DT_x^n for arbitrarily large m and n, but the behavior of these derivatives in turn depend on how the orbit of x interacts with the critical set. This almost seems like circular reasoning, but can in fact be achieved through inductive arguments. In dimensions greater than one, the inductive character of the analysis is both more prominent and more essential than in 1-dimension.

What Benedicks and Carleson did in [BC2] was to identify and control – for a positive measure set of parameters – a critical set $\mathcal{C}$ as described above. We stress that their inductive procedure goes through only on a positive measure set of parameters. Furthermore they showed that for certain orbits approaching this set, the loss of hyperbolicity is $\sim \text{dist}(f^n x,\ \mathcal{C})$, and that subsequent recovery is guaranteed.

Building on this machinery, Benedicks and I constructed SRB measures for Hénon maps corresponding to these "good" parameters.

Theorem 4 ([BC], [BY1]). *Let $\{T_{a,b}\}$ be the Hénon family. Then for each sufficiently small b, there is a positive measure set Δ_b such that for each $a \in \Delta_b$, $T = T_{a,b}$ admits an SRB measure μ. This SRB measure is unique; its support is all of Λ; and (T, μ) is isomorphic to a Bernoulli shift.*

As a corollary to this theorem and to Theorem 1, we have a positive Lebesgue measure set in $\mathbb{R}^2$ consisting of points the statistics of whose future trajectories are governed by μ. If for instance one is to pick a point in this set and to plot its first N iterates for some sufficiently large N, then the resulting picture is essentially that of μ. It follows from our proof of Theorem 4 that this set of generic points fills up a large part of the basin of Λ; we believe (but have not yet proved) that it in fact fills up the entire basin up to a set of measure zero.

In [BC2] the analysis is focused mostly on the "bad set" $\mathcal{C}$. Part of the proof of Theorem 4 consists of adapting and globalizing these ideas to unstable manifolds. We then prove the existence of μ by pushing forward Lebesgue measure m on a piece of unstable leaf γ. A key observation is that it is only necessary to consider a positive percentage of these pushed-forward measures. Roughly speaking we show that for a positive density set of integers n, there are subsets γ_n of γ with $m(\gamma_n)$ bounded away from 0 such that for each n,

(i) $|DT^n|_{\gamma_n}| \geq c\lambda^n$ for some $\lambda > 1$;
(ii) $DT^n(\gamma_n)$ is the union of (many) roughly parallel curves of a fixed length.

An SRB measure is then extracted from the Ceasaro averages of $T_*^n(m|_{\gamma_n})$.

While the result above is stated only for the Hénon family, it holds for families with similar qualitative properties, such as those that appear in certain homoclinic bifurcations [MV].

We close this discussion by remarking that one cannot expect *all* attractors – or even all attractors with the general appearance of the Hénon attractors – to admit SRB measures. Periodic sinks are easily created near homoclinic tangencies

[N], and the presence of sinks substantially complicates the dynamical picture. Nonhyperbolic periodic points are also not condusive to the existence of invariant measures with smooth conditional measures on unstable manifolds [HY]. The question of existence of SRB measures in general is not one that is likely to be resolved in the near future.

It seems, though, that the time has come to attempt the following type of questions: given a "typical" or "generic" 1-parameter family of dynamical systems that are hyperbolic on large parts of their phase spaces without being uniformly hyperbolic everywhere, is it reasonable to expect that a positive measure set of them will admit SRB measures? (This is the "attractor" or "dissipative" version; one could also formulate similar questions for the positivity of Lyapunov exponents for conservative systems.) I will come back with some brief remarks on this in Section 4.

§*3. Decay of correlations for Hénon maps*

Independent identically distributed random variables are "chaotic" in the sense that it is impossible to predict the future from knowledge of the past, yet their distributions obey very simple limit laws. One might wonder if the same is true for processes coming from chaotic dynamical systems. For example, if f has an attractor Λ and μ is its SRB measure, what can be said about the random variables $\{\varphi \circ f^i\}_{i=0,1,2,\cdots}$ where φ is a reasonable function on Λ?

I would like to report on some recent results in this direction.

Theorem 5 [BY2]. *Let $\{T_{a,b}\}$ be the Hénon family, and let $T = T_{a,b}$ be any one of the maps in Theorem 4 shown to admit an SRB measure μ. Let $\mathcal{H}_\beta$ denote the set of Hölder continuous functions on Λ with exponent β. Then there exists $\tau < 1$ such that for all $\varphi, \psi \in \mathcal{H}_\beta$, there is a constant $C = C(\varphi, \psi)$ such that*

$$\left| \int \varphi \cdot (\psi \circ T^n) d\mu - \int \varphi d\mu \cdot \int \psi d\mu \right| \leq C\tau^n \quad \forall n \geq 1.$$

The main ideas of our proofs are as follows. Given that "horseshoes" are building blocks of uniformly hyperbolic systems, the following seems to be a natural generalization to the nonuniform setting: Let Δ_0 be a rectangular lattice obtained by intersecting local stable and unstable manifolds. Suppose that Δ_0 intersects unstable manifolds in positive Lebesgue measure sets, and that it is the disjoint union of a countable number of "s-subrectangles" $\Delta_{0,i}$ each one of which is mapped under some power of T, say under T^{R_i}, hyperbolically onto a "u-subrectangle" of Δ_0. (A subset $X \subset \Delta_0$ is called an "s-subrectangle" of Δ_0 if $X \cap \gamma = \Delta_0 \cap \gamma$ for every local stable leaf γ used to define Δ_0.) Let $R(x) = R_i$ for $x \in \Delta_{0,i}$. Then we may regard the dynamics of T as something like the discrete time version of a special flow built under the return time function R over a uniformly hyperbolic "horseshoe" with infinitely many branches.

For the "good" Hénon maps in Theorem 4, we show that sets with the properties of Δ_0 above are easily constructed. Furthermore, because of the rapid recovery after each visit to the critical set $\mathcal{C}$, the return time function R has the property that $\mu\{R > n\} < C\theta^n$ for some $\theta < 1$. This enables us to show that there is a gap in the spectrum of the Perron-Frobenius operator, proving exponential decay of correlations. (A similar tower construction is used in [Y2].)

Using this spectral property of the Perron-Frobenius operator we obtain also the Central Limit Theorem for $\{\varphi \circ T^i\}_{i=0,1,2,\cdots}$:

Theorem 6 [BY2]. *Let (T, μ) be as in Theorem 5, and let $\varphi \in \mathcal{H}_\beta$ be a function with $\int \varphi d\mu = 0$ and $\varphi \neq \psi \circ T - \psi$. Then*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ T^i \quad \overset{distribution}{\longrightarrow} \mathcal{N}(0, \sigma)$$

where $\mathcal{N}(0, \sigma)$ is the normal distribution and $\sigma > 0$ is given by

$$\sigma \ = \ \lim_{n \to \infty} \left[\frac{1}{n} \int (\sum_{i=0}^{n-1} \ \varphi \circ T^i)^2 d\mu \right]^{1/2} .$$

§*4. Final remarks*

The proofs of Theorems 4, 5 and 6 involve technical estimates specific to the Hénon maps, but I would like to point out that the ideas behind them are not model-specific and may be quite general.

Very roughly speaking, the existence and mixing properties of SRB measures seem to be related to the rates at which arbitrarily small pieces of unstable manifolds grow to a fixed size (which is more than just the existence of a positive Lyapunov exponent pointwise). To formulate something more precisely, one could look for a set with the properties of Δ_0 in the last section, and study the return time function R. If R is integrable with respect to Lebesgue measure on W^u- leaves, then an SRB measure exists; and if in addition to that, R has an exponentially decaying tail estimate as in Section 3, then the system has the exponential mixing property provided all powers of the map are ergodic.

In general, I doubt that it is possible to determine the nature of R from the overall appearance of a dynamical system. If, however, there is a recognizable "bad set" – in the sense that away from this set the map is uniformly hyperbolic (with no discontinuities), and when an orbit gets near it there is a quantifiable loss in hyperbolicity followed by a "recovery period" – then, as observed in [BY2], there are often natural candidates for Δ_0, and the character of the return time function R is directly related to the rate of recovery after each encounter with the "bad set". In particular, if the recovery is "exponential" (meaning it takes $\sim log\frac{1}{\delta}$ iterates to recover fully from a loss $\sim \delta$) then R has an exponentially decaying tail estimate.

Obvious examples that fit into this "bad set – recovery" scenario are large classes of piecewise uniformly hyperbolic maps, including certain billiards, where the "bad set" is the set of singularity curves (see also the recent preprint [Li]), and quadratic maps of the interval whose critical orbits carry positive Lyapunov exponents (see Section 3). It is less obvious *a priori* that the Hénon maps fit into this category; indeed the various notions there have to be interpreted with a bit more care. The rate of recovery is exponential in these examples, but not in e.g. [HY].

It is certainly not the case that all nonuniformly hyperbolic systems have recognizable "bad sets", nor am I suggesting a generic theorem that can be applied to all "bad set – recovery" type scenarios. I wish only to point out that many of the known nonuniform examples belong in this category, and I hope that the methods discussed here will shed some light on the ergodic properties of these systems – as well as on other systems with similar characteristics.

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