Real Variables Fall 2011 (Young) HW 6 Due Oct 24

1. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. Prove that
(a) $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist for all $c \in (a, b)$;
(b) f has at most a countable number of discontinuities.

2. For $x \in [-1, 1]$, let

$$f(x) = x \sin \frac{1}{x} \quad \text{for} \quad x \neq 0; \qquad f(0) = 0 .$$

Determine if f is (i) continuous, (ii) differentiable, and (iii) of bounded variation. What if $x \sin \frac{1}{x}$ is replaced by $x^2 \sin \frac{1}{x}$?

3. Let $E \subset \mathbb{R}$ be such that $m^*(E) < \infty$, and let $\mathcal{I}$ be a cover of E in the sense of Vitali. Prove that there is a countable subcover $\{I_n\}$ of $\mathcal{I}$ consisting of pairwise disjoint intervals such that

$$m^*(E \setminus \cup_1^\infty I_n) = 0 .$$

4. Let $C \subset [0, 1]$ be the middle third Cantor set. Define $f|_{[0,1]\setminus C}$ by

$$f|_{(\frac{1}{3},\frac{2}{3})} = \frac{1}{2} , \quad f|_{(\frac{1}{9},\frac{2}{9})} = \frac{1}{4}, \quad f|_{(\frac{7}{9},\frac{8}{9})} = \frac{3}{4} ,$$

$$f|_{(\frac{1}{27},\frac{2}{27})} = \frac{1}{8}, \quad f|_{(\frac{7}{27},\frac{8}{27})} = \frac{3}{8} , \quad f|_{(\frac{19}{27},\frac{20}{27})} = \frac{5}{8} , \qquad \text{and so on .}$$

(a) Prove that f can be extended to a continuous function on $[0, 1]$ (called the *Cantor ternary function*).
(b) Notice that $f' = 0$ a.e., so $\int_0^1 f'(x)dx \neq f(1) - f(0)$.
(c) What is $f(C)$?
(d) Give an example of a measurable function g with the property that the inverse images of Lebesgue measurable sets under g are not necessarily measurable.
Hint: Show that $g = f^{-1}$ is well defined on $[0, 1] \setminus D$ where D is a countable set.