Real Variables Fall 2011 (Young) HW 9 Due Nov 14

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. For each Borel set A, define

$$\mu(A) = m(f^{-1}(A))$$

where m is Lebesgue measure on $\mathbb{R}$. Prove that μ is a measure on $(\mathbb{R}, \mathcal{B})$.

2. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $\mathcal{A}$ be an algebra that generates $\mathcal{B}$. Prove that give any $B \in \mathcal{B}$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(B \Delta A) < \varepsilon$.

3. Let $(X, \mathcal{A}, \mu_0)$ be a finitely additive measure on an algebra. The following condition on μ_0 is often referred to as "continuity from above at $\emptyset$":

$$(**) \qquad \text{if } A_n \in \mathcal{A},\ n = 1, 2, \ldots, \text{ are such that } A_n \downarrow \emptyset, \text{ then } \mu_0(A_n) \downarrow 0 \ .$$

Prove using the Caratheodory Extension Theorem that if $\mu_0(X) < \infty$, then under condition (**) μ_0 extends to a measure on $\sigma(\mathcal{A})$.

4. Given a measure space $(X, \mathcal{B}, \mu)$, prove that its completion $(X, \bar{\mathcal{B}}, \bar{\mu})$ is well defined. More precisely, let $\bar{\mathcal{B}}$ be the σ-algebra generated by

$$\mathcal{B} \cup \{A : A \subset B \text{ for some } B \in \mathcal{B} \text{ with } \mu(B) = 0\} \ .$$

Prove that there is a unique measure $\bar{\mu}$ on $\bar{\mathcal{B}}$ with $\bar{\mu}(B) = \mu(B)$ for all $B \in \mathcal{B}$.

5. Let $\mathcal{B}_R$ denote the σ-algebra on $\mathbb{R}^n$ generated by rectangles of the form $A_1 \times A_2 \times \cdots \times A_n$ where each $A_i = [a_i, b_i)$, and let $\mathcal{B}$ be the Borel σ-algebra, i.e. the σ-algebra of subsets of $\mathbb{R}^n$ generated by open sets. Prove that $\mathcal{B}_R = \mathcal{B}$.