Real Variables Fall 2011 (Young) HW 5 Due Oct 17

1. Let $f : D \to \mathbb{R}$ be an integrable function where D is any measurable set. Prove that given $\varepsilon > 0$, there is a simple function φ (with $m\{\varphi \neq 0\} < \infty$) such that

$$\int_D |f - \varphi| \; < \; \varepsilon \; .$$

2. Prove the following slight generalization of the Dominated Convergence Theorem: Let $f_n \to f$ a.e. Suppose there is a sequence of integrable functions g_n with the properties that
(i) $|f_n| \leq g_n$ for all n,
(ii) $g_n \to g$ a.e., and $\int g_n \to \int g$.
Then $\int f_n \to \int f$.

3. Let $f_n : D \to \mathbb{R}$ be nonnegative measurable functions, and assume $f_n \to f$ a.e. Suppose that $\int_D f_n \to \int_D f$, and these integrals are finite. Prove that for every measurable subset $E \subset D$,

$$\int_E f_n \to \int_E f \; .$$

4. Let $f_n : D \to \mathbb{R}$ be measurable. Compare the L^1 convergence and convergence in measure of f_n under each of the following two conditions:
(a) $|f_n| \leq M$ for all n;
(b) $\int |f_n| \leq M$.

5. Let f_n and f be integrable functions, and assume $f_n \to f$ a.e. Prove that

$$\int |f_n - f| \to 0 \quad \text{if and only if} \quad \int |f_n| \to \int |f| \; .$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable, and let g be defined by

$$g(x) := \int_{(-\infty, x)} f \; .$$

Prove that $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.