Probab. Th. Rel. Fields 80, 217–240 (1988)

Probability Theory and Related Fields

Entropy Formula for Random Transformations

F. Ledrappier[1] and L.-S. Young[2]*

[1] Laboratoire de Probabilités, Université Pierre et Marie Curie (Paris VI), 4, place Jussieu, Tour 56, 3ème Etage, F-75252 Paris Cedex 05, France

[2] Department of Mathematics, University of Arizona, Tucson AZ 85721, USA

Summary. We exhibit random strange attractors with random Sinai-Bowen-Ruelle measures for the composition of independent random diffeomorphisms.

We consider in this paper compositions of independent random diffeomorphisms of a compact manifold M. This set-up occurs in the theory of stochastic differential equations (see [Ki] and [Ku]). It has been used as a model for studying the effect of noise on deterministic dynamical systems (see e.g., [Y]). It can also be thought of as a random walk on Diff M, the group of diffeomorphisms of M. Our aim here is to communicate some results on the ergodic theory of these random maps.

Let (Ω, ν) be a probability space which is identified with Diff M, and let μ be a stationary probability measure for the associated one-point Markov process, i.e., μ satisfies $\mu = \int f_\omega \mu \, \nu(d\omega)$. We shall refer to this process as $\mathfrak{X} = \mathfrak{X}(M, \nu; \mu)$. For ease of reference, we shall call the composition of a sequence of maps a *composed* map. Given $\mathfrak{X}$, one can associate with typical x in M Lyapunov exponents $\lambda_i(x)$, with multiplicity $m_i(x)$, $i = 1, \ldots, r(x)$. These numbers describe the asymptotic growth rates of the derivation at x of the typical composed map (see [Ki]).

If all the exponents are negative and the stationary measure is ergodic, then the dynamics of $\mathfrak{X}$ is the dynamics of a random sink: a typical composed map send most of the measure into the neighborhood of a finite random set, usually a single point. In other words each of the "sample measures" is supported on a finite set ("sample measures" are the conditional measures of μ given past sequences of maps [Y]. They are also called statistical equilibrium in [Le J]). Furthermore, the two-point Markov process associated to $\mathfrak{X}$ is transient on $M \times M$ minus the diagonal.

In this paper we are interested in the case when the largest exponent is positive. Under the hypothesis that the stationary measure is absolutely continuous with respect to Lebesgue, we exhibit what could be called a random strange attractor.

---

* The research of this author is partially supported by the National Science Foundation and the Sloan Foundation.

Theorem B. *Assume $\mu \ll$ Lebesgue and $\lambda_i > 0$ for some i. Then the sample measures have the Sinai-Bowen-Ruelle (SBR) property.*

Roughly speaking, we say that a measure has the SBR property if locally there exists a k-dimensional foliation (k is the total multiplicity of the positive exponents) such that the measure is the product of a transverse measure and a smooth measure along the leaves of this foliation. Theorem B reflects the fact that a typical composed map expands and smooths along some k-dimensional objects. It extends to the random context a classical property of hyperbolic attractors [Bo, S] and of measure perserving diffeomorphisms [Pe]. This property is believed to hold for a large class of attracting sets (see [ER]).

One consequence of Theorem B is that the two-point process associated with $\mathfrak{X}$ is positively recurrent off the diagonal. Furthermore, in the case when no exponent is zero, we deduce dimensional properties for the sample measures, and also mixing properties of the underlying process (see Theorem C).

If the stationary measure μ itself is invariant under v-a.e. map f_ω (or equivalently if the sample measures are all equal to μ, or else if $\Sigma \lambda_i m_i = 0$ μ-a.e.), then Theorem B can be proved directly by "relativizing" the arguments used for a single diffeomorphism, and the (relative) entropy is given by Pesin's formula (see [Ki], [KS]). The bulk of our work here consists in showing that this entropy formula holds when the stationary measure is smooth – without requiring that the maps individually preserve a smooth measure.

Theorem A. *Assume $\mu \ll$ Leb., and let h be the entropy of $\mathfrak{X}(M, v; \mu)$. Then $h = \int \Sigma \lambda_i^+ m_i d\mu$, where $\alpha^+ = \max(\alpha, 0)$.*

Theorem B is deduced from this entropy formula using the characterization of (relative) SBR-measures by a (relative) variational principle. This paper is organized as follows: Sect. 1 contains some definitions and background information. In Sect. 2 we review the random version of some basic concepts from the ergodic theory of a single transformation. Much of the material in these two sections can be found in [Ki]. Precise statements of our results are given is Sect. 3. The next two sections are devoted to the proof of Theorem A, while in Sect. 6 we indicate how the proofs of Theorems B and C are obtained. We include in the appendix a brief explanation of how the theory of stochastic flows is related to our study.

Part I

1. Preliminaries

(1.1) *General Setting*

Let M be a C^∞ compact Riemannian manifold and let $\mathrm{Diff}^2(M)$ denote the group of C^2 diffeomorphisms of M onto itself. Let $(\Omega, \mathfrak{F})$ be a probability space which we identify with $\mathrm{Diff}^2(M)$ together with the Borel σ-algebra generated by its C^2-topology. For $\omega \in \Omega$, we denote the corresponding diffeomorphism by f_ω.

Let ν be a probability on $(\Omega, \mathfrak{F})$. For technical reasons we shall assume throughout this paper that

$$\begin{aligned}&\int \mathrm{Log}^+ |f_\omega|_{C^2}\, \nu(d\omega) < \infty\\ &\int \mathrm{Log}^+ |f_\omega^{-1}|_{C^2}\, \nu(d\omega) < \infty\end{aligned} \tag{$*$}$$

where $|f|_{C^2}$ denotes the C^2-norm of $f \in \mathrm{Diff}^2(M)$.

We are concerned with the ergodic theory associated with the successive application of randomly chosen maps in $\mathrm{Diff}^2(M)$. These maps will be independent and identically distributed with law ν. More precisely, let $\Omega^{\mathbb{Z}}$ denote the bi-infinite product of Ω with itself and let $\nu^{\mathbb{Z}}$ denote the corresponding product measure of ν. For each $\underline{\omega} = \ldots \omega_{-1}\omega_0\omega_1 \ldots \in \Omega^{\mathbb{Z}}$, we define for $n > 0$

$$f_{\underline{\omega}}^n = f_{\omega_{n-1}} \circ \ldots f_{\omega_1} \circ f_{\omega_0}$$

and

$$f_{\underline{\omega}}^{-n} = f_{\omega_{-n}}^{-1} \circ \ldots \circ f_{\omega_{-1}}^{-1}.$$

Our goal is to study the asymptotic behavior of these compositions as $n \to \infty$ for $\nu^{\mathbb{Z}}$-a.e. $\underline{\omega}$. This set-up will be referred to as $\mathfrak{X}(M, \nu)$ in the rest of this paper.

(1.2) *Stationary and Sample Measures*

(1.2.1) **Definition.** A Borel probability measure μ on M is called a *stationary* measure for $\mathfrak{X}(M, \nu)$ if

$$\mu = \int f_\omega \mu \;\nu(d\omega).$$

Let $\mathfrak{M}(\mathfrak{X})$ denote the set of stationary measures for $\mathfrak{X}(M, \nu)$. Since M is compact and $x \to \int \delta_{f_\omega x} \nu(d\omega)$ is continuous, $\mathfrak{M}(\mathfrak{X})$ is nonempty. In fact, $\mathfrak{M}(\mathfrak{X})$ is a compact convex set with respect to the weak topology. Its extreme points are called ergodic. We will return to this notion of ergodicity later.

One view of $\mathfrak{X}(M, \nu)$ with stationary measure μ-which we will henceforth abbreviate as $\mathfrak{X}(M, \nu; \mu)$ – is the following skew product: let $\tau : \Omega^{\mathbb{N}} \circlearrowleft$ be the shift operator, i.e., if $(\underline{\omega}^+)_n$ denotes the n^{th} coordinate of $\underline{\omega}^+ \in \Omega^{\mathbb{N}}$, then $(\tau\underline{\omega}^+)_n = (\underline{\omega}^+)_{n+1}$. It is easy to verify that $F^+ : \Omega^{\mathbb{N}} \times M \circlearrowleft$ defined by

$$F^+(\underline{\omega}^+, x) = (\tau\underline{\omega}^+, f_{\omega_0} x)$$

preserves the measure $\nu^{\mathbb{N}} \times \mu$.

We shall consider an invertible extension of $(F^+, \nu^{\mathbb{N}} \times \mu)$ as follows: let $F : \Omega^{\mathbb{Z}} \times M \circlearrowleft$ be defined by

$$F(\underline{\omega}, x) = (\tau\underline{\omega}, f_{\omega_0} x).$$

Clearly, there exists a unique F-invariant probability measure on $\Omega^{\mathbb{Z}} \times M$, the projection of which on $\Omega^N \times M$ is the measure $\nu^N \times \mu$. We denote this probability measure by μ^*. In most of our arguments, we shall be working with (F, μ^*). We shall identify $(F^+, \nu^{\mathbb{N}} \times \mu)$ with the action of F on the decreasing σ-algebra generated by $\{x, \underline{\omega}_n, n \geq 0\}$. Sometimes we also condition with respect to the invariant sub-σ-algebra generated by $\{\underline{\omega}_n, n \in \mathbb{Z}\}$, which amounts to choosing $\underline{\omega}$ in $\Omega^{\mathbb{Z}}$ and studying the action of $\{f_{\underline{\omega}}^n, n \in \mathbb{Z}\}$ on M. This leads to:

(1.2.2) **Definition.** *Given $(M, \nu; \mu)$ and the associated measure μ^* on $\Omega^{\mathbb{Z}} \times M$. We call the family of conditional measures of μ^* on M-fibers the family of sample measures.*

(1.2.3) **Proposition.** *Let μ be a stationary measure for $\mathfrak{X}(M, \nu)$. Then the family of sample measures $\{\mu_{\underline{\omega}}, \underline{\omega} \in \Omega^{\mathbb{Z}}\}$ is the essentially unique measurable family of probability measures on M with the following three properties:*

(1) $f_{\omega_0} \mu_{\underline{\omega}} = \mu_{\tau \underline{\omega}}$ $\nu^{\mathbb{Z}}$-a.e.

(2) $\mu_{\underline{\omega}}$ *depends only on* $\underline{\omega}_n$, $n < 0$,

(3) $\int \mu_{\underline{\omega}} \nu^{\mathbb{Z}}(d\underline{\omega}) = \mu$.

Moreover, for $\nu^{\mathbb{Z}}$-a.e. $\underline{\omega}$, $f^n_{\tau^{-n}\underline{\omega}} \mu \to \mu_{\underline{\omega}}$ *as* $n \to \infty$.

Proof. Property (1) is the invariance relation of μ^*, properties (2) and (3) express that μ^* projects on $\Omega^{\mathbb{N}} \times M$ into the measure $\nu^{\mathbb{N}} \times \mu$.

Conversely if $\{\mu_{\underline{\omega}}, \omega \in \Omega^{\mathbb{Z}}\}$ satisfies (1), (2), (3), the measure $\mu_{\underline{\omega}}(dx) \nu^{\mathbb{Z}}(d\underline{\omega})$ is F-invariant and projects on $\Omega^{\mathbb{N}} \times M$ into $\nu^{\mathbb{N}} \times \mu$.

Finally, the limit property is the approximation of conditional measures with respect to $\{\underline{\omega}_m, m \in \mathbb{Z}\}$ by conditional measures with respect to $\{\underline{\omega}_m, m \geqq -n\}$.

Sample measures are studied in e.g., [LeJ] and [Y].

(1.3) *Other representation of* $\mathfrak{X}(M, \nu; \mu)$

We mention three more views of the object $\mathfrak{X}(M, \nu; \mu)$. The first one is the Markov process with state space $\Omega \times M$, initial distribution $\nu \times \mu$, transition probabilities

$$Q(\Gamma | (\omega, x)) = \nu\{\omega' : (\omega', f_\omega x) \in \Gamma\}$$

The standard representation of this process on $(\Omega \times M) \times (\Omega \times M) \times \ldots$ with the shift operator in clearly isomorphic to $F_+ : (\Omega^{\mathbb{N}} \times M, \nu^{\mathbb{N}} \times \mu)$.

Another representation of $\mathfrak{X}(M, \nu; \mu)$ is the Markov process with state space M, initial distribution μ, and transition probabilities

$$P(E|x) = \nu\{\omega : f_\omega x \in E\}.$$

This is a factor of the process above. Some readers may prefer to view Theorem C, for instance, in this context.

A further representation of $\mathfrak{X}(M, \nu; \mu)$ is the two-point Markov process with state space $M \times M$ and transition probabilities given by $P_2(E \times F/(x, y) = \nu\{\omega : f_\omega x \in E$ and $f_\omega y \in F\}$. The importance of this process comes from the fact that for continuous time models, it describes unambiguously the original process on Diff M ([Ba]). From Proposition 1.2.3 follows:

(1.3.1) **Proposition.** *Let μ be a stationary measure for $\mathfrak{X}(M, \nu)$, $\{\mu_{\omega}, \omega \in \Omega^{\mathbb{Z}}\}$ the family of sample measures, $(M \times M, P_2)$ the two-point Markov process. Then the measure μ^2 defined by*

$$\mu^2(E \times F) = \int \mu_{\underline{\omega}}(E) \mu_{\underline{\omega}}(F) \nu^{\mathbb{Z}}(d\underline{\omega})$$

is P_2-invariant and satisfies

$$(\mu\times\mu)\cdot P_2^n\to\mu^2 \quad as \quad n\to\infty\,.$$

(1.3.2) **Corollary.** *Let $\mathfrak{X}(M,\nu;\mu)$ be such that the sample measures are continuous. Then, the two-point Markov process is recurrent outside the diagonal.*

This is clear since Proposition 1.3.1 gives us an invariant probability measure μ^2 such that

$$\mu^2(\text{Diagonal})=\int(\mu_{\underline{\omega}}\times\mu_{\underline{\omega}})(\text{Diagonal})\;\nu^{\mathbb{Z}}(d\underline{\omega})=0\,.$$

(1.4) *Ergodicity and Uniqueness of Stationary Measures*

Let $\mathfrak{X}(M,\nu;\mu)$ be as always.

(1.4.1) **Proposition.** *If $A\subset\Omega^{\mathbb{N}}\times M$ is an F^+-invariant subset, then there is a Borel subset $B\subset M$ with the property that*

$$(\nu^{\mathbb{N}}\times\mu)(A\Delta(\Omega^{\mathbb{N}}\times B))=0\,.$$

Proof. See [B] for a proof in the Markov chain setting.

The fact that F^+-invariant sets (and hence F-invariant sets) correspond essentially to subsets of M leads to the following characterization of ergodicity.

(1.4.2) **Proposition.** *The following are equivalent:*

(1) *μ is ergodic, i.e., it is an extreme point of $\mathfrak{M}(\mathfrak{X})$;*

(2) *M cannot be written as a disjoint union $C_1\cup C_2$ where for $i=1,2$ we have*

$$f_\omega C_i\subset C_i$$

for ν-a.e. ω;

(3) *$F^+:(\Omega^{\mathbb{N}}\times M,\;\nu^{\mathbb{N}}\times\mu)$ is ergodic;*

(4) *$F:(\Omega^{\mathbb{N}}\times M,\;\mu^*)$ is ergodic.*

Our next remark concerns a situation of particular interest.

(1.4.3) *Remark.* Suppose $P(\cdot|x)$ as defined in (1.3) is $\ll$Leb. for every $x\in M$. Then:

(1) Every stationary measure μ is $\ll$Leb.

(2) If there does not exist two disjoint Borel subsets C_1, C_2 of M, both with positive Lebesgue measure, s.t.

$$f_\omega C_i=C_i$$

up to sets of Leb. measure 0 for ν-a.e. ω, then μ is unique. In other words, with the hypotheses above, the family $\{\mu_{\underline{\omega}}\}$ is completely determined by $\mathfrak{X}(M,\nu)$.

2. Basic Concepts from the Ergodic Theory of a Single Diffeomorphism

(2.1) *Entropy*

We define a notion of "fiber entropy" for the skew product (F,μ^*). Let $\mathfrak{a}$ be the σ-algebra of subsets of $\Omega^{\mathbb{Z}}\times M$ generated by cylinders of $\Omega^{\mathbb{Z}}$, and let P be a measurable partition of $\Omega^{\mathbb{Z}} \times M$ with $H_{\mu^*}(P) < \infty$. We define

$$h_{\mu^*}(F; P|\mathfrak{a}) = \lim_{n\to\infty} \frac{1}{n} H_{\mu^*}\left(\bigvee_0^{n-1} F^{-i}P|\mathfrak{a}\right).$$

The limit on the right exists by a relativized version of the Shannon-Breiman-McMillan Theorem.

(2.1.1) **Definition.** The entropy of $\mathfrak{X}(M, \nu)$ with stationary measure μ is defined to be

$$\operatorname*{Sup}_{\substack{P \text{ with} \\ H_{\mu^*}(P)<\infty}} h_{\mu^*}(F; P|\mathfrak{a}).$$

This number is denoted by $h(\mathfrak{X}(M, \nu; \mu))$, or simply h.

In some ways it is more natural to define the entropy of random maps by taking a partition Q on M with small diameter, a typical $\underline{\omega} \in \Omega^{\mathbb{Z}}$, and to compute entropy as approximately

$$h_\mu(f_{\underline{\omega}}; Q) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_0^{n-1} (f_{\underline{\omega}}^i)^{-1} Q\right).$$

(2.1.2) **Proposition.** *Let Q be a partition of M with $H_\mu(Q) < \infty$. Then for a.e. $\underline{\omega}$,*

$$h_\mu(f_{\underline{\omega}}; Q) = h_{\mu^*}(F; \tilde{Q}|\mathfrak{a})$$

where $\tilde{Q} = \{\Omega^{\mathbb{Z}} \times A,\ A \in Q\}$. In particular,

$$\operatorname*{Sup}_Q h_\mu(f_{\underline{\omega}}; Q) = h.$$

See [Ki] for details.

(2.2) *Lyapunov Exponents*

For $x \in M$, let T_xM denote the tangent space to M at x. We consider (F, μ^*) and view $(\underline{\omega}, x) \to T_xM$ as a bundle over $\Omega^{\mathbb{Z}} \times M$. Oseledec's Theorem then tells us that there is a measurable splitting of this bundle into

$$E_1(\underline{\omega}, x) \oplus \ldots \oplus E_{r(\underline{\omega}, x)}(\underline{\omega}, x)$$

such that at μ^*-a.e. $(\underline{\omega}, x)$, if $v \neq 0 \in E_i(\underline{\omega}, x)$, then

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Log} |Df_{\underline{\omega}}^{\pm n} v| = \pm \lambda_i(\underline{\omega}, x).$$

(See e.g. [Led 3] Théorème I.4.2).

Moreover, the functions $(\underline{\omega}, x) \to r(\underline{\omega}, x)$, $\lambda_i(\underline{\omega}, x)$ and $\dim E_i(\underline{\omega}, x)$ $i = 1, \ldots, r((\underline{\omega}, x)$, are measurable and are constant along orbits of F.

(2.2.1) **Definition.** The numbers $\lambda_i(\underline{\omega}, x)$ with their respective multiplicities $m_i(\underline{\omega}, x)$ are called the Lyapunov exponents of $\mathfrak{X}(M, \nu; \mu)$.

Just as with entropy, it may be natural to define exponents by looking at $f_{\underline{\omega}}^n$ for $n \geqq 0$ and considering μ-a.e. x.

(2.2.2) *Remark*

(1) The set of Lyapunov exponents for F^+ is identical to that of F, with the obvious correspondence.
(2) It follows from Proposition 1.4.1 that there are functions $r, \lambda_i, m_i: M \to \mathbb{R}$ s.t. for μ^*-a.e. $(\underline{\omega}, x)$, $r(\underline{\omega}, x) = r(x)$, $\lambda_i(\underline{\omega}, x) = \lambda_i(x)$ and $\dim E_i(\underline{\omega}, x) = m_i(x)$.

(2.3) *Stable and Unstable Manifolds*

Let $(\underline{\omega}, x) \in \Omega^{\mathbb{Z}} \times M$ be s.t. $\lambda_i(x) > 0$ for some i.

(2.3.1) **Definition**

$$W^u(\underline{\omega}, x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f_{\underline{\omega}}^{-n} x, f_{\underline{\omega}}^{-n} y) < 0 \right\}$$

and

$$W^s(\underline{\omega}, x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f_{\underline{\omega}}^{n} x, f_{\underline{\omega}}^{n} y) < 0 \right\}$$

are called respectively the unstable manifold and the stable manifold of F at $(\underline{\omega}, x)$.

At μ^*-a.e. $(\underline{\omega}, x)$ with $\lambda_i(x) > 0$ for some i, $W^u(\underline{\omega}, x)$ is a $\left(\sum_{\lambda_i > 0} \dim E_i(\underline{\omega}, x)\right)$-dimensional C^2 immersed submanifold of M. It is tangent at x to the subspace $\bigoplus_{\lambda_i > 0} E_i(\underline{\omega}, x)$. Analogous properties hold for $W^s(\underline{\omega}, x)$. This is proved by first constructing local versions of stable and unstable manifolds. See [C1] or [Rue].

When using properties of μ, it is important that no reference be made to the past. (Otherwise μ becomes $\mu_{\underline{\omega}}$!). Hence we stress the following dependence.

(2.3.2) *Remark*. The definition of $W^s(\underline{\omega}, x)$ and many of its properties depend only on x and ω_n for $n \geqq 0$. By contrast, $W^u(\underline{\omega}, x)$ depends on x and ω_n, $n < 0$.

(2.4) *Inequalities Relating Entropy and Exponents*

Let $\mathfrak{X}(M, v; \mu)$ be given.

(2.4.1) **Proposition.** (*Ruelle's inequality*).

$$h \leqq \int \Sigma \lambda_i^+ m_i d\mu .$$

See [Ki] for a proof.

(2.4.2) **Proposition.** *Suppose* $\mu \ll \text{Leb}$. *Then*

(1) $\Sigma \lambda_i^+ m_i \leqq 0 \quad \mu\text{-a.e.}$,

(2) $\Sigma \lambda_i^+ m_i = 0 \quad \mu\text{-a.e.}$ *iff* $f_\omega \mu = \mu$ *for* v-a.e. ω.

See [Ki].

3. Precise Statements of Results

We assume throughout that $\mathfrak{X}(M, \nu)$ is as defined in (1.1) with the C^2-norms of the f_ω's satisfying condition $(*)$ in (1.1), and that μ is a stationary measure for $\mathfrak{X}(M, \nu)$.

(3.1) *Entropy Formula*

Let h, λ_i and m_i be as defined In Sect. 2.

Theorem A. *Suppose* $\mu \ll$ Leb. *Then*

$$h = \int \Sigma \lambda_i^+ m_i d\mu$$

where $\alpha^+ = \max(\alpha, 0)$.

(3.2) *Geometric Properties of Sample Measures*

Let $\{\mu_{\underline{\omega}}\}$ be the sample measures associated with μ as defined in (1.2). To be completely accurate in the definitions and statements to follow, we set $W^u(\underline{\omega}, x) = \{x\}$ when $\lambda_i(x) \leqq 0$ for all i. If η is a partition of $\Omega^{\mathbb{Z}} \times M$, we let $\eta_{\underline{\omega}}$ denote the restriction of η to the fiber $\{\underline{\omega}\} \times M$ and consider it a partition of M.

(3.2.1) **Definition.** A measurable partition η of $\Omega^{\mathbb{Z}} \times M$ is said to be *subordinate to* W^u if for μ^*-a.e. $(\underline{\omega}, x)$, $\eta_{\underline{\omega}}(x) \subset W^u(\underline{\omega}, x)$ and contains an open neighborhood of x in $W^u(\underline{\omega}, x)$, this neighborhood being taken in the submanifold topology of $W^u(\underline{\omega}, x)$.

Confusing σ-algebras with their corresponding partitions, recall that $\mathfrak{a}$ is the partition of $\Omega^{\mathbb{Z}} \times M$ into sets of the form $\{\underline{\omega}\} \times M$. Let η be a partition subordinate to W^u. Then μ^* disintegrates into a canonical system of conditional measures on elements of $\eta \vee \mathfrak{a}$. (See e.g., [Ro]). We denote this by $\{\mu^{*\eta \vee \mathfrak{a}}_{(\underline{\omega}, x)}\}$. Identifying $\{\underline{\omega}\} \times M$ with M, we have $\mu^{*\eta \vee \mathfrak{a}}_{(\underline{\omega}, x)} = (\mu_{\underline{\omega}})^{\eta_{\underline{\omega}}}_x$ for μ^*-a.e. $(\underline{\omega}, x)$.

(3.2.2) **Definition.** We say that μ^*, or equivalently $\{\mu_{\underline{\omega}}\}$, has *absolutely continuous conditional measures on* W^u*-manifolds* if for every measurable partition η subordinate to W^u, $\mu^{*\eta \vee \mathfrak{a}}_{(\underline{\omega}, x)} \ll \lambda_{W^u(\underline{\omega}, x)}$ for μ^*-a.e. $(\underline{\omega}, x)$. Here $\lambda_{W^u(\underline{\omega}, x)}$ denotes the Riemannian measure on $W^u(\underline{\omega}, x)$ that comes from its inherited Riemannian structure as a submanifold of M.

Theorem B. *Suppose* $\mu \ll$ Leb. *Then* $\{\mu_{\underline{\omega}}\}$ *has absolutely continous conditional measures on* W^u*-manifolds.*

In the case of a single diffeomorphism, invariant measures with this property are sometimes called Sinai-Bowen-Ruelle measures. We generalize another geometric property of these measures. (See [Led^2]).

(3.2.3) **Definition.** Let X be a compact metric space, let m be a finite Borel measure on X, and let $B(x, \varepsilon)$ denote the ε-ball about x. Then the dimension of m, written $\dim(m)$, is defined to be α if for m-a.e. x,

$$\lim_{\varepsilon \to 0} \frac{\operatorname{Log} m B(x, \varepsilon)}{\operatorname{Log} \varepsilon}$$

exists and equals α.

We remark that finite Borel measures in general do not have a well defined dimension.

Corollary to Theorem B. *Suppose* $\mu \ll \mathrm{Leb.}$ *and is ergodic. Assume also that* $\lambda_i \neq 0 \ \forall i$. *Then* $\dim(\mu_{\underline{\omega}})$ *exists for* $\nu^{\mathbb{Z}}$-a.e. $\underline{\omega}$ *and* $\underline{\omega} \to \dim(\mu_{\underline{\omega}})$ *is constant* a.e.

(3.3) *Ergodic Properties of* $\mathfrak{X}(M, \nu; \mu)$

Theorem C. *Suppose* $\mu \ll \mathrm{Leb.}$ *and that for* μ-a.e. x, $\lambda_i(x) \neq 0 \forall i$. *Then*

(1) (F, μ^*) *has at most a countable number of ergodic components;*

(2) *each ergodic components is isomorphic to the product of a Bernoulli shift and a finite system.*

In view of Proposition 1.4.1 we have

Corollary to Theorem C. *Suppose* $\mu \ll \mathrm{Leb.}$, *is ergodic, and* $\lambda_i \neq 0 \ \forall i$. *Then either*

(1) (F, μ^*) *is Bernoulli (and hence mixing)*

or

(2) $\exists n \in \mathbb{Z}^+$, *a permutation* $\sigma: \{1, \ldots, n\} \circlearrowleft$, *and mutually disjoints subsets* $A_1, \ldots, A_n$ *of* M *each having positive Lebesgue mesure*, s.t.

$$f_\omega A_i = A_{\sigma(i)}$$

for ν-a.e. ω, *and* $F^n|(\Omega^{\mathbb{Z}} \times A_i)$ *is Bernoulli.*

Part II. Proofs

As with Pesin's formula for diffeomorphisms, Theorem A is valid because of the existence of a smooth invariant measure. However, μ is invariant only when one averages over all past histories. So to use the smoothness of μ, our constructions must essentially depend only on $f^n_{\underline{\omega}}$ for $n \geq 0$. We take a slightly different approach than in [M] or [Pe], for both of these proofs, as they stand, involve some knowledge of backward iterates.

The discussion above does not apply to Theorems B and C, which concern the $\mu_{\underline{\omega}}$'s. In fact, as we shall see, Theorems B and C follows readily from standard techniques for diffeomorphisms once Theorem A is proved.

4. Technical preparations for the proof of Theorem A

In (4.1) and (4.2) we write down the "random version" of some results that are known for maps. These proofs do not contain any new elements and will be omitted. In (4.3) we construct a partition subordinate to W^s that will be used for estimating entropy.

To stress the fact that all the constructions here are independent of the past, we will work exclusively with $F^+ : (\Omega^{\mathbb{N}} \times M, \ \nu^{\mathbb{N}} \times \mu)$ in this section.

(4.1) *One-Sided Lyapunov Charts and Stable Manifolds*

Let $\Lambda_0 = \{(\underline{\omega}^+, x) \in \Omega^{\mathbb{N}} \times M : (\underline{\omega}^+, x)$ is regular with respect to $f^n_{\underline{\omega}^+}$ in the sense of Oseledec$\}$. For $(\underline{\omega}^+, x) \in \Lambda_0$, we let

$$\lambda(\underline{\omega}^+, x) = -\left(\max_{\lambda_i < 0} \lambda_i(x)\right)$$

and

$$E^s(\underline{\omega}^+, x) = \bigoplus_{\lambda_i < 0} E_i(\underline{\omega}^+, x).$$

For fixed $\lambda > 0$ and $0 \leqq k \leqq \dim M$, we let

$$\Lambda(\lambda, k) = \{(\underline{\omega}^+, x) \in \Lambda_0 : \lambda(\underline{\omega}^+, x) \geqq \lambda,\ \dim E^s(\underline{\omega}^+, x) = k\}$$

and consider these F^+-invariant sets one at a time. For definiteness we assume $0 < k < \dim M$.

One sided charts for endomorphisms are treated in [KS], to which we refer the reader for details. We state the "random version":

Let $\varepsilon > 0$ be a number very small compared to λ. Then there is a measurable function $l : \Lambda(\lambda, k) \to \mathbb{R}^+$ s.t. for $(\underline{\omega}^+, x) \in \Lambda(\lambda, k)$ and $n \geqq 0$, we have

(1) (a) $v \in E^s(\underline{\omega}^+, x) \Rightarrow |Df^n_{\underline{\omega}^+}(x) v| \leqq l(\underline{\omega}^+, x) e^{-n(\lambda - \varepsilon)} |v|$;

(b) $v \in E^s(\underline{\omega}^+, x)^\perp \Rightarrow |Df^n_{\underline{\omega}^+}(x) v| \geqq l(\underline{\omega}^+, x)^{-1} e^{-\varepsilon n} |v|$

(where $V^\perp$ = orthogonal complement of V);

(c) $\measuredangle(Df^n_{\underline{\omega}^+}(x) E^s(\underline{\omega}^+, x),\ Df^n_{\underline{\omega}^+}(x)(E^s(\underline{\omega}^+, x))^\perp)$

$$\geqq l(\underline{\omega}^+, x)^{-1} e^{-\varepsilon n};$$

and

(2) There is a chart $\{\phi_n(\underline{\omega}^+, x)\}$ s.t. $\phi_n(\underline{\omega}^+, x)$ is a diffeomorphism from a neighborhood of $f^n_{\underline{\omega}^+} x$ in M onto the $l(\underline{\omega}^+, x)^{-1} e^{-\varepsilon n}$ – disk centered at 0 in $\mathbb{R}^{\dim M}$. These charts have the following properties:

(a) (i) $\phi_0(\underline{\omega}^+, x) x = 0$,

(ii) $D(\phi_0(\underline{\omega}^+, x))(x) E^s(\underline{\omega}^+, x) = \mathbb{R}^k \times \{0\}$,

(iii) $D(\phi_0(\underline{\omega}^+, x))(x)(E^s(\underline{\omega}^+, x))^\perp = \{0\} \times \mathbb{R}^{\dim M - k}$;

(b) If $\tilde{f}_n(\underline{\omega}^+, x) = \phi_{n+1}(\underline{\omega}^+, x) \circ f_{\omega_n} \circ \phi_n(\underline{\omega}^+, x)^{-1}$ is defined wherever it makes sense, then

(i) $\tilde{f}_n(\underline{\omega}^+, x)(0) = 0$

(ii) $D\tilde{f}_n(\underline{\omega}^+, x)(0) = \begin{pmatrix} A_n(\underline{\omega}^+, x) & 0 \\ 0 & B_n(\underline{\omega}^+, x) \end{pmatrix}$

where $A_n(\underline{\omega}^+, x) : \mathbb{R}^k \to \mathbb{R}^k$ satisfies

$$\|A_n(\underline{\omega}^+, x) v\| \leqq e^{-(\lambda - \varepsilon)} \|v\|,$$

$B_n(\underline{\omega}^+, x) : \mathbb{R}^{\dim M - k} \to \mathbb{R}^{\dim M - k}$ satisfies

$$\|B_n(\underline{\omega}^+, x) v\| \geqq e^{-\varepsilon} \|v\|,$$

($\|\cdot\|$ = Euclidean norm),

(iii) $L(\tilde{f}_n(\underline{\omega}^+, x) - D\tilde{f}_n(\underline{\omega}^+, x)(0)) < \varepsilon$

where $L(\cdot) =$ Lipschitz constant

(iv) $L(D\tilde{f}_n(\underline{\omega}^+, x)) \leqq l(\underline{\omega}^+, x)$;

(c) For Z in the domain of $\phi_n(\underline{\omega}^+, x)$ and $v \in T_Z M$,

$$l(\underline{\omega}^+, x)^{-1} e^{-\varepsilon n} \leqq \frac{|v|}{\|D\phi_n(\underline{\omega}^+, x)(Z)v\|} \leqq l(\underline{\omega}^+, x) e^{\varepsilon n}.$$

(4.1.1) *Remark.* We point out that without uniform bounds on the first and second derivatives of f_ω, $\omega \in \Omega$, special care has to be taken to arrange for (2)(b)(iii) and (iv). This can be done by incorporating the function

$$l_1(\underline{\omega}^+, x) = \max_{n \geqq 0} |f_{\tau^n \omega}|_{C^2} e^{-\varepsilon n}$$

into the usual definition of l.

Using the charts $\phi_n(\underline{\omega}^+, x)$, $n = 0, 1, 2, \ldots$, and standard graph transform methods, we construct a local stable manifold at $(\underline{\omega}^+, x)$. These and other results are summarized below.

Let $\Lambda = \{(\underline{\omega}^+, x) \in \Lambda(\lambda, k) : l(\underline{\omega}^+, x) \leqq l_0\}$ for some fixed l_0. We assume $(\nu^{\mathbb{N}} \times \mu)\Lambda > 0$, so that if $\Lambda_{\underline{\omega}^+} = \{x \in M : (\underline{\omega}^+, x) \in \Lambda\}$ then $\mu\Lambda_{\underline{\omega}^+} > 0$ for a set of $\underline{\omega}^+$ with positive $\nu^{\mathbb{N}}$-measure.

(4.1.2) **Proposition.** *Let Λ be as above*

(1) *If $\Lambda_{\underline{\omega}^+} \neq \emptyset$, then $x \to E^s(\underline{\omega}^+, x)$ is uniformly continuous on $\Lambda_{\underline{\omega}^+}$ for each fixed $\underline{\omega}^+$.*

(2) *$\exists \alpha = \alpha(\lambda, k, l_0)$ s.t. for each $(\underline{\omega}^+, x) \in \Lambda$, there is an embedded disk $W^s_\alpha(\underline{\omega}^+, x)$ in M with the following properties:*

(a) *$W^s_\alpha(\underline{\omega}^+, x) = \{y \in M : d^s(x, y) \leqq \alpha\}$ where*

$$d^s = \textit{distance along } W^s(\underline{\omega}^+, x);$$

(b) *$\exp_x^{-1} W^s_\alpha(\underline{\omega}^+, x)$ is part of the graph of a function*

$$g_{(\underline{\omega}^+, x)} : E^s(\underline{\omega}^+, x) \to E^s(\underline{\omega}^+, x)^\perp$$

with

$$g(\underline{\omega}^+, x)(0) = 0$$

and

$$|Dg_{(\omega^+, x)}| \leqq \frac{1}{1000}$$

(c) *d^s-radius of $f^n_{\underline{\omega}^+} W^s_\alpha(\underline{\omega}^+, x) \leqq e^{-n(\lambda - \varepsilon)}$ for all sufficiently large n.*

The map $x \to W_\alpha(\underline{\omega}^+, x)$ is uniformly continuous on each $\Lambda_{\underline{\omega}^+}$.

(4.2) *Absolute Continuity of the Stable Foliation*

Let $\Lambda \subset \Lambda(\lambda, k)$ be as in (4.1), and let $\underline{\omega}^+$ with $\Lambda_{\underline{\omega}^+} \neq \emptyset$ be fixed for now. For $x \in \Lambda_{\underline{\omega}^+}$ and $\delta > 0$, let $\mathfrak{n}_\delta(\Lambda_{\underline{\omega}^+}, x) = \{y \in \Lambda_{\underline{\omega}^+} : d(x, y) < \delta\}$. Suppose T_1 and T_2 are codimension k disks embedded in M in a small-neighborhood of x. Assume that they are transverse to $W_\alpha^s(\underline{\omega}^+, x)$ and each intersects $W_\alpha^s(\underline{\omega}^+, x)$ in exactly one point. Then by Proposition 4.1.2 they intersect $W_\alpha^s(\underline{\omega}^+, y)$ in the same way for all $y \in \mathfrak{n}_\delta(\Lambda_{\underline{\omega}^+}, x)$ provided δ is sufficiently small. One can then define the Poincaré map θ from a subset of T_1 to T_2 by sliding along $\{W_\alpha^s(\underline{\omega}^+, y), y \in \mathfrak{n}_\delta(\Lambda_{\underline{\omega}^+}, x)\}$. Recall that θ is called absolutely continuous if θ^{-1} of sets of Lebesgue measure zero in T_2 are sets of Lebesgue measure zero in T_1.

(4.2.1) **Proposition.** *Let Λ be as before. Then $\exists \delta_0,\ \delta_1 : \Omega^{\mathbb{N}} \to \mathbb{R}^+$ s.t. for a.e. $(\underline{\omega}^+, x) \in \Lambda$, if T_1 and T_2 are $\exp_x$-images of small disks in T_xM that are roughly parallel to $E^s(\underline{\omega}^+, x)^\perp$ and are $\leqq \delta_1(\underline{\omega}^+)$ distance away from it, then the Poincaré map from T_1 to T_2 by sliding along $W_\alpha^s(\underline{\omega}^+, y), y \in \mathfrak{n}_{\delta_0(\underline{\omega}^+)}(\Lambda_{\underline{\omega}^+}, x)$ is absolutely continuous.*

We refer the reader to [BN] or [KS] for more precise estimates and for a detailed proof. The main difference between our situation here and that in these papers is that the modulus of continuity of $x \to W_\alpha^s(\underline{\omega}^+, x)$, $x \in \Lambda_{\underline{\omega}^+}$, may a priori depend on $\underline{\omega}^+$. Hence we have written δ_0 and δ_1 as functions of $\underline{\omega}^+$.

This absolute continuity property of the stable foliation will be used as follows. Let ξ be a measurable partition of $\Omega^{\mathbb{N}} \times M$ subordinate to W^s, i.e. for $(\nu^{\mathbb{N}} \times \mu)$-a.e. $(\underline{\omega}^+, x)$, $\xi_{\underline{\omega}^+}(x) \subset W^s(\underline{\omega}^+, x)$ and contains an open neighborhood of x in $W^s(\underline{\omega}^+, x)$.

(4.2.2) **Corollary.** *Assume $\mu \ll$ Leb. and let m be a probability measure on $\Omega^{\mathbb{N}} \times M$ s.t. $m \ll \nu^{\mathbb{N}} \times \mu$. Let ξ be a measurable partition subordinate to W^s. Then there is a measurable function $\varrho : \Omega^{\mathbb{N}} \times M \to \mathbb{R}$ s.t. m-a.e. We have*

$$m_x^{\xi_{\underline{\omega}^+}} = \varrho(\underline{\omega}^+, x)\, \lambda_{(\underline{\omega}^+, x)}$$

where $\lambda_{(\underline{\omega}^+, x)}$ denotes the Riemannian measure on $W^s(\underline{\omega}^+, x)$.

See [Pe] for a proof of a similar result.

(4.3) *Construction of a Partition*

Let $\mathfrak{a}^+$ denote the σ-algebra of subsets of $\Omega^{\mathbb{N}} \times M$ generated by cylinders of $\Omega^{\mathbb{N}}$, and let $\sigma(\xi)$ be the σ-algebra generated by elements of the partition ξ. For $X \subset \Omega^{\mathbb{N}} \times M$, we write $W^s(X) = U\{W^s(\underline{\omega}^+, x), (\underline{\omega}^+, x) \in X\}$.

(4.3.1) **Proposition.** *There exists a measurable partition ξ on*

$$\Delta = \bigcup_{\substack{\lambda > 0 \\ 0 < k < \dim M}} W^s(\Lambda(\lambda, k)) \text{ s.t.}$$

(1) $\mathfrak{a}^+ \subset \sigma(\xi)$,
(2) *ξ is subordinate to W^s,*
(3) *ξ is decreasing, i.e., $(F^+)^{-1}\xi < \xi$.*

Our construction essentially follows that in [LS], but since there are some differences we will go over the entire argument.

Proof. It suffices to write $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$, where each Δ_n satisfies $(F^+)^{-1}\Delta_n = \Delta_n$ and $W^s(\Delta_n) = \Delta_n$, and so construct a partition ξ_n on each Δ_n. For then we can set $\xi = \xi_n$ on $\Delta_n - \left(\bigcup_{i=1}^{n-1} \Delta_i\right)$. These Δ_n's will be specified in the course of the proof.

We fix $\lambda>0$, $0<k<\dim M$, $l_0>0$, and let $\Lambda=\Lambda(\lambda,k)\cap\{l\leqq l_0\}$ be as in (4.1). We assume $(v^{\mathbb{N}}\times\mu)\Lambda>0$ and let α be s.t. $W^s_\alpha(\underline{\omega}^+,x)$, $(\underline{\omega}^+,x)\in\Lambda$, have the properties stated in Proposition 4.1.2.

Next we choose r_1, r_2 and β with $0<\beta\ll r_1<r_2<\frac{1}{2}\alpha$, and let $p_1,\dots,p_{N_1}$ be a β-dense subset of M. Letting $Gr(m,n)$ denote the space of n-dimensional subspaces of $\mathbb{R}^m$, we choose a cover $\{Q_1,\dots,Q_{N_2}\}$ of $Gr(\dim M,k)$ by balls of radius $\frac{1}{1000}$. For each p_i and Q_j we will construct one of the ξ_n's mentioned in the first paragraph of the proof. So let us fix $p=\text{some}\,p_i$ and $Q=\text{some}\,Q_j$.

A number $r\in[r_1,r_2]$ has to be chosen carefully. We will come back to this later. Using this r, we now define a partition η on $\Omega^{\mathbb{N}}\times M$ with $\sigma(\eta)\supset\mathfrak{a}^+$ by specifying $\eta_{\underline{\omega}^+}$ for each $\underline{\omega}^+$. First let

$$S=S(\Lambda,p,Q)=\{(\underline{\omega}^+,x)\in\Lambda: x\in B(p,\beta)\quad\text{and}\quad E^s(\underline{\omega}^+,x)\in Q\}.$$

(Here $B(p,\beta)$ denotes the β-ball centered at p in M, and using local coordinates around p, $E^s(\underline{\omega}^+,x)\in Q$ makes sense.). If $S_{\underline{\omega}^+}=S\cap(\{\underline{\omega}^+\}\times M)=\emptyset$, we let $\eta_{\underline{\omega}^+}=\{M\}$. If not, we define for $x\in S_{\underline{\omega}^+}$

$$\eta_{\underline{\omega}^+}(x)=W^s_\alpha(\underline{\omega}^+,x)\cap B(p,r)$$

and let

$$\eta_{\underline{\omega}^+}=\{\eta_{\underline{\omega}^+}(x),x\in S_{\underline{\omega}^+}\}\cup\left\{M-\bigcup_{x\in S_{\underline{\omega}^+}}\eta_{\underline{\omega}^+}(x)\right\}.$$

We leave it to the reader to verify that η so defined is a measurable partition.

Now let

$$\tilde{\xi}=\mathfrak{a}^+\vee\bigvee_{n=0}^{\infty}(F^+)^{-n}\eta,$$

i.e.,

$$\xi_{\underline{\omega}^+}(x)=\bigcap_{n\geqq 0}(f^n_{\underline{\omega}^+})^{-1}(\eta_{\tau^n\underline{\omega}^+}(f_{\underline{\omega}^+}x)).$$

One checks easily that if $\Delta_n=W^s\{(\underline{\omega}^+,x):(F^+)^n(\underline{\omega}^+,x)\in S(\Lambda,p,Q)\text{ i.o. for }n\geqq 0\}$, then $\xi_n=\hat{\xi}|\Delta_n$ satisfies $(\xi_n)_{\underline{\omega}^+}(x)\subset W^s(\underline{\omega}^+,x)$.

It remains to choose r in such a way that for a.e. $(\underline{\omega}^+,x)\in\Delta_n$, $\exists\,\delta(\underline{\omega}^+,x)>0$ s.t. $W^s_{\delta(\underline{\omega}^+,x)}(\underline{\omega}^+,x)\subset\xi_{\underline{\omega}^+}(x)$. We need only to do this for a.e. $(\underline{\omega}^+,x)\in\Lambda\cap\Delta_n$ and here the situation differs slightly from that in [LS].

By Proposition 4.1.2, $\delta(\underline{\omega}^+,x)$ exists if

$$d^s(f^n_{\underline{\omega}^+}x,\partial B(p,r))>e^{-n\lambda}$$

for all large n. We define the γ-boundary of $B(p,r)$ to be

$$\partial_\gamma B(p,r)=\{x\in M: r-\gamma<d(x,p)<r+\gamma\}.$$

Then the condition above is satisfied for $(\underline{\omega}^+,x)$ if

$$f^n_{\underline{\omega}^+}x\notin\partial_{2e^{-n\lambda}}B(p,r)$$

for all large n. Or, equivalently,

$$x\notin(f^n_{\underline{\omega}^+})^{-1}\partial_{2e^{-n\lambda}}B(p,r).$$

The Borel-Cantelli lemma tells us that this holds for μ-a.e. $x \in \Lambda_{\underline{\omega}^+}$ if we can show that

$$\sum_{n=0}^{\infty} \mu[(f^n_{\underline{\omega}^+})^{-1} \partial_{2e^{-n\lambda}} B(p,r)] < \infty . \tag{**}$$

To arrange for (**), we fix a number a with $e^{-\lambda} < a < 1$ and define

$$m_{\underline{\omega}^+} = \frac{1}{1-a} \sum_{n=0}^{\infty} a^n (f^n_{\underline{\omega}^+} \mu) .$$

Then $\{m_{\underline{\omega}^+}, \underline{\omega}^+ \in \Omega^{\mathbb{N}}\}$ is a measurable family of probability measures on M. For $r \in [r_1, r_2]$, let

$$C(\underline{\omega}^+, x) = \sup_{n \geqq 0} e^{n\lambda} m_{\underline{\omega}^+} \partial_{e^{-n\lambda}} B(p,r) .$$

Then for each $\underline{\omega}^+$, $C(\underline{\omega}^+, x) < \infty$ for Leb.-a.e. r (see [LS]). So by Fubini's theorem, Leb.-a.e. $r \in [r_1, r_2]$ has the property that $C(\underline{\omega}^+, x) < \infty$ for a.e. $\underline{\omega}^+$. Select one such r.

Now for a.e. $\underline{\omega}^+$,

$$\begin{aligned}(f^n_{\underline{\omega}^+} \mu) \partial_{2e^{-n\lambda}} B(p,r) &\leqq a^{-n}(1-a) m_{\underline{\omega}^+} \partial_{2e^{-n\lambda}} B(p,r) \\ &\leqq 2(1-a)\, C(\underline{\omega}^+, r) \left(\frac{e^{-\lambda}}{a}\right)^n\end{aligned}$$

which is summable over n. Hence (**) holds.

This completes the proof of Prop. 4.3.1.

5. Proof of Theorem A

(5.1) *Notations and Overall Strategy*

First, by Proposition 2.4.2(i), $\mu\{x : \lambda_i(x) > 0\ \forall i\} = 0$. The set $\{x : \lambda_i(x) \leqq 0\ \forall i\}$ can have positive μ-measure, but nothing needs to be proved for these invariant sets. This together with Proposition 1.4.1 allows us to assume without loss of generality that for μ-a.e. x,

$$0 < \dim E^s < \dim M .$$

Our proof will be carried out using the 2-sided skew product $F : (\Omega^{\mathbb{Z}} \times M, \mu^*)$. For the convenience of the reader we now make a list of the notations that will be used, including some that have been introduced before:

- $\mathfrak{a}$ $= \sigma$-algebra of subsets of $\Omega^{\mathbb{Z}} \times M$ generated by ω_n, $n \in \mathbb{Z}$
- $\mathfrak{a}^+$ $=$ sub-σ-algebra of $\mathfrak{a}$ generated by ω_n, $n \geqq 0$
- $\mathfrak{B}$ $= \sigma$-algebra of subset of $\Omega^{\mathbb{Z}} \times M$ generated by Borel subsets of M
- $\mathfrak{I}$ $=$ sub-σ-algebra of $\mathfrak{a} \vee \mathfrak{B}$ consisting of F-invariant sets
- I $=$ information function (with respect to μ^*)

- E = expectation (with respect to μ^*)
- λ = Riemannian measure on M
- $\lambda_{(\underline{\omega},x)}$ = Riemannian measure on $W^s(\underline{\omega},x)$

If m is a probability on $\mathfrak{a} \vee \mathfrak{B}$ and $\mathfrak{C}$ is a sub-algebra of $\mathfrak{a} \vee \mathfrak{B}$, then

- $m^{\mathfrak{C}}_{(\underline{\omega},x)}$ = conditional measure given $\mathfrak{C}$
- $m|\mathfrak{C}$ = m restricted to $\mathfrak{C}$

In particular, $m|\mathfrak{B}$ is identified with a measure on M.

Let ξ^+ be a partition on $\Omega^{\mathbb{N}} \times M$ of the type constructed in (4.3), and let ξ be the partition on $\Omega^{\mathbb{Z}} \times M$ defined by

$$\xi(\underline{\omega},x) = \xi^+(\underline{\omega}^+,x)$$

where $\underline{\omega}^+$ denotes the positive coordinates of $\underline{\omega}$. That is, ξ is a measurable partition with

(1) $\mathfrak{a}^+ \subset \sigma(\xi) \subset \mathfrak{a}^+ \vee \mathfrak{B}$,

(2) ξ is subordinate to W^s, i.e. $\xi_{\underline{\omega}}(x) \subset W^s(\underline{\omega},x)$ etc. for μ^*-a.e. $(\underline{\omega},x)$,

(3) $F\xi > \xi$.

Entropy will be estimated via

$$\lim_{n\to\infty} \int \frac{1}{n} I(F^n\xi \,|\, \xi \vee F^n\mathfrak{a}^+)\,d\mu^*.$$

To see that this gives the correct number, suppose for now that $\int I(F\xi|\xi \vee F\mathfrak{a}^+)\,d\mu^* < \infty$. Then

$$\begin{aligned}
&\frac{1}{n} I(F^n\xi|\xi \vee F^n\mathfrak{a}^+)(\underline{\omega},x)\\
&= \frac{1}{n}\sum_1^n I(F^i\xi|F^{i-1}\xi \vee F^n\mathfrak{a}^+)(\underline{\omega},x)\\
&= \frac{1}{n}\sum_1^n I(F\xi|\xi \vee F^{n-i+1}\mathfrak{a}^+)F^{-i+1}(\underline{\omega},x)\\
&\xrightarrow{L^1} E(I(F\xi|\xi \vee \mathfrak{a})|\mathfrak{I}).
\end{aligned}$$

The integral of this limit is $\leqq h$ by a standard argument. The L^1-convergence is valid because our integrability assumption above guaranties that

$$\int \sup_{k\geqq 0} I(F\xi|(\xi \vee F\mathfrak{a}^+) \vee F^k\mathfrak{a}^+)\,d\mu^* < \infty.$$

What we need to do then is to show:

$$\int I(F\xi|\xi \vee F\mathfrak{a}^+)\,d\mu^* < \infty$$

and

$$\lim_{n\to\infty} \int \frac{1}{n} I(F^n\xi\,|\,\xi \vee F^n\mathfrak{a}^+)d\mu^* \geqq \int \Sigma\lambda_i^+ m_i d\mu\,.$$

Both of these estimates will come out of the same arguments.

(5.2) *Entropy Computation (minus a few details)*

By definition,

$$I(F^n\xi\,|\,\xi \vee F^n\mathfrak{a}^+)(\underline{\omega}, x)$$

$$= -\log \mu^{*\xi \vee F^n\mathfrak{a}^+}_{(\underline{\omega}, x)}(F^n\xi)(\underline{\omega}, x)\,.$$

Observe that

(i) $F^n\mathfrak{a}^+$ is the σ-algebra generated by ω_k, $k \geqq -n$, so that

$$\mu^{*F^n\mathfrak{a}^+}_{\underline{\omega}}\,|\,\mathfrak{B} = f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}}\mu$$

$$= (f^n_{\tau^{-n}\underline{\omega}})\,\mu\,.$$

(ii) $\mu^{*\xi \vee F^n\mathfrak{a}^+}_{(\underline{\omega}, x)}(F^n\xi)(\underline{\omega}, x)$

$= (f^n_{\tau^{-n}\underline{\omega}}\mu)^{\xi_{\underline{\omega}}}_x (f^n_{\tau^{-n}\underline{\omega}}\xi_{\tau^{-n}\underline{\omega}})\,.$

(iii) Since $f^n_{\tau^{-n}\underline{\omega}}\mu \ll \mu \ll \lambda$, Proposition 4.2.2 says we can define

$$X = \frac{d(f^n_{\tau^{-n}\underline{\omega}}\mu)^{\xi_{\underline{\omega}}}_x}{d\lambda_{(\underline{\omega}, x)}}$$

and

$$Y = \frac{d(f^n_{\tau^{-n}\underline{\omega}}\mu)^{f^n_{\tau^{-n}\underline{\omega}}\xi_{\tau^{-n}\underline{\omega}}}_x}{d\lambda_{(\underline{\omega}, x)}}$$

Moreover, we have

(iv) $(f^n_{\tau^{-n}\underline{\omega}}\mu)^{\xi_{\underline{\omega}}}_x (f^n_{\tau^{-n}\underline{\omega}}\xi_{\tau^{-n}\underline{\omega}}) = \dfrac{X}{Y}.$

We compute X and Y separately.

Since $f^n_{\tau^{-n}\underline{\omega}}\mu \ll \mu$ for a.e. $\underline{\omega}$, we can write

$$X = \frac{d(f^n_{\tau^{-n}\underline{\omega}}\mu)^{\xi_{\underline{\omega}}}_x}{d\mu^{\xi_{\underline{\omega}}}_x} \cdot \frac{d\mu^{\xi_{\underline{\omega}}}_x}{d\lambda_{(\underline{\omega}, x)}}.$$

The first term $= \dfrac{d(f^n_{\tau^{-n}\underline{\omega}}\mu)}{d\mu} \cdot \dfrac{1}{Z_n}$ where

$$Z_n(\underline{\omega}, x) = \int \frac{d(f^n_{\tau^{-n}\underline{\omega}}\mu)}{d\mu}(y)\,\mu^{\xi_{\underline{\omega}}}_x(dy)$$

is the normalizing factor. Writing

$$\varrho(\underline{\omega}, x)=\frac{d\mu_x^{\xi_\omega}}{d\lambda_{(\underline{\omega}, x)}},$$

we have

$$X=\frac{d(f_{\tau^{-n}\underline{\omega}}^n \mu)}{d\mu} \cdot \frac{1}{Z_n} \cdot \varrho \tag{1}$$

Next, by the change of variables formula we have

$$Y(\underline{\omega}, x)=\frac{d\mu_{(f_{\tau^{-n}\underline{\omega}}^n)^{-1} x}^{\xi_{\tau^{-n}\underline{\omega}}}}{d\lambda_{F^{-n}(\underline{\omega}, x)}} \cdot \frac{1}{J_n^s F^{-n}(\underline{\omega}, x)}$$

where

$$J_n^s(\underline{\omega}, x)=|\operatorname{Jac}(f_{\underline{\omega}}^n)(x) \mid E^s(\underline{\omega}, x)| .$$

Thus

$$Y=\varrho \circ F^{-n} \cdot \frac{1}{J_n^s \circ F^{-n}} . \tag{2}$$

Combining (1) and (2) we have

$$\frac{1}{n} I(F^n \xi \mid \xi \vee F^n \mathfrak{a}^+)=A_n+B_n+C_n+D_n$$

where

$$A_n(\underline{\omega}, x)= -\frac{1}{n} \log \frac{d(f_{\tau^{-n}\underline{\omega}}^n \mu)}{d\mu}(x)$$

$$B_n(\underline{\omega}, x)= -\frac{1}{n} \log \frac{\varrho(\underline{\omega}, x)}{\varrho \circ F^{-n}(\underline{\omega}, x)},$$

$$C_n(\underline{\omega}, x)= -\frac{1}{n} \log J_n^s F^{-n}(\underline{\omega}, x)$$

and

$$D_n(\underline{\omega}, x)=\frac{1}{n} \log Z_n(\underline{\omega}, x) .$$

We state two lemmas, the proofs of which are postponed to the next subsection.

(5.3.2) **Lemma.** (a) $A_1 \in L^1(\mu^*)$;

(b) $\int A_n d\mu^*=\int \Sigma \lambda_i m_i d\mu \ \forall n$.

(5.3.3) **Lemma.** (a) $D_1 \in L^1(\mu^*)$;

(b) $\int D_n d\mu^* \geqq 0 \ \forall n$.

Assuming these lemmas we now complete the proof. Note that

$$I(F\xi \mid \xi \vee F\mathfrak{a}^+)=A_1+B_1+C_1+D_1$$

and is always $\geqq 0$. We know from the lemmas above that $A_1, D_1 \in L^1(\mu^*)$. Also, $\int C_n d\mu^*=-\int \Sigma \lambda_i^- m_i$ (where $a^-=\min(a, 0)$) for all n. This forces $B_1^- \in L^1(\mu^*)$, and by a standard argument (which we have stated as Lemma 5.3.1) we have $B_1 \in L^1(\mu^*)$ as well.

In fact, it follows from Lemma 5.3.1 that $\int B_1 \, d\mu^* = 0$. Since $B_n = \frac{1}{n}\sum_0^{n-1} B_1 \circ F^{-i}$, we have $\int B_n d\mu^* = 0$ for all n. This together with part (b) of Lemmas 5.3.2 and 5.3.3 gives

$$\int \frac{1}{n} I(F^n \xi \mid \xi \vee F^n \alpha^+) \geqq \int \Sigma \lambda_i m_i - \int \Sigma \lambda_i^- m_i$$

$$= \int \Sigma \lambda_i^+ m_i .$$

Letting $n \to \infty$, we arrive at the desired conclusion.

(5.3) *Details*

(5.3.1) **Lemma.** *Let* $f:(X,m) \circlearrowleft$ *be a measure preserving transformation. Let* $g: X \to \mathbb{R}^+$ *be measurable and define*

$$G = -\log \frac{g}{g \circ f} .$$

If $G^- \in L^1(m)$, *then* $G \in L^1(m)$ *and* $\int G\, dm = 0$.

Proof. Exercise.

(5.3.2) **Lemma**

(a) *Let*

$$\phi(\underline{\omega}, x) = \log \frac{df_{\omega_{-1}}\mu}{d\mu}(x) .$$

Then $\phi \in L^1(\mu^*)$ *and* $\int \phi \, d\mu^* = -\int \Sigma \lambda_i m_i d\mu$.

(b) $\int -\frac{1}{n} \log \frac{d(f^n_{\tau^{-n}\underline{\omega}}\mu)}{d\mu} \, d\mu^* = \int \Sigma \lambda_i m_i d\mu .$

Proof. First,

$$\int \phi^- d\mu^* = \int \left[\int \left(\log \frac{df_{\omega_{-1}}\mu}{d\mu} \right)^- df_{\omega_{-1}}\mu \right] d\nu(\omega_{-1})$$

$$= \int \left[\int \left(\frac{df_{\omega_{-1}}\mu}{d\mu} \log \frac{df_{\omega_{-1}}\mu}{d\mu} \right)^- d\mu \right] d\nu(\omega_{-1})$$

$$> -\infty .$$

Let us write

$$\varrho = \frac{d\mu}{d\lambda}, \quad J = |\mathrm{Jac}\,(f_{\omega_0})|$$

and let

$$\Pi : \Omega^{\mathbb{Z}} \times M \to M$$

be projection onto the second factor. Then

$$\phi=\log\left(\frac{\bar{\varrho}\circ\Pi\circ F^{-1}}{J\circ F^{-1}}\cdot\frac{1}{\bar{\varrho}\circ\Pi}\right).$$

Since $\int\phi^- d\mu^*>-\infty$ and $\int\log J=\int\Sigma\lambda_i m_i$, we have

$$\left(\log\frac{\bar{\varrho}\circ\Pi F^{-1}}{\bar{\varrho}\circ\Pi}\right)^-\in L^1(\mu^*)$$

and Lemma 5.3.1 applies to give (a).

To prove (b), observe that

$$-\frac{1}{n}\log\frac{df^n_{\tau^{-n}\underline{\omega}}\mu}{d\mu}(x)=-\frac{1}{n}\sum_0^{n-1}\phi\circ F^{-i}(\underline{\omega},x).$$

(5.3.3) **Lemma.** *Let*

$$Z_n(\underline{\omega},x)=\int\frac{d(f^n_{\tau^{-n}\underline{\omega}}\mu)}{d\mu}(y)\mu_x^{\xi_{\underline{\omega}}}(dy).$$

Then (a) $\log Z_1\in L^1(\mu^*)$;
(b) $\int\log Z_n d\mu^*\geqq 0$.

Proof. Observe that

$$Z_n(\underline{\omega},x)=\frac{d(f^n_{\tau^{-n}\underline{\omega}}\mu|\xi_{\underline{\omega}})}{d(\mu|\xi_{\underline{\omega}})}(x),$$

so that

$$\begin{aligned}\frac{1}{n}\log Z_n(\underline{\omega},x)&=\frac{1}{n}\sum_0^{n-1}\log\frac{d(f^{i+1}_{\tau^{-(i+1)}\underline{\omega}}\mu|\xi_{\underline{\omega}})}{d(f^i_{\tau^{-i}\underline{\omega}}\mu|\xi_{\underline{\omega}})}\\&=\frac{1}{n}\sum_0^{n-1}X_i\circ F^{-i}(\underline{\omega},x)\end{aligned}$$

where

$$X_i(\underline{\omega},x)=\log\frac{d(f_{\omega_{-1}}\mu|\eta^i_{\underline{\omega}})}{d(\mu|\eta^i_{\underline{\omega}})}(x)$$

and

$$\eta^i=\mathfrak{a}^+\vee F^{-i}\xi.$$

Note that $\int X_i d\mu^*$ can be written as

$$\int\left[\int\theta\left(E_\mu\left(\frac{df_{\omega_{-1}}\mu}{d\mu}\middle|\eta^i_{\underline{\omega}}\right)\right)d\mu\right]d\nu^{\mathbb{N}}(\omega_0\omega_1\ldots)$$

where $\theta(t)=t\log t$. Using the convexity of θ and the fact that $\mathfrak{a}^+<\sigma(\eta^i)<\mathfrak{a}^+\vee\mathfrak{B}$, we have

$$0\leqq\int X_i d\mu^*\leqq\int\log\frac{df_{\omega_{-1}}\mu}{d\mu}d\mu^*,$$

this last integral being $<\infty$ by Lemma 5.3.2.

6. Consequences of Theorem A

Recall that for a single diffeomorphism preserving a Borel probability measure, Pesin's entropy formula implies a certain geometric property of the measure which in turn leads to other ergodic properties of the system. Indeed, in the case of a single diffeomorphism, Theorem B and C and their corollaries have been shown to be consequences of Theorem A (see Part I of [LY], [Led 1] and [Pe]). To complete this paper, we will adapt these proofs to the fiber transformations associated with (F, μ^*). This is fairly straightforward for the reader familiar with the techniques in the original papers. A few of the modifications needed are pointed out in this section.

(6.1) *Proof of Theorem B*

We recall the main steps in Part I of [LY], referring the reader to [LY] for details. Following [LY], we first reduce to the ergodic case.

(6.1.1) **Lemma.** *Consider the σ-algebra $\mathfrak{B}^u$ of measurable subsets A of $\Omega^{\mathbb{Z}} \times M$ such that for $v^{\mathbb{Z}}$-a.e. $\underline{\omega}$, the section $A_{\underline{\omega}}$ is a union of global unstable manifolds. Let $\mathfrak{N}$ be the σ-algebra of μ^*-negligible subsets and $\mathfrak{J}$ be the σ-algebra of F-invariant subsets of $\Omega^{\mathbb{Z}} \times M$. Then*

$$\mathfrak{J} \subset \sigma(\mathfrak{B}^u, \mathfrak{N}).$$

Proof. Let $C(M)$ denote the set of continuous real-valued functions on M. For $g \in C(M)$, let $\hat{g} : \Omega^{\mathbb{Z}} \times M \to \mathbb{R}$ be defined by

$$\hat{g}(\underline{\omega}, x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f_{\underline{\omega}}^{-i} x).$$

Clearly, $\hat{g}$ is $\mathfrak{B}^u$-measurable. It follows from Proposition 1.4.1 that the family $\{\hat{g}, g \in C(M)\}$ generates the σ-algebra of F-invariant sets (up to μ^*-negligible sets).

Lemma 6.1.1 allows us to assume that (F, μ^*) is ergodic, simplifying the proof a little. Next, we observe that if ξ is an increasing partition subordinate to W^u and satisfies

$$H_{\mu^*}(\xi | F\xi \vee \mathfrak{a}) = \Sigma \lambda_i^+ m_i,$$

then the argument in (6.1) of [LY] exactly as it is proves that μ^* is absolutely continuous on W^u. As with the diffeomorphism case, we will need to show that h can be attained using partitions subordinate to W^u.

Now the definition of entropy says that there exists finite entropy partitions P on $\Omega^{\mathbb{Z}} \times M$ s.t. $H_{\mu^*}(P^+ | FP^+ \vee \mathfrak{a})$ is arbitrarily near h. Using relativized versions of Lemma 3.1.2 and Lemma 3.2.1 in [LY], it suffices to construct

1. a partition P on $\Omega^{\mathbb{Z}} \times M$ with $H_\mu(P) < \infty$,
2. an increasing partition ξ on $\Omega^{\mathbb{Z}} \times M$ subordinate to W^u such that if

$$\eta_1 = \xi \vee P^+ \quad \text{and} \quad \eta_2 = P^+$$

then

$$H_{\mu^*}(\eta_1 | F\eta_1 \vee \mathfrak{a}) \quad \text{and} \quad H_{\mu^*}(\eta_2 | F\eta_2 \vee \mathfrak{a})$$

are arbitrarily close. The constructions of P and ξ follow essentially those in Sects. 2 and 3 of [LY]. The proof that these two entropies are close in our case is identical to that in Sect. 5 of [LY].

Perhaps a remark concerning Lyapunov charts is in order. In [LY] we used the fact that chart maps are uniformly bounded to simplify some of our estimates (second to last paragraph of (2.1) in [LY]). This is true using any of the standard changes of coordinates in the diffeomorphism case* but does not necessarily hold for random maps without a uniform bound on their C^1-norms. One can, however, get this bound for random maps using new norms defined as in Remark 2 at the end of the Appendix in [LY]. Remark 4.1.1 of this paper also applies.

(6.2) *Proof of Corollary to Theorem B*

The idea is as follows: Suppose m is a Borel probability measure on a manifold M and suppose W^u and W^s are two "foliations" on M defined m-a.e. and satisfying (i)–(iv) below:

(i) W^u and W^s are transversal to each other m-a.e. with $\dim W^u + \dim W^s = \dim M$;

(ii) m has absolutely continuous conditional measures on W^u in the sense of Definition 3.2.2;

(iii) W^s as a foliation is absolutely continuous in the sense of Proposition 4.2.1 and

(iv) the conditional measures of m on local leaves of W^s have a well defined dimension δ^s in the sense of Definition 3.2.3.

Then m has a well defined dimension and

$$\dim(m) = \dim W^u + \delta^s .$$

See [Led 2] for a proof.

It follows from Theorem B and (4.2) that when none of the exponents is zero, the scenario above applies to a.e. $\mu_{\underline{\omega}}$ – provided we can prove (iv). In the case of a single diffeomorphism this is Proposition 7.3.1 in [LY]. The proof here is completely parallel.

(6.3) *Proof of Theorem C*

First we explain why μ has at most a countable number of ergodic components. As noted before, the σ-algebra $\mathfrak{J}$ is contained in both $\sigma(\mathfrak{B}^u, \mathfrak{N})$ and $\sigma(\mathfrak{B}^s, \mathfrak{N})$, and a.e. $\mu_{\underline{\omega}}$ has properties (i), (ii) and (iii) in (6.2). These two observations put together imply that if $A \in \mathfrak{J}$ has positive μ^*-measure, then $A_{\underline{\omega}}$ has positive Lebesgue measure for a.e. $\underline{\omega}$.

We assume from now on that μ is ergodic. Consider the relative Pinsker σ-algebra Π of (F, μ^*), i.e., the σ-algebra of subsets A of $\Omega^{\mathbb{Z}} \times M$ with the property that

$$h_{\mu^*}(F, \{A, A^c\} | \mathfrak{a}) = 0 .$$

* The statement in [LY] that this followed from (ii) and (iii) of (2.1) is obviously inaccurate

Since there exists generating increasing and decreasing partitions subordinate to W^u and W^s respectively, Π is contained in both $\sigma(\mathfrak{B}^u, \mathfrak{N}, \mathfrak{a})$ and $\sigma(\mathfrak{B}^s, \mathfrak{N}, \mathfrak{a})$. The same reasoning as above then tells us that Π is a countable extension of $\mathfrak{a}$. It is easy to see that ergodic countable extensions are in fact finite extensions. (See e.g., [Pa].)

Next we quote the theorem from [Rud], which says that any ergodic finite extension of a Bernoulli shift is either a Bernoulli shift itself or it is the product of a Bernoulli shift and a finite system. In the latter case there is a partition $\{A_1, \ldots, A_k\}$ of M s.t. F permutes the sets $\{\Omega^{\mathbb{Z}} \times A_i\}$ and that $F^k|(\Omega^{\mathbb{Z}} \times A_i)$ is Bernoulli (1.4 again). Thus we need only to consider the case where $F: (\Omega^{\mathbb{Z}} \times M, \Pi, \mu^*)$ is Bernoulli.

To complete the argument we need to show that $F: (\Omega^{\mathbb{Z}} \times M, \mathfrak{a} \vee \mathfrak{B}, \mu^*)$ is relatively Bernoulli with respect to Π. The proof uses [T] and runs parallel to that in [Pe] or [Led 1].

Appendix

We describe here two classes of examples to which the results of this paper apply.

I. *Stochastic flows*

Let M be a compact manifold. It is well known that if X is a C^∞ vector field on M then X generates a flow $\{\Phi_t: M \to M, t \in \mathbb{R}\}$ the asymptotic properties of which are captured by the iteration of the diffeomorphism Φ_1.

We describe the analogous situation for stochastic differential equation. Let $X_0, X_1, \ldots, X_m$ be C^∞ vector fields on M and let $B_t = (B_t^1, \ldots, B_t^m)$ be a standard m-dimensional Brownian motion defined on some probability space $(\Omega, \mathfrak{F}, P)$. Consider the SDE

$$d\xi_t = X_0\, dt + \sum_{i=1}^{m} X_i(\xi_t) \circ dB_t^i \tag{1}$$

(where $\circ$ denotes the Stratonovich integral). It is proved that the solutions to (1) are Markov processes that can be represented by $\{\Phi_{t,\omega}: M \to M, t \geqq 0, \omega \in \Omega\}$ where $\Phi_{t,\omega} \in \mathrm{Diff}^\infty(M)$ for each t and ω, $\Phi_{t,\omega}$ varies continuously with t for fixed ω, and the transition probabilities $P_t(\cdot|x)$ are given by the distributions of $\omega \to \Phi_{t,\omega}(x)$. We refer the reader to [E] or [Ku] for the precise meaning of a "solution" to (1) and for the theorem we have just quoted.

Since the X_i's are time-independent, the law of this stochastic semi-flow from time s to time t $(s < t)$ depends only on the number $t - s$. Thus if ν is the distribution of $\{\Phi_{1,\omega}, \omega \in \Omega\}$ in $\mathrm{Diff}(M)$, then for fixed $n \in \mathbb{Z}^+$ the random diffeomorphisms $\Phi_{n,\omega}, \omega \in \Omega$, are simply products of n independent diffeomorphisms with law ν. In other words, we are in the situation of $\mathfrak{X}(M, \nu)$ as defined in (1.1).

Standard arguments in the subject show that the derivative conditions (*) in (1.1) are satisfied here. (See [C 1] or [Ki].)

Furthermore, the kernels $P_1(\cdot|x)$ are the time-one kernels of the PDE associated with (1), namely

$$u_t = Au$$

where

$$A = X_0 + \frac{1}{2} \sum_{i=1}^{m} X_i^2 .$$

If the operator A is elliptic – and generically this is the case when m is sufficiently large – then the $P_1(\cdot|x)$'s have C^∞ densities with respect to Lebesgue and the result of this paper apply. If M is connected and A is elliptic, then the stationary probability measure μ is in fact unique (see Remark 1.4.3, [IK]).

We mention also that computation of Lyapunov exponents (and hence entropy and dimension of sample measures) in terms of the generating vector fields have been successfully carried out for certain stochastic flows. (See e.g., [Le J] and the references in [C 2]).

k-Parameter Families of Maps

Let D^k be a disk in $\mathbb{R}^k$ and let ν be a probability measure on D^k s.t. $\nu \ll$ Leb. Let $G: D^k \to \mathrm{Diff}^2(M)$ be a smooth map. Then G induces a probability $\hat{\nu}$ on $\mathrm{Diff}^2(M)$ with which we can define $\mathfrak{X}(M, \hat{\nu})$. Using the implicit function theorem, one verifies that $P(\cdot|x)$ is $\ll$Leb. if the map $\phi_x: D^k \to M$ defined by $\phi_x(a) = G(a)x$ has the property that $D\phi_x(a)$ is surjective at ν-a.e. a.

We remark that this condition is easy to meet for all x when k is large, and that it holds for open sets of G's because M is compact and the rank of $D\phi_x(a)$ is lower semi-continuous in x and a.

Acknowledgements. We wish to thank the Universität Bielefeld, I.H.E.S., the Université Paris VI, and the University of Warwick for their hospitality.

References

[B] Breiman, L.: Probability. Reading, Massachusetts: Addison Wesley 1968

[Ba] Baxendale, P.: Brownian motion in the diffeomorphism group I. Compositio Mathematica **53**, 19–50 (1984)

[BN] Brin, M., Nitecki, Z.: Absolute continuity of stable foliation in Hilbert space. In preparation

[Bo] Bowen, R.: Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lect. Notes Math., vol. 470. Berlin Heidelberg New York: Springer 1975

[C 1] Carverhill, A.: Flows of stochastic dynamical systems: ergodic theory. Stochastics **14**, 273–317 (1985)

[C 2] Carverhill, A.: Survey: Lyapunov exponents for stochastic flows on manifolds. In: Arnold, L., Wihstutz, V. (eds.) Lyapunov Exponents. Proceedings, Bremen 1984. (Lect. Notes Math., vol. 1186, pp. 292–307) Berlin Heidelberg New York: Springer 1986

[E] Elworthy, K.D.: Stochastic differential equations on manifolds. L.M.S. Lecture Notes. Cambridge: Cambridge University Press 1982

[ER] Eckmann, J.P., Ruelle, D.: Ergodic theory of chaos and strange attractors. Rev. Modern Phys. **57**, 617–656 (1985)

[IK] Ichihara, K., Kunita, H.: A classification of the second order degenerate eliptic operators and its probabilistic characterization. Z. Wahrscheinlichkeitstheor. Verw. Geb. **30**, 253–254 (1974)

[Ki] Kifer, Y.: Ergodic theory of random transformations. Progress in probability and statistics. Basel Boston: Birkhäuser 1986

[KS] Katok, A., Strelcyn, J.M.: Smooth maps with singularities: invariant manifolds, entropy and billiards. Lect. Notes Math., vol. 1222. Berlin Heidelberg New York: Springer 1986

[KU] Kunita, H.: Stochastic differential equations and stochastic flow of diffeomorphisms. In: Hennequin, P.L. (ed.) Ecole d'Eté de Probabilités de Saint-Flour XII, 1982 (Lect. Notes Math., vol. 1097, pp. 144–303) Berlin Heidelberg New York: Springer 1984

[Led 1] Ledrappier, F.: Propriétés ergodiques des mesures de Sinai. Publ. Math., Inst. Hautes Etud. Sci. **59**, 163–188 (1984)

[Led 2] Ledrappier, F.: Dimension of invariant measures. Proceedings of the conference ergodic theory and related topics II. Georgenthal 1986. Teubner-Texte zur Mathematik **94**, 116–124 (1987)

[Led 3] Ledrappier, F.: Quelques propriétés des exposants caractéristiques. Lect. Notes Math., vol. 1097, pp. 305–396. Berlin Heidelberg New York: Springer 1984

[Le J] Le Jan, Y.: On isotropic Brownian motion. Z. Wahrscheinlichkeitstheor. Verw. Geb. **70**, 609–620 (1985)

[LS] Ledrappier, F., Strelcyn, J.-M.: A proof of the estimation from below in Pesin entropy formula. Ergodic Theory Dyn. Syst. **2**, 203–219 (1982)

[LY] Ledrappier, F., Young, L.-S.: The metric entropy of diffeomorphisms. Ann. Math. **122**, 509–574 (1985)

[M] Mañé, R.: A proof of Pesin's formula. Ergodic Theory Dyn. Syst. **1**, 95–102 (1981)

[Pa] Parry, W.: Entropy and generators in ergodic theory. Math. Lecture Notes Series. New York: Benjamin 1969

[Pe] Pesin, Ia.B.: Lyapunov characteristic exponents and smooth ergodic theory. Russ. Math. Surv. **32**, 55–114 (1977)

[Ro] Rohlin, V.A.: On the fundamental ideas of measure theory. Transl., I. Ser., Am. Math. Soc. **10**, 1–54 (1962)

[Rud] Rudolph, D.: If a finite extension of a Bernoulli shift has no finite rotation factors, it is Bernoulli. Isr. J. Math. **30**, 193–206 (1978)

[Rue] Ruelle, D.: Ergodic theory of differentiable dynamical systems. Publ. Math., Inst. Hautes Etud. Sci. **50**, 27–58 (1979)

[S] Sinai, Ya.: Classical dynamical systems with countable Lebesgue spectrum II. Izv. Akad. Nauk Arm. SSR, Mat. **30**, 15–68 (1966); Transl., II. Ser., Am. Math. Soc. **68**, 34–88 (1968)

[T] Thouvenot, J.P.: Remarques sur les systèmes dynamiques donnés avec plusieurs facteurs. Isr. J. Math. **21**, 215–232 (1975)

[Y] Young, L.-S.: Stochastic stability of Hyperbolic Attractors. Ergodic Theory Dyn. Syst. **6**, 31-319 (1986)

Received June 29, 1987; in revised form June 13, 1988