Real Variables Fall 2011 (Young) HW 8 Due Nov 7

1. Let f be a bounded measurable function on $[0,1]$. Prove that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty \ .$$

2. Let $X \subset \mathbb{R}$ be a measurable set. Prove
   (a) Minkowski Inequality for $L^\infty(X)$, and
   (b) the completeness of $L^\infty(X)$.

3. Let $C[0,1]$ be the space of all continuous functions on $[0,1]$, and let $\|\cdot\|$ be the "sup norm", i.e.

$$\|f\| \ = \ \max_{x\in[0,1]} |f(x)| \ .$$

Prove that $C[0,1]$ is a Banach space.

4. Are continuous functions dense in $L^\infty([a,b])$? Prove or give a counterexample.

5. Let $f_\alpha(x) = x^{-\alpha}$, $\alpha \geq 0$.
   (a) For which p is $f_\alpha \in L^p([0,1])$?
   (b) For which p is $f_\alpha \in L^p([1,\infty))$?

6. Let $f_n, f \in L^p$, $1 \leq p < \infty$, be such that $f_n \to f$ a.e. Prove that $f_n \to f$ in L^p if and only if $\|f_n\| \to \|f\|$.

7. Deduce the following slightly more general version of Jensen's Inequality from the one in Royden (p.115): Let $f : [a,b] \to \mathbb{R}$ be integrable, and left $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then

$$\varphi(\text{Avg}(f)) \leq \text{Avg}(\varphi(f)) \qquad \text{where} \qquad \text{Avg}(g) := \frac{1}{b-a}\int_a^b g \ .$$