

Real Variables Fall 2011 (Young) HW 4 Due Oct 10

(No class on Oct 10. Please leave hw in my mailbox by 7pm.)

1. Let C be a Cantor set of positive measure as constructed in HW2, Problem 5, and let $f = \chi_C$. Given $\varepsilon > 0$, describe explicitly how you would construct a continuous function $g : [0, 1] \rightarrow \mathbb{R}$ with $m\{f \neq g\} < \varepsilon$.

2. Let $D \subset \mathbb{R}$ be a measurable set with $m(D) < \infty$, and let $f : D \rightarrow \mathbb{R}$ be a nonnegative measurable function. Prove that $f = 0$ a.e. if and only if $\int_D f = 0$.

3. Let $f : D \rightarrow \mathbb{R}$ be measurable and bounded, and assume $m(D) < \infty$. Prove that

$$\left| \int_D f \right| \leq \int_D |f|.$$

4. (a) Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function, and suppose $g = f$ a.e. Is g necessarily (i) Riemann integrable, (ii) Lebesgue integrable? Prove your assertions.

(b) Let $f = \chi_C$ be as in Problem 1. Prove that f is not Riemann integrable. (It is clearly Lebesgue integrable.)

5. In the Bounded Convergence Theorem, can the condition that there is a uniform bound for all the functions involved be replaced by “there exist $M_n, M > 0$ such that $|f_n| \leq M_n$ for all n and $|f| \leq M$ ”? Prove or give a counterexample.

6. Let $\{f_n\}$ be a sequence of nonnegative measurable functions converging a.e. to f . Prove that if $f_n \leq f$ a.e. for every n , then $\int f_n \rightarrow \int f$.

7. Prove the following generalization of Fatou’s Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions, then

$$\int \liminf f_n \leq \liminf \int f_n.$$